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A Second-Order Lagrangian Time Discretization with Variable Steps for Convection-Diffusion Problems

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Abstract We study a variable-step BDF2 time discretization along the trajectories of fluid particles for convection-diffusion problems. Let $r_n := \tau_n/\tau_{n-1}$ $(n \geq 2)$ denote the consecutive time-step ratio corresponding to the n-th time-step size τ_n, and let $R \in (0, 1+\sqrt{3})$ be a fixed constant. Under several assumptions, including a uniform bound $r_n \leq R$, we prove stability and convergence with second-order accuracy for the resulting time-discrete scheme in the $\ell^\infty(0,T;H_0^1(\Omega))$-norm.

Keywords. Lagrangian time discretization, variable-step BDF2, second-order, convergence.

1 Introduction

In the numerical analysis of time-dependent partial differential equations on non-uniform time grids, the variable BDF2 (the backward difference formula of degree two) method is an effective second-order time integrator. For a final time $T > 0$, let $0 = t_0 < \cdots < t_{N_T} = T$, $\tau_n := t_n - t_{n-1}$, and $r_n := \tau_n/\tau_{n-1}$ be the non-uniform time grid, time-step size, and consecutive time-step ratio, respectively. The conventional ODE theory establishes zero-stability of variable-step multistep methods on variable grids, and in particular of variable BDF formulas, under suitable restrictions on the ratios $\{r_n\}$, cf., e.g., [8, 10]. Motivated by these considerations, the variable BDF2 analysis for parabolic evolution equations typically assumes a uniform bound $r_n \leq R$ $(n \geq 2)$ with R, a method-dependent stability threshold. Under such ratio constraints, energy techniques yield $\ell^\infty(0,T;L^2(\Omega))$-type stability and second-order error estimates on nonuniform time grids for linear parabolic equations [2, 3], for semilinear parabolic equations [9], and for semilinear parabolic equations including reaction-diffusion equations as a special case [19]. Moreover, recent work further improves admissible ratio bounds for parabolic equations with self-adjoint elliptic part by employing refined multiplier arguments [1].

The convection-diffusion equation serves as a fundamental model in fluid dynamics and is often viewed as a simplified prototype for transport processes arising in incompressible flow simulations [13, 18]. Consequently, developing robust numerical schemes for this equation is a

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prerequisite for simulating complex fluid flows, as it provides essential insights into the behavior of transport phenomena driven by prescribed velocity fields [13].

However, in convection-dominated regimes, standard Eulerian Galerkin-type discretizations may suffer from spurious oscillations [6,7]. To address this issue, characteristic-based approaches, especially the Lagrange–Galerkin method, have been widely adopted [4,5,13,15–17]. This method incorporates the method of characteristics into the finite element framework by approximating the material derivative along characteristic trajectories [13, 15]. Such a Lagrangian treatment effectively decouples advection from the time evolution and typically reduces the core difficulty to solving diffusion-(reaction) type subproblems, which yields strong stability even for high-Péclet-number flows [14, 15, 17].

In this paper, we propose a second-order Lagrangian time discretization with variable steps for convection-diffusion problems and establish the theoretical foundation of the proposed method. Relying on the variable-step BDF theory on nonuniform grids [8, 10, 11] and the stability and error analyses for parabolic equations [1, 3, 9, 19], we rigorously prove stability and convergence of the scheme in $\ell^\infty(0,T;H^1_0(\Omega))$-norm under the standard step-variation restriction expressed by bounded step-size ratios. We note that our upper bound of r_n is larger than that in the zero-stability theory; $R_0 = 1+\sqrt{2}$ for the zero-stability and in our analysis $R_1 = 1+\sqrt{3}$. This is obtained by explicitly exploiting the ellipticity of the elliptic operator.

As far as we know, there are no theoretical results for the Lagrangian BDF(-type) time discretization with variable steps. For flow problems, especially with a high Péclet number, we need to take into account another constraint on the time-step size, the so-called CFL condition, which makes numerical schemes complicated. The Lagrangian time discretization has, at least for convection-diffusion problems, the significant property of being CFL-free. Although we consider only a semi-discretization in this paper, based on the existing literature on Lagrangian approaches, we expect that the Lagrangian time discretization relaxes the usual CFL restriction arising from the convection term, even when variable time steps are employed.

The remainder of this paper is organized as follows. Section 2 is devoted to the formulation of a Lagrangian time discretization with variable time steps. In Section 3, we state the main results concerning stability and convergence. Section 4 provides detailed proofs of the main results. Concluding remarks are given in Section 5.

2 A Lagrangian time discretization with variable steps

The function spaces and notation used throughout this paper are defined as follows. Let Ω be a bounded domain in $\mathbb{R}^d$ with $d = 1, 2$, or 3, and let $T > 0$ be a fixed final time. We use the Lebesgue spaces $L^p(\Omega)$ and the Sobolev spaces $W^{k,p}(\Omega)$, $H^k(\Omega)\ (= W^{k,2}(\Omega))$, $H^1_0(\Omega)$, and $W^{1,\infty}_0(\Omega) := \{f \in W^{1,\infty}(\Omega) \mid f = 0 \text{ on } \partial\Omega\}$, for $k \in \mathbb{N}\cup\{0\}$ and $p \in [1,\infty]$. For a normed space X, its norm and, if it exists, its seminorm are denoted by $\|\cdot\|_X$ and $|\cdot|_X$, respectively. For simplicity, we write $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$. For $m \in \mathbb{N}\cup\{0\}$ and a Banach space X, we define the function spaces $H^m(X) := H^m(0,T;X)$ and $C(X) := C([0,T];X)$. When no confusion arises, we omit the domains $(0,T)$ and Ω. For example, we write $C(W^{1,\infty})$ instead of $C([0,T];W^{1,\infty}(\Omega))$. For $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$, we introduce the function space $Z^m(t_0,t_1)$ defined by

$$Z^m(t_0,t_1) := \bigcap_{j=0}^{m} H^j(t_0,t_1;H^{m-j}(\Omega))$$

with the norm

$$\|f\|_{Z^m(t_0,t_1)} := \left(\sum_{j=0}^{m} \|f\|^2_{H^j(t_0,t_1;H^{m-j}(\Omega))} \right)^{1/2},$$

and set $Z^m := Z^m(0,T)$. We use c and c_u to denote generic positive constants, independent of $\{\tau_n\}_n$, and independent of and dependent on the velocity u, respectively. A superscript $'$ (prime) is often used to distinguish between two constants, such as c and c'.

