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On Certain Manifolds which are Tangentially Homotopy Equivalent

Dedicated to Professor Nobuo Shimada on his 60th birthday

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Abstract. It is shown that simply connected smooth m-mainfolds with a certain homotopy and tangential property are very frequently diffeomorphic modΘ_m if they are tangentially homotopy equivalent.

1. Introduction

Two manifolds M_1, M_2 are called tangentially homotopy equivalent if there exists a homotopy equivalence $f : M_1 \to M_2$ such that $f^*(\tau M_2)$ is stably equivalent to τM_1, where τM_i, $i = 1,2$, are their tangent bundles. In this paper, we are concerned with closed smooth m-manifolds M which are simply connected and satisfy the following hypotheses :

(H_1) M − (a point) has the homotopy type of a bouquet of spheres $(\vee_{i=1}^{r} S_i^p) \vee (\vee_{j=1}^{r} S_j^q)$, where $0<p<q$, $p+q=m$.

(H_2) M is p-parallelizable (that is, M is parallelizable on its p-skeleton of a triangulation).

We study whether such two manifolds M_1, M_2 as above which are tangentially homotopy equivalent are diffeomorphic modΘ_m (that is, $M_1 = M_2 \# \Sigma$ for some Σ of Θ_m) or not. We treat with the following cases :

(A) $m=2n+1$, $p=n$, $n\geqq 2$,
(B) $m=2n$, $p=n-1$, $n\geqq 4$,
(C) $m=2n-1$, $p=n-2$, $n\geqq 6$,
(D) $m=2n-2$, $p=n-3$, $n\geqq 8$,
(E) $m=2n-3$, $p=n-4$, $n\geqq 10$,
(F) $m=2n-4$, $p=n-5$, $n\geqq 12$.

Here, q is kept as $q=n+1$ in every case to advance our argument in a unifying way.

One of the authors showed the following in [25] and partly in [4] and [5] .

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THEOREM 1. *Let* M_1, M_2 *be closed smooth* m*-manifolds satisfying the hypotheses* (H_1), (H_2). *In the cases* (A), (B), *and* (C), *let* M_1, M_2 *be homotopy equivalent if* $n+1 \not\equiv 0 \pmod 4$ *and tangentially homotopy equivalent if* $n+1 \equiv 0 \pmod 4$. *Then,* M_1, M_2 *are diffeomorphic* mod Θ_m.

In this paper, by a further investigation, we have the following results.

THEOREM 2. *Let* M_1, M_2 *be closed smooth* m*-manifolds satisfying the hypotheses* (H_1), (H_2). *In the cases* (D), (E), *and* (F), *let* M_1, M_2 *be homotopy equivalent if* $n+1 \not\equiv 0 \pmod 4$ *and tangentially homotopy equivalent if* $n+1 \equiv 0 \pmod 4$. *Then,* M_1, M_2 *are diffeomorphic* mod Θ_m *unless* $n=2^l-1$ *for* $l \geqq 4$.

Remark. If we consider the case $p=q$, then the maifolds are n-connected and $(2n+2)$-dimensional. If M_1, M_2 are n-connected and $(2n+2)$-dimensional $(n \geqq 2)$, then the conclusion of Theorem 1 holds also without the hypotheses corresponding to (H_1), (H_2). This fact was shown by [14] and partly by [22], but we can show it also using the handlebody theory of [23] and studying the homomorphism $J: \pi_n(SO_{n+1}) \rightarrow \pi_{2n+1}(S^{n+1})$.

THEOREM 3. *Let* $n=2^l-1$ $(l \geqq 4)$. *Then, in each of the cases* (D), (E), *and* (F), *there exist certain closed smooth* m*-manifolds* M_1, M_2 *satisfying the hypotheses* (H_1), (H_2) *which form a counter example for the conclusion of Theorem 2. Furthermore, if* (M', M), (M'', M) *are such counter examples and* $\operatorname{rank} H_p(M)=1$, *then* M', M'' *are diffeomorphic* mod Θ_m.

We denote the generators of $\pi_{n+3}(S^n) \cong Z_3+Z_8 (n \geqq 5)$ by $\alpha_1(n)$, ν_n respectively ([21]) and denote the oriented generator of $\pi_n(S^n)$ by ι_n.

Let $n=2^l-1 (l \geqq 4)$. Let $B^{(i)}$ be the total space of the $(n-i+1)$-sphere bundle over the $(n+1)$-sphere which corresponds to the non-trivial torsion element of $\pi_n(SO_{n-i+2}) \cong Z_2+Z$, $i=4, 5, 6$. In the proof of Theorem 3, it is shown that for $i=4, 5$, and 6, $B^{(i)}$ is tangentially homotopy equivalent to $S^{n-i+1} \times S^{n+1}$ but they are not diffeomorphic mod Θ_{2n-i+2}. Here, we note that $i=4, 5$, and 6 correspond to the cases (D), (E), and (F) respectively.

On the other hand, it is known that the Whitehead product $[\nu_{n-2}, \iota_{n-2}]=0$ for $n=2^l-1$ $(l \geqq 3)$ (*Cf.* [15], [17]). Therefore, by Theorem (1.2) and Remarks of [2], there exists an exotic $(n+1)$-sphere Σ^{n+1} which can be imbedded in R^{2n-1} with non-trivial normal bundle for $n=2^l-1$ $(l \geqq 4)$. (In [2], it was shown that such an exotic sphere exists for $n=15$). Then, $S^{n-3} \times \Sigma^{n+1}$ is not diffeomorphic to $S^{n-3} \times S^{n+1}$ mod Θ_{2n-2}. In fact, if $S^{n-3} \times \Sigma^{n+1} = S^{n-3} \times S^{n+1}$ mod Θ_{2n-2}, we have

$$\Sigma^{n+1} \subset S^{n-3} \times \Sigma^{n+1} - * = S^{n-3} \times S^{n+1} - * \subset S^{n-3} \times S^{n+1} \subset R^{2n-1}.$$

Hence, Σ^{n+1} can be imbedded in R^{2n-1} with trivial normal bundle, and this yields a contradiction by Lemma (1.1) of [2]. Since $S^{n-3}\times\Sigma^{n+1}$ is tangentially homotopy equivalent to $S^{n-3}\times S^{n+1}$, we have an example of Theorem 3 for the case (D). Similar arguments also hold for the pairs $(S^{n-4}\times\Sigma^{n+1}, S^{n-4}\times S^{n+1})$, $(S^{n-5}\times\Sigma^{n+1}, S^{n-5}\times S^{n+1})$, and therefore, the pairs form the examples of Theorem 3 for the cases (E), (F), respectively. Then, by the latter half of Theorem 3, $S^{n-i+1}\times\Sigma^{n+1}$ is diffeomorphic to $B^{(i)}$ "mod Θ_{2n-i+2}" for $i=4$, 5, and 6. However, more precisely, we can show that they are just diffeomorphic.

COROLLARY 4. *Let $n=2^l-1 (l\geqq 4)$. For such an exotic sphere Σ^{n+1} as above, $S^{n-i+1}\times\Sigma^{n+1}$ is not diffeomorphic to $S^{n-i+1}\times S^{n+1} \bmod \Theta_{2n-i+2}$ but just diffeomorphic to $B^{(i)}$, for $i=4$, 5, and 6.*

The proofs of the above results are given in § 4. The proof of Theorem 2 includes that of Theorem 1, and consequently Theorem 1 is to be proved additionally. In this paper, manifolds are oriented and homotopy equivalences and diffeomorphisms are orientation preserving.