We consider a convection-diffusion problem to find $\phi\colon \Omega \times (0,T) \to \mathbb{R}$ such that

$$\mathcal{D}_t\phi - \nu\Delta\phi = f \qquad \text{in } \Omega \times (0,T), \tag{2.1a}$$

$$\phi = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{2.1b}$$

$$\phi = \phi_0 \qquad \text{in } \Omega, \text{at } t = 0, \tag{2.1c}$$

where $\mathcal{D}_t$ denotes the material derivative defined by

$$\mathcal{D}_t := \frac{\partial}{\partial t} + u \cdot \nabla,$$

for a given velocity field $u\colon \Omega\times(0,T) \to \mathbb{R}^d$, $\nu > 0$ is a diffusion coefficient, and $f\colon \Omega\times(0,T) \to \mathbb{R}$ and $\phi_0\colon \Omega \to \mathbb{R}$ are given functions.

We consider a time-discretization for problem (2.1). Let $\{t_n\}_{n=0}^{N_T} \subset [0,T]$ with $0 = t_0 < t_1 < \cdots < t_{N_T} = T$ be a non-uniform time grid. Let $\tau_n := t_n - t_{n-1}$ $(1 \le n \le N_T)$ be the n-th time step size and $\tau := \max\{\tau_n;\ n = 1,\ldots,N_T\}$ the maximum. We introduce two parameters r_n and α_n defined by

$$r_n := \frac{\tau_n}{\tau_{n-1}}, \qquad \alpha_n := \frac{\tau_n}{\tau_n + \tau_{n-1}} = \frac{r_n}{1 + r_n},$$

which are the consecutive time-step ratio and its normalized counterpart, respectively. For a function g defined in $\Omega \times (0,T)$, we denote simply $g^n := g(\cdot,t_n)$. For $k = 0,1,2$, we introduce the k-th characteristic/upwind mappings $X^n_{-k}\colon \Omega \to \mathbb{R}^d$ with respect to the velocity field u at $t = t^n$ defined by

$$X^n_{-k}(x) := x - (t_n - t_{n-k})u^n(x),$$

where X^n_0 is nothing but the identity map, which is used in the analysis later. In the following, we use the symbol "$\circ$" to represent the composition of functions, i.e., for a function ψ defined in Ω,

$$[\psi \circ X^n_{-k}](x) := \psi(X^n_{-k}(x)).$$

Let ϕ^0_τ $(\approx \phi^0 = \phi_0)$ and ϕ^1_τ $(\approx \phi^1)$ be assumed to be given. Now, we present a second-order Lagrangian time discretization with variable steps for (2.1): find $\{\phi^n_\tau\colon \Omega \to \mathbb{R}; n = 2,\ldots,N_T\}$ such that, for $n = 2,\ldots,N_T$,

$$\widetilde{D}_2\phi^n_\tau - \nu\Delta\phi^n_\tau = f^n, \qquad \text{in } \Omega, \tag{2.2a}$$

$$\phi^n_\tau = 0, \qquad \text{on } \partial\Omega, \tag{2.2b}$$

where $\widetilde{D}_2$ is an approximation of $\mathcal{D}_t$ defined by, in general, for $\{\phi^n\colon \Omega \to \mathbb{R};\ n = 0,\ldots,N_T\}$ and $n = 2,\ldots,N_T$,

$$\begin{aligned}\widetilde{D}_2\phi^n &:= \frac{2\tau_n + \tau_{n-1}}{\tau_n(\tau_n + \tau_{n-1})}(\phi^n - \phi^{n-1} \circ X^n_{-1}) - \frac{\tau_n}{\tau_{n-1}(\tau_n + \tau_{n-1})}(\phi^{n-1} \circ X^n_{-1} - \phi^{n-2} \circ X^n_{-2}) \\ &= \frac{2\tau_n + \tau_{n-1}}{\tau_n(\tau_n + \tau_{n-1})}\phi^n - \frac{\tau_n + \tau_{n-1}}{\tau_n\tau_{n-1}}\phi^{n-1} \circ X^n_{-1} + \frac{\tau_n}{\tau_{n-1}(\tau_n + \tau_{n-1})}\phi^{n-2} \circ X^n_{-2}.\end{aligned}$$

Remark 1. *$\widetilde{D}_2\phi^n$ can be also written as, with r_n or α_n,*

$$\begin{aligned}\widetilde{D}_2\phi^n &= \frac{2r_n+1}{\tau_n(r_n+1)}(\phi^n\circ X_0^n - \phi^{n-1}\circ X_{-1}^n) - \frac{r_n}{\tau_{n-1}(r_n+1)}(\phi_\tau^{n-1}\circ X_{-1}^n - \phi_\tau^{n-2}\circ X_{-2}^n)\\ &= \frac{1+\alpha_n}{\tau_n}(\phi^n\circ X_0^n - \phi^{n-1}\circ X_{-1}^n) - \frac{\alpha_n}{\tau_{n-1}}(\phi_\tau^{n-1}\circ X_{-1}^n - \phi_\tau^{n-2}\circ X_{-2}^n).\end{aligned}$$

Remark 2. *The well-known BDF1 and BDF2 discretizations with variable steps for $\partial/\partial t$ are given by*

$$\begin{aligned}D_1\phi^n &:= \frac{1}{\tau_n}(\phi^n - \phi^{n-1}),\\ D_2\phi^n &:= \frac{2\tau_n+\tau_{n-1}}{\tau_n+\tau_{n-1}}D_1\phi^n + \frac{\tau_n}{\tau_n+\tau_{n-1}}D_1\phi^{n-1} = (1+\alpha_n)D_1\phi^n - \alpha_n D_1\phi^{n-1}.\end{aligned}$$

The discrete operator $\widetilde{D}_2$ is an approximation of $\mathcal{D}_t$ by using the idea of BDF2 along the trajectory of a fluid particle, and it is, therefore, obvious that $\widetilde{D}_2 = D_2$ when $u = 0$.

At the end of this section, we provide a weak formulation to problem (2.2). Let $\phi_\tau^0, \phi_\tau^1 \in \Psi := H_0^1(\Omega)$ be given. The weak formulation is to find $\{\phi_\tau^n\}_{n=2}^{N_T} \subset \Psi$ such that, for $n = 2, \ldots, N_T$,

$$\big(\widetilde{D}_2\phi_\tau^n, \psi\big) + a(\phi_\tau^n, \psi) = (f^n, \psi) \qquad \forall\psi\in\Psi, \tag{2.3}$$

where $a(\cdot,\cdot)$ is the bilinear form defined by

$$a(\phi,\psi) := \nu(\nabla\phi, \nabla\psi).$$

Remark 3. *Problem* (2.3) *is uniquely solvable by the Lax–Milgram theorem, if $\phi_\tau^0, \phi_\tau^1 \in L^2(\Omega)$, $f \in C([0,T]; L^2(\Omega))$, $u \in C([0,T]; W_0^{1,\infty}(\Omega)^d)$, and $\tau|u|_{C(W^{1,\infty})} \le 1/8$, where Proposition 7 below ensures that all upwind points of $x\in\Omega$ by the mappings X_{-k}^n ($k = 1, 2, n = 2, \ldots, N_T$) are in Ω and that their Jacobian values are close to* 1*. See also Remark 6 below.*

3 Main results

We start this section by setting hypotheses for the velocity u and reviewing previous results.

Hypothesis 4 (condition for the velocity)**.** *The function u satisfies $u \in C([0,T]; W_0^{1,\infty}(\Omega)^d)$.*

Hypothesis 5 (condition for steps)**.** *The set of step sizes $\{\tau_n\}_{n=1}^{N_T}$ satisfies the condition $(\tau_n + \tau_{n-1})|u^n|_{W^{1,\infty}(\Omega)} \le 1/4$ for $n = 2, \ldots, N_T$.*

Remark 6. *$\tau|u|_{C(W^{1,\infty})} \le 1/8$ is a sufficient condition for Hypothesis 5.*

Proposition 7 ([15, 18])**.** *(i) Under Hypothesis 4 and $(\tau_n + \tau_{n-1})|u^n|_{W^{1,\infty}(\Omega)} < 1$, it holds that $X_{-k}^n(\Omega) = \Omega$ for $n = 2, \ldots, N_T$ and $k = 1, 2$. (ii) Under Hypotheses 4 and 5, it holds that $1/2 \le \det(\nabla X_{-k}^n)(x) \le 3/2$ for $n = 2, \ldots, N_T$ and $k = (0,)1, 2$.*

3.1 Stability

For a given series of functions $\{\phi_\tau^n\}_{n=0}^{N_T}$, we define the following norms, for $i < j$,

$$\|\phi_\tau\|_{\ell^\infty(t_i,t_j;L^2)} := \max\{\|\phi_\tau^n\|;\ i \le n \le j\},$$

$$\|\phi_\tau\|_{\ell^2(t_i,t_j;L^2)} := \left\{ \sum_{n=i}^{j} \tau_n \|\phi_\tau^n\|^2 \right\}^{1/2}, \qquad \|\phi_\tau\|_{\ell^2_\alpha(t_i,t_j;L^2)} := \left\{ \sum_{n=i}^{j} \alpha_n \tau_n \|\phi_\tau^n\|^2 \right\}^{1/2},$$

and set $\|\phi_\tau\|_{\ell^\infty(L^2)} := \|\phi_\tau\|_{\ell^\infty(t_1,T;L^2)}$ and $\|\phi_\tau\|_{\ell^2(L^2)} := \|\phi_\tau\|_{\ell^2(t_1,T;L^2)}$. Let $R_1 := 1+\sqrt{3}$. After setting a hypothesis, we present stability results.

Hypothesis 8. *For a fixed number $R \in (1, R_1)$, the set of the consecutive time-step ratios $\{r_n\}_{n=2}^{N_T}$ satisfies the condition $r_n \le R$ for $n = 2, \ldots, N_T$.*

Theorem 9 (stability for the time discretization)**.** *Let $f \in C([0,T]; L^2(\Omega))$ and $\phi_\tau^0, \phi_\tau^1 \in \Psi$ be given. Suppose that Hypotheses 4, 5, and 8 hold true. Let $\phi_\tau = \{\phi_\tau^n\}_{n=2}^{N_T}$ be a solution to problem* (2.3)*. Then, there exists a constant $C_1 > 0$ independent of $\{\tau_n\}_n$ and N_T such that*

$$\begin{aligned}\sqrt{\nu}\|\nabla\phi_\tau\|_{\ell^\infty(t_2,T;L^2)} &+ \|D_1\phi_\tau\|_{\ell^2_\alpha(t_2,T;L^2)} \\ &\le C_1\big[\sqrt{\nu}\|\nabla\phi_\tau^0\| + \sqrt{\nu}\|\nabla\phi_\tau^1\| + \sqrt{\tau_1}\|D_1\phi_\tau^1\| + \|f\|_{\ell^2(t_2,T;L^2)}\big]. \end{aligned} \tag{3.1}$$