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2. Handlebodies of type 0

Let W be a handlebody of $\mathscr{H}(p+q+1, r, q)$ $(2p>q>1)$ and let $(H; \phi, \alpha)$ be the invariant system defined in [23] which characterizes W up to diffeomorphism. W is called of *type* 0 if the pairing $\phi: H\times H\rightarrow\pi_q(S^{p+1})$ $(H=H_q(W))$ is trivial. Then, by Theorem 1 of [23], the map $\alpha: H\rightarrow\pi_{q-1}(SO_{p+1})$ is a homomorphism which takes the values in $i_*(\pi_{q-1}(SO_p))$, where $i_*: \pi_{q-1}(SO_p)\rightarrow\pi_{q-1}(SO_{p+1})$ is induced from the inclusion $i: SO_p\rightarrow SO_{p+1}$. If W is of type 0, then for an arbitrary basis $\{g_1, g_2, \cdots, g_r\}$ of H, W can be represented as a boundary connected sum $\bar{A}_1\natural\ \bar{A}_2\natural\ \cdots\natural\ \bar{A}_r$ up to diffeomorphism, where each $\bar{A}_i$ is the $(p+1)$-disk bundle over the q-sphere with the characterstic element $\alpha(g_i)$. (See p. 21 of [6]).

Henceforth, let W_1, W_2 be handlebodies of $\mathscr{H}(p+q+1, r, q)$ $(2p>q>1)$ and let $(H_1; \phi_1, \alpha_1)$, $(H_2; \phi_2, \alpha_2)$ be the invariant systems of W_1, W_2 respectively. By Theorem 2 of [23], we have

THEOREM 2.1. *Let W_1, W_2 be of type 0. Then, W_1, W_2 are diffeomorphic if and only if there exists an isomorphism $h: H_1\rightarrow H_2$ such that $\alpha_1=\alpha_2\circ h$.*

Let $P: \pi_q(S^p)\rightarrow\pi_{p+q-1}(S^p)$ be the homomorphism defined by $P(x)=[x, \iota_p]$, the Whitehead product with ι_p. We define the homomorphism $\lambda: i_*\pi_{q-1}(SO_p)\rightarrow J\pi_{q-1}(SO_p)/\mathrm{Im}P$ by $\lambda(i_*\xi)=\{J\xi\}$. We note that λ can be identified with $-J\mid i_*\pi_{q-1}(SO_p)$, the restriction of $-J$ to $i_*\pi_{q-1}(SO_p)$, under the isomorphism $J\pi_{q-1}(SO_p)/\mathrm{Im}P\cong J(i_*\pi_{q-1}(SO_p))$ (*cf.* [9]). Then, by Theorem 1 of [6], we have

THEOREM 2.2. *Let W_1, W_2 be of type 0 and let $p\neq q$. Then, ∂W_1, ∂W_2 are homotopy equivalent if and only if there exists an isomorphism $h: H_1\to H_2$ such that $\lambda\circ\alpha_1=\lambda\circ\alpha_2\circ h$. Here, λ may be replaced by the homomorphism $J: \pi_{q-1}(SO_{p+1})\to\pi_{p+q}(S^{p+1})$.*

Remark. In Theorem 2.2, if $f: \partial W_1\to\partial W_2$ is a homotopy equivalence, then h can be taken as $h=(i_2)_*\circ f_*\circ(i_1)_*^{-1}$, where $(i_k)_*: H_q(\partial W_k)\to H_q(W_k)$, $k=1,2$, are isomorphisms induced from the inclusions $i_k: \partial W_k\to W_k$, $k=1,2$. Conversely, if $\lambda\circ\alpha_1=\lambda\circ\alpha_2\circ h$ for some isomorphism $h: H_1\to H_2$, then there exists a homotopy equivalence $f:\partial W_1\to\partial W_2$ such that $h=(i_2)_*\circ f_*\circ(i_1)_*^{-1}$. These facts are easily seen from the proof of Theorem 1 of [6].