Remark 10. *The value of R_1 is larger than the zero-stability bound for variable BDF2, $R_0 = 1+\sqrt{2}$, since our analysis explicitly exploits the ellipticity of the elliptic operator $-\Delta$.*

Theorem 11 (error estimates for the time discretization)**.** *Let $f \in C([0,T]; L^2(\Omega))$ be given. Suppose that Hypothesis 4 holds true. Let ϕ be the solution to problem* (2.1)*. Assume $\phi \in Z^3$. Let $\{t_n\}_{n=0}^{N_T}$ be a set of nonuniform time grids satisfying Hypotheses 5 and 8. Let $\phi_\tau^0, \phi_\tau^1 \in \Psi$ be given with $\phi_\tau^0 = \phi_0$. Let $\phi_\tau = \{\phi_\tau^n\}_{n=2}^{N_T}$ be the solution to problem* (2.3)*. Then, there exists a constant $C_2 > 0$ independent of τ and N_T such that*

$$\begin{aligned}\sqrt{\nu}\|\nabla(\phi-\phi_\tau)\|_{\ell^\infty(t_2,T;L^2)} &+ \|D_1(\phi-\phi_\tau)\|_{\ell^2_\alpha(t_2,T;L^2)} \\ &\le C_2\big[\sqrt{\nu}\|\nabla(\phi^1-\phi_\tau^1)\| + \sqrt{\tau_1}\|D_1(\phi^1-\phi_\tau^1)\| + \tau^2\|\phi\|_{Z^3}\big]. \end{aligned} \tag{3.2}$$

The next corollary holds immediately from (3.2).

Corollary 12. *If the first step ϕ_τ^1 is provided sufficiently accurately, then the error of the time discretization* (2.3) *is of order $O(\tau^2)$ in $\ell^\infty(0,T;H_0^1)$-norm.*

4 Proofs

For the proof of Theorem 9, we prepare two lemmas, cf. [9, 12] for the proofs.

Lemma 13 ([12, Lemma 6])**.** *Under Hypotheses 4 and 5, we have the following inequalities for the mappings X_{-k}^n ($k \in \{1,2\}, n \in \{2,\ldots,N_T\}$):*

$$\|\psi - \psi\circ X_{-k}^n\| \le \bar{c}_u^n (t_n - t_{n-k})\|\nabla\psi\| \qquad \forall\psi \in H^1(\Omega),$$

for a positive constant $\bar{c}_u^n$, depending on $\|u^n\|_{L^\infty(\Omega)}$ and independent of $\{\tau_n\}_n$.

Remark 14. *The constant $\bar{c}_u^n$ depends on $\|u^n\|_{L^\infty(\Omega)}$, which is bounded by $\|u\|_{C(L^\infty)}$. Hence, there exists a positive constant $\bar{c}_u$ independent of n such that $\bar{c}_u^n \le \bar{c}_u$.*

Lemma 15 ([9, Lemma 1])**.** *Let $a_n, b_n, c_n, \lambda_n \ge 0$ with $\{c_n\}_{n\ge2}$ being non-decreasing. Suppose that the inequalities*

$$a_n + b_n \le \sum_{j=2}^{n-1}\lambda_j a_j + c_n, \qquad n = 2, 3, \cdots$$

hold true. Then, it holds that, for $n = 2, 3, \ldots,$

$$a_n + b_n \le c_n \prod_{j=2}^{n-1} \big(1 + \lambda_j\big) \le c_n \exp\Big(\sum_{j=2}^{n-1} \lambda_j\Big).$$

4.1 Proof of Theorem 9

Noting that

$$\begin{aligned}\widetilde{D}_2\phi_\tau^n &= D_2\phi_\tau^n - \frac{\tau_n + \tau_{n-1}}{\tau_n\tau_{n-1}}(\phi_\tau^{n-1}\circ X_{-1}^n - \phi_\tau^{n-1}) + \frac{\tau_n}{\tau_{n-1}(\tau_n + \tau_{n-1})}(\phi_\tau^{n-2}\circ X_{-2}^n - \phi_\tau^{n-2})\\ &= D_2\phi_\tau^n - \frac{1 + r_n}{\tau_n}(\phi_\tau^{n-1}\circ X_{-1}^n - \phi_\tau^{n-1}) + \frac{r_n}{\tau_n + \tau_{n-1}}(\phi_\tau^{n-2}\circ X_{-2}^n - \phi_\tau^{n-2}),\end{aligned}$$

we rewrite (2.3) as

$$\begin{aligned}(D_2\phi_\tau^n, \psi) + a(\phi_\tau^n, \psi) = (f^n, \psi) &+ \frac{1 + r_n}{\tau_n}(\phi_\tau^{n-1}\circ X_{-1}^n - \phi_\tau^{n-1}, \psi)\\ &- \frac{r_n}{\tau_n + \tau_{n-1}}(\phi_\tau^{n-2}\circ X_{-2}^n - \phi_\tau^{n-2}, \psi) \qquad \forall\psi\in\Psi,\end{aligned}$$

which implies that, for $\psi = 2\tau_n D_1\phi_\tau^n \in \Psi$,

$$\begin{aligned}&2\tau_n(D_2\phi_\tau^n, D_1\phi_\tau^n) + 2\tau_n a(\phi_\tau^n, D_1\phi_\tau^n)\\ &= 2\tau_n(f^n, D_1\phi_\tau^n) + 2(1 + r_n)(\phi_\tau^{n-1}\circ X_{-1}^n - \phi_\tau^{n-1}, D_1\phi_\tau^n)\\ &- \frac{2\tau_n r_n}{\tau_n + \tau_{n-1}}(\phi^{n-2}\circ X_{-2}^n - \phi_\tau^{n-2}, D_1\phi_\tau^n). \qquad (4.1)\end{aligned}$$

Each term in (4.1) is evaluated as follows:

$$\begin{aligned}2\tau_n(D_2\phi_\tau^n, D_1\phi_\tau^n) &= 2\tau_n\big(D_1\phi_\tau^n + \alpha_n(D_1\phi_\tau^n - D_1\phi_\tau^{n-1}), D_1\phi_\tau^n\big)\\ &= 2\tau_n\|D_1\phi_\tau^n\|^2 + \alpha_n\tau_n\big(\|D_1\phi_\tau^n\|^2 - \|D_1\phi_\tau^{n-1}\|^2 + \|D_1\phi_\tau^n - D_1\phi_\tau^{n-1}\|^2\big)\\ &\ge 2\tau_n\|D_1\phi_\tau^n\|^2 + \alpha_n\tau_n\big(\|D_1\phi_\tau^n\|^2 - \|D_1\phi_\tau^{n-1}\|^2\big), \qquad (4.2a)\\ 2\tau_n a(\phi_\tau^n, D_1\phi_\tau^n) &= 2\nu\big(\nabla\phi_\tau^n, \nabla(\phi_\tau^n - \phi_\tau^{n-1})\big)\\ &= \nu\big(\|\nabla\phi_\tau^n\|^2 - \|\nabla\phi_\tau^{n-1}\|^2 + \|\nabla(\phi_\tau^n - \phi_\tau^{n-1})\|^2\big)\\ &\ge \nu\big(\|\nabla\phi_\tau^n\|^2 - \|\nabla\phi_\tau^{n-1}\|^2\big), \qquad (4.2b)\\ 2\tau_n(f^n, D_1\phi_\tau^n) &\le \frac{\tau_n}{\epsilon_0}\|f^n\|^2 + \epsilon_0\tau_n\|D_1\phi_\tau^n\|^2, \qquad (4.2c)\end{aligned}$$

$$\begin{aligned}\big|2(1 + r_n)\big(\phi_\tau^{n-1}\circ X_{-1}^n - \phi_\tau^{n-1}, D_1\phi_\tau^n\big)\big| &\le \frac{(1 + r_n)^2}{\epsilon_1\tau_n}\|\phi_\tau^{n-1}\circ X_{-1}^n - \phi_\tau^{n-1}\|^2 + \epsilon_1\tau_n\|D_1\phi_\tau^n\|^2\\ &\le \frac{(\bar{c}_u^n)^2(1 + r_n)^2\tau_n}{\epsilon_1}\|\nabla\phi_\tau^{n-1}\|^2 + \tau_n\epsilon_1\|D_1\phi_\tau^n\|^2, \qquad (4.2d)\\ \Big|\frac{2r_n\tau_n}{\tau_n + \tau_{n-1}}(\phi^{n-2}\circ X_{-2}^n - \phi_\tau^{n-2}, D_1\phi_\tau^n)\Big| &\le \frac{r_n^2\tau_n}{\epsilon_2(\tau_n + \tau_{n-1})^2}\|\phi_\tau^{n-2}\circ X_{-2}^n - \phi_\tau^{n-2}\|^2 + \tau_n\epsilon_2\|D_1\phi_\tau^n\|^2\\ &\le \frac{(\bar{c}_u^n)^2 r_n^2\tau_n}{\epsilon_2}\|\nabla\phi_\tau^{n-2}\|^2 + \tau_n\epsilon_2\|D_1\phi_\tau^n\|^2, \qquad (4.2e)\end{aligned}$$

for any constants $\epsilon_k > 0$ $(k = 0, 1, 2)$, where Lemma 13 has been employed in (4.2d) and (4.2e). Combining (4.2) with (4.1) and introducing a number $\epsilon_* := \epsilon_0 + \epsilon_1 + \epsilon_2 > 0$, we have

$$(2 - \epsilon_*)\tau_n\|D_1\phi_\tau^n\|^2 + \alpha_n\tau_n\big(\|D_1\phi_\tau^n\|^2 - \|D_1\phi_\tau^{n-1}\|^2\big) + \nu\|\nabla\phi_\tau^n\|^2 - \nu\|\nabla\phi_\tau^{n-1}\|^2$$

$$\leq \frac{\tau_n}{\epsilon_0}\|f^n\|^2 + \frac{(\bar{c}_u^n)^2(1+r_n)^2\tau_n}{\epsilon_1}\|\nabla\phi_\tau^{n-1}\|^2 + \frac{(\bar{c}_u^n)^2 r_n^2\tau_n}{\epsilon_2}\|\nabla\phi_\tau^{n-2}\|^2. \tag{4.3}$$

Let $N \in \{2, \dots, N_T\}$ be any fixed number. By summing up for $n = 2, 3, \dots, N$ in (4.3), we have

$$\begin{aligned}
&(2+\alpha_N-\epsilon_*)\tau_N\|D_1\phi_\tau^N\|^2 + \sum_{n=2}^{N-1}\big[(2+\alpha_n-\epsilon_*)\tau_n - \alpha_{n+1}\tau_{n+1}\big]\|D_1\phi_\tau^n\|^2 - \alpha_2\tau_2\|D_1\phi_\tau^1\|^2 \\
&\quad + \nu\|\nabla\phi_\tau^N\|^2 - \nu\|\nabla\phi_\tau^1\|^2 \\
&\leq \sum_{n=2}^{N}\frac{\tau_n}{\epsilon_0}\|f^n\|^2 + \sum_{n=2}^{N}\frac{(\bar{c}_u^n)^2(1+r_n)^2\tau_n}{\epsilon_1}\|\nabla\phi_\tau^{n-1}\|^2 + \sum_{n=2}^{N}\frac{(\bar{c}_u^n)^2 r_n^2\tau_n}{\epsilon_2}\|\nabla\phi_\tau^{n-2}\|^2.
\end{aligned}$$

Noting $\frac{r^2}{1+r} \leq \frac{R^2}{1+R} < \frac{R_1^2}{1+R_1} = 2$ $(r \in [0, R])$ and $\alpha_{n+1}\tau_{n+1} = \frac{r_{n+1}^2}{1+r_{n+1}}\tau_n \leq \frac{R^2}{1+R}\tau_n$, and choosing $\epsilon_i = \frac{1}{3}(2 - \frac{R^2}{1+R}) =: \epsilon_\star > 0$ $(i = 0, 1, 2)$, i.e., $\epsilon_* = 2 - \frac{R^2}{1+R} = 3\epsilon_\star > 0$, we have

$$(2+\alpha_n-\epsilon_*)\tau_n - \alpha_{n+1}\tau_{n+1} \geq \tau_n\Big(2+\alpha_n-\epsilon_*-\frac{R^2}{1+R}\Big) \geq \alpha_n\tau_n,$$

which implies that

$$\begin{aligned}
&\nu\|\nabla\phi_\tau^N\|^2 + \sum_{n=2}^{N}\alpha_n\tau_n\|D_1\phi_\tau^n\|^2 \\
&\leq \nu\|\nabla\phi_\tau^1\|^2 + \alpha_2\tau_2\|D_1\phi_\tau^1\|^2 \\
&\quad + \frac{1}{\epsilon_\star}\Bigg[\sum_{n=2}^{N}\tau_n\|f^n\|^2 + \bar{c}_u^2(1+R)^2\sum_{n=1}^{N-1}\tau_{n+1}\|\nabla\phi_\tau^n\|^2 + \bar{c}_u^2R^2\sum_{n=0}^{N-2}\tau_{n+2}\|\nabla\phi_\tau^n\|^2\Bigg] \\
&\leq \nu\|\nabla\phi_\tau^1\|^2 + \alpha_2\tau_2\|D_1\phi_\tau^1\|^2 + \frac{1}{\epsilon_\star}\Bigg[\sum_{n=2}^{N}\tau_n\|f^n\|^2 + \bar{c}_u^2[(1+R)^2+R^3]\sum_{n=0}^{N-1}\tau_{n+1}\|\nabla\phi_\tau^n\|^2\Bigg] \\
&\leq \sum_{n=0}^{N-1}c_\nu\tau_{n+1}\nu\|\nabla\phi_\tau^n\|^2 + \nu\|\nabla\phi_\tau^1\|^2 + \alpha_2\tau_2\|D_1\phi_\tau^1\|^2 + \frac{1}{\epsilon_\star}\sum_{n=2}^{N}\tau_n\|f^n\|^2,
\end{aligned} \tag{4.4}$$

where we have used the inequality $\bar{c}_u^n \leq \bar{c}_u$ for a positive constant $\bar{c}_u$, cf. Remark 14, for the first inequality, and the relation $\tau_{n+2} \leq R\tau_{n+1}$ for the second inequality, and $c_\nu := \bar{c}_u^2[(1+R)^2 + R^3]/(\nu\epsilon_\star)$. From Lemma 15 with

$$\begin{aligned}
&\lambda_n = c_\nu\tau_{n+1}, \quad a_N = \nu\|\nabla\phi_\tau^N\|^2, \quad b_N = \sum_{n=2}^{N}\alpha_n\tau_n\|D_1\phi_\tau^n\|^2, \\
&c_N = c_\nu\tau_1\nu\|\nabla\phi_\tau^0\|^2 + (c_\nu\tau_2+1)\nu\|\nabla\phi_\tau^1\|^2 + \alpha_2\tau_2\|D_1\phi_\tau^1\|^2 + \frac{1}{\epsilon_\star}\sum_{n=2}^{N}\tau_n\|f^n\|^2,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\nu\|\nabla\phi_\tau^N\|^2 + \sum_{n=2}^{N}\alpha_n\tau_n\|D_1\phi_\tau^n\|^2 \\
&\leq \Bigg(c_\nu\tau_1\nu\|\nabla\phi_\tau^0\|^2 + (c_\nu\tau_2+1)\nu\|\nabla\phi_\tau^1\|^2 + \alpha_2\tau_2\|D_1\phi_\tau^1\|^2 + \frac{1}{\epsilon_\star}\sum_{n=2}^{N}\tau_n\|f^n\|^2\Bigg)\prod_{n=2}^{N-1}\big(1+c_\nu\tau_{n+1}\big)
\end{aligned}$$

$$\leq c \exp(c_\nu T) \left(\nu \|\nabla \phi_\tau^0\|^2 + \nu \|\nabla \phi_\tau^1\|^2 + \tau_2 \|D_1 \phi_\tau^1\|^2 + \sum_{n=2}^{N} \tau_n \|f^n\|^2 \right)$$
$$\leq c' \exp(c_\nu T) \left(\nu \|\nabla \phi_\tau^0\|^2 + \nu \|\nabla \phi_\tau^1\|^2 + \tau_1 \|D_1 \phi_\tau^1\|^2 + \|f\|^2_{\ell^2(t_2, t_N; L^2)} \right),$$

for some positive constants c and c', which completes the proof with $C_1 = \sqrt{2c'} \exp(c_\nu T/2)$ by the arbitrariness of N.

4.2 Proof of Theorem 11

Let ϕ be the solution to problem (2.1) and ϕ_τ be the solution to problem (2.2) for given ϕ_τ^0 $(= \phi_0), \phi_\tau^1 \in \Psi$. Let $e^n := \phi^n - \phi_\tau^n$ and $\eta^n := \mathcal{D}_t \phi^n - \widetilde{D}_2 \phi^n$. The equation for the error $\{e^n\}_{n=2}^{N_T} \subset \Psi$ is given as follows:

$$\big(\widetilde{D}_2 e^n, \psi\big) + a(e^n, \psi) = (\eta^n, \psi) \qquad \forall \psi \in \Psi, \tag{4.5}$$

where $e^0 (= 0), e^1 \in \Psi$ are given.

For $x \in \Omega$ and $n \in \{0, \ldots, N_T\}$, we introduce the differential operator $\bar{\mathcal{D}}_t^n(x)$ with the velocity $u^n(x)$, which is a version of $\mathcal{D}_t$ and often used in the sequel, defined by

$$\bar{\mathcal{D}}_t^n(x) := \frac{\partial}{\partial t} + u^n(x) \cdot \nabla,$$

that is, for $(\tilde{x}, \tilde{t}) \in \Omega \times (0, T)$,

$$\big[\bar{\mathcal{D}}_t^n(x) \phi\big](\tilde{x}, \tilde{t}) = \Big[\Big(\frac{\partial}{\partial t} + u^n(x) \cdot \nabla\Big) \phi\Big](\tilde{x}, \tilde{t}) = \frac{\partial \phi}{\partial t}(\tilde{x}, \tilde{t}) + u^n(x) \cdot \nabla \phi(\tilde{x}, \tilde{t}).$$

Lemma 16. *(truncation error) Assume $\phi \in Z^3$. Suppose that Hypotheses 4, 5, and 8 hold true. Then, there exists a positive constant $\tilde{c}_u^n$ depending on $\|u^n\|_{L^\infty(\Omega)}$ and independent of $\{\tau_n\}_n$ such that*

$$\|\eta^n\| \leq \tilde{c}_u^n \max\{\tau_n, \tau_{n-1}\}^{3/2} \|\phi\|_{Z^3(t_{n-2}, t_n)} \leq \tilde{c}_u^n \tau^{3/2} \|\phi\|_{Z^3(t_{n-2}, t_n)}, \qquad n \in \{2, \cdots, N_T\}.$$

Proof. Let $n \in \{2, \cdots, N_T\}$ be fixed arbitrarily. Let $y_{-k}^n(x, s) := x - u^n(x)(1 - s)(t_n - t_{n-k})$ and $t_{-k}^n(s) := t_{n-k} + s(t_n - t_{n-k})$ be a trajectory of a fluid particle for $k = (0,)1, 2$. Applying the identity

$$g(0) = g(1) - g'(1) + \frac{1}{2} g^{(2)}(1) - \frac{1}{2} \int_0^1 s^2 g^{(3)}(s)\, ds$$

for $g(s) = \phi(y_{-k}^n(\cdot, s), t_{-k}^n(s))$ and changing a variable from s to $t = t_{-k}^n(s)$, we have the following expressions of $(\phi^{n-k} \circ X_{-k}^n)(x)$,

$$\begin{aligned}\big(\phi^{n-k} \circ X_{-k}^n\big)(x) &= \phi^n(x) - (t_n - t_{n-k})(\mathcal{D}_t \phi^n)(x) + \frac{1}{2}(t_n - t_{n-k})^2 (\mathcal{D}_t^2 \phi^n)(x) \\ &\quad - \frac{1}{2} \int_{t_{n-k}}^{t_n} (t - t_{n-k})^2 \big[\bar{\mathcal{D}}_t^n(x)^3 \phi\big]\big(x - u^n(x)(t_n - t), t\big)\, dt,\end{aligned} \tag{4.6}$$

for $k = 1, 2$, which imply that

$$\eta^n(x) = (\mathcal{D}_t \phi^n)(x) - (\widetilde{D}_2 \phi^n)(x)$$

$$
\begin{aligned}
&= (\mathcal{D}_t\phi^n)(x) - \Big[\frac{2\tau_n+\tau_{n-1}}{\tau_n(\tau_n+\tau_{n-1})}\phi^n - \frac{\tau_n+\tau_{n-1}}{\tau_n\tau_{n-1}}\phi^{n-1}\circ X_{-1}^n + \frac{\tau_n}{\tau_{n-1}(\tau_n+\tau_{n-1})}\phi^{n-2}\circ X_{-2}^n\Big](x)\\
&= -\frac{\tau_n+\tau_{n-1}}{2\tau_n\tau_{n-1}}\int_{t_{n-1}}^{t_n}(t-t_{n-1})^2\big[\bar{\mathcal{D}}_t^n(x)^3\phi\big]\big(x-u^n(x)(t_n-t),t\big)\,dt\\
&\quad+\frac{\tau_n}{2\tau_{n-1}(\tau_n+\tau_{n-1})}\int_{t_{n-2}}^{t_n}(t-t_{n-2})^2\big[\bar{\mathcal{D}}_t^n(x)^3\phi\big]\big(x-u^n(x)(t_n-t),t\big)\,dt. \qquad (4.7)
\end{aligned}
$$

We evaluate $\|\eta\|$ from (4.7) as follows:

$$
\begin{aligned}
\|\eta^n\|^2 &\le \int_\Omega \frac{1}{2}\Big[\frac{\tau_n+\tau_{n-1}}{\tau_n\tau_{n-1}}\int_{t_{n-1}}^{t_n}(t-t_{n-1})^2\big[\bar{\mathcal{D}}_t^n(x)^3\phi\big]\big(x-u^n(x)(t_n-t),t\big)\,dt\Big]^2 dx\\
&\quad+\int_\Omega \frac{1}{2}\Big[\frac{\tau_n}{\tau_{n-1}(\tau_n+\tau_{n-1})}\int_{t_{n-2}}^{t_n}(t-t_{n-2})^2\big[\bar{\mathcal{D}}_t^n(x)^3\phi\big]\big(x-u^n(x)(t_n-t),t\big)\,dt\Big]^2 dx\\
&=: I_1+I_2. \qquad (4.8)
\end{aligned}
$$

Let $\tilde{c}_{u,1}^n := c(1+R)^2(1+\|u^n\|_{L^\infty(\Omega)})^3$, $\tilde{c}_{u,2}^n := cR^2(1+\|u^n\|_{L^\infty(\Omega)})^3$, and $\tilde{c}_u^n := \max\{\tilde{c}_{u,1}^n,\ 8\tilde{c}_{u,2}^n\}^{1/2}$, to be used in what follows. From the following estimates of I_1 and I_2

$$
\begin{aligned}
I_1 &= \int_\Omega \frac{1}{2\tau_n^2}\Big(\frac{\tau_n+\tau_{n-1}}{\tau_{n-1}}\Big)^2\Big[\int_{t_{n-1}}^{t_n}(t-t_{n-1})^2\big[\bar{\mathcal{D}}_t^n(x)^3\phi\big]\big(x-u^n(x)(t_n-t),t\big)\,dt\Big]^2 dx\\
&\le \frac{(1+r_n)^2}{2\tau_n^2}\int_\Omega\Big(\int_{t_{n-1}}^{t_n}(t-t_{n-1})^4\,dt\Big)\Big(\int_{t_{n-1}}^{t_n}\big[\bar{\mathcal{D}}_t^n(x)^3\phi\big]\big(x-u^n(x)(t_n-t),t\big)^2\,dt\Big)dx\\
&= \frac{(1+R)^2\tau_n^3}{10}\int_{t_{n-1}}^{t_n}\Big(\int_\Omega\big[\bar{\mathcal{D}}_t^n(x)^3\phi\big]\big(x-u^n(x)(t_n-t),t\big)^2\,dx\Big)dt\\
&\le c(1+R)^2\tau_n^3\int_{t_{n-1}}^{t_n}\Big(\int_\Omega\big[\bar{\mathcal{D}}_t^n(x)^3\phi\big]\big(y,t\big)^2\,dy\Big)dt \quad (\text{by } y=x-u^n(x)(t_n-t),\ \text{Prop. 7})\\
&\le \tilde{c}_{u,1}^n\tau_n^3\|\phi\|_{Z^3(t_{n-1},t_n)}^2, \qquad (4.9)\\
I_2 &\le \int_\Omega \frac{1}{2}\Big(\frac{r_n}{\tau_n+\tau_{n-1}}\Big)^2\Big(\int_{t_{n-2}}^{t_n}(t-t_{n-2})^4\,dt\Big)\Big(\int_{t_{n-2}}^{t_n}\big[\bar{\mathcal{D}}_t^n(x)^3\phi\big]\big(x-u^n(x)(t_n-t),t\big)^2\,dt\Big)dx\\
&= \frac{r_n^2(\tau_n+\tau_{n-1})^3}{10}\int_{t_{n-2}}^{t_n}\Big(\int_\Omega\big[\bar{\mathcal{D}}_t^n(x)^3\phi\big]\big(x-u^n(x)(t_n-t),t\big)^2\,dx\Big)dt\\
&\le cR^2(\tau_n+\tau_{n-1})^3\int_{t_{n-2}}^{t_n}\Big(\int_\Omega\big[\bar{\mathcal{D}}_t^n(x)^3\phi\big](y,t)^2\,dy\Big)dt\\
&\le \tilde{c}_{u,2}^n(\tau_n+\tau_{n-1})^3\|\phi\|_{Z^3(t_{n-2},t_n)}^2, \qquad (4.10)
\end{aligned}
$$

we finally obtain

$$
\begin{aligned}
\|\eta^n\|^2 &\le I_1+I_2 \le \tilde{c}_{u,1}^n\tau_n^3\|\phi\|_{Z^3(t_{n-1},t_n)}^2 + \tilde{c}_{u,2}^n(\tau_n+\tau_{n-1})^3\|\phi\|_{Z^3(t_{n-2},t_n)}^2)\\
&\le (\tilde{c}_u^n)^2\max\{\tau_n,\tau_{n-1}\}^3\|\phi\|_{Z^3(t_{n-2},t_n)}^2 \le (\tilde{c}_u^n)^2\tau^3\|\phi\|_{Z^3(t_{n-2},t_n)}^2,
\end{aligned}
$$

which completes the proof. □

Remark 17. *$\tilde{c}_u^n$ depends on $\|u^n\|_{L^\infty(\Omega)}$, which is bounded by $\|u\|_{C(L^\infty)}$. Hence, there exists a positive constant $\tilde{c}_u$ independent of n such that $\tilde{c}_u^n \le \tilde{c}_u$.*

Proof of Theorem 11. Applying Theorem 9 for (4.5) with $\phi_\tau = e\ (=\phi-\phi_\tau)$ and $f=\eta$, we have

$$
\sqrt{\nu}\|\nabla e\|_{\ell^\infty(t_2,T;L^2)} + \|D_1 e\|_{\ell^2_\alpha(t_2,T;L^2)} \le C_1\big[\sqrt{\nu}\|\nabla e^1\| + \sqrt{\tau_1}\|D_1 e^1\| + \|\eta\|_{\ell^2(t_2,T;L^2)}\big]. \qquad (4.11)
$$

Since Lemma 16, together with Remark 17, provides the estimate

$$\|\eta\|^2_{\ell^2(t_2,T;L^2)} \leq \sum_{n=2}^{N_T} (\tilde{c}_u^n)^2 \tau^4 \|\phi\|^2_{Z^3(t_{n-2},t_n)} \leq 2\,\tilde{c}_u^2 \tau^4 \|\phi\|^2_{Z^3(0,T)},$$

we obtain the desired result (3.2) by combining the above with (4.11). □

5 Conclusions

In this paper, we proposed a second-order Lagrangian time discretization with variable time steps for convection-diffusion problems and established its stability and convergence in the $\ell^\infty(H_0^1)$ norm. The admissible upper bound for the step-size ratio obtained in our analysis exceeds that of the classical zero-stability theory, thanks to an explicit use of the ellipticity of the underlying operator. To the best of our knowledge, this work provides the first theoretical results for Lagrangian BDF2 time discretizations with variable time steps. Although our analysis is restricted to a semi-discretization, existing studies indicate that Lagrangian time discretizations relax the usual CFL restriction arising from the convection term, thereby supporting their effectiveness for convection-dominated problems.

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