COROLLARY 2.3. *Let W_1, W_2 be of type 0 and let $p\neq q$. We assume that $J: \pi_{q-1}(SO_{p+1})\to\pi_{p+q}(S^{p+1})$ is monic on $i_*(\pi_{q-1}(SO_p))$. Then, W_1, W_2 are diffeomorphic if and only if ∂W_1, ∂W_2 are homotopy equivalent.*

By a similar argument of Theorem 9.1 of [4], we have

THEOREM 2.4. *Let W_1, W_2 be of type 0 and let $p\neq q$. If ∂W_1, ∂W_2 are diffeomorphic* mod Θ_{p+q}, *then W_1, W_2 are diffeomorphic.*

The following is easily obtained from Lemma 1.1 of [11]. (Cf. Lemma 9.2 of [4]).

LEMMA 2.5. *If $q=4t$ $(t>0)$, there is a commutative diagram*

$$\begin{array}{ccc} H_q(W) & \xrightarrow{\quad\alpha\quad} & \pi_{q-1}(SO_{p+1}) \\ {\scriptstyle \langle P_t(W),\ \rangle}\downarrow & & \downarrow{\scriptstyle i^s_*} \\ Z & \xleftarrow{\quad c\times\quad} & Z\cong\pi_{q-1}(SO), \end{array}$$

where $i^s: SO_{p+1}\to SO$ is the inclusion map, $P_t(W)$ is the t-th Pontrjagin class, and

$$c=\pm\begin{cases} 2(2\cdot t-1)! & \text{if } t \text{ is odd,} \\ (2t-1)! & \text{if } t \text{ is even.}\end{cases}$$

By Lemma 2.5, immediately we have

LEMMA 2.6 *Let $q=4t$ $(t>0)$, $p\neq q$, and let W_1, W_2 be of type 0 if $p=q-1$. If there* *exists a tangential homotopy equivalence* $f:\partial W_1 \to \partial W_2$, *then* $i_*^s \circ \alpha_1 = i_*^s \circ (\alpha_2 \circ h)$ *for the isomorphism* $h=(i_2)_* \circ f_* \circ (i_1)_*^{-1} : H_1 \to H_2$.

Here, we should note that $(i_k)_* : H_k(\partial W_k) \to H_k(W_k)$ is an isomorphism also for $p=q-1$ if W_k is of type 0, since ϕ_k coincides with the intersection number pairing, $k=1, 2$.

THEOREM 2.7. *Let* $q=4t$ $(t>0)$, $p \neq q$, *and let* W_1, W_2 *be of type* 0 *if* $p=q-1$. *Assume that* $i_*^s : \pi_{q-1}(SO_{p+1}) \to \pi_{q-1}(SO)$ *is monic when* $p<q-1$. *Then,* W_1, W_2 *are diffeomorphic if* ∂W_1, ∂W_2 *are tangentially homotopy equivalent.*

Proof. Let $f:\partial W_1 \to \partial W_2$ be the tangential homotopy equivalence. By Lemma 2.6, we have $\alpha_1=\alpha_2 \circ h$ for $h=(i_2)_* \circ f_* \circ (i_1)_*^{-1}$. Here, we note that i_*^s is monic on $i_*\pi_{q-1}(SO_p)$ $\subset \pi_{q-1}(SO_{p+1}))$ when $p=q-1$ (cf. Table 3 of [25]). If $p \geqq q$, Then ϕ_1, ϕ_2 are trivial by definition. If $p=q-1$, then ϕ_1, ϕ_2 are also trivial by assumption. If $p<q-1$, then we have $\phi_1=\phi_2 \circ (h \times h)$ by proposition 1 of [7] . Thus, $(H_1 ; \phi_1, \alpha_1)$ is isomorphic to $(H_2 ; \phi_2, \alpha_2)$, and so W_1, W_2 are diffeomorphic by Theorem 2 *of* [23] .

THEOREM 2.8. *Let* W_1, W_2 *be of type* 0 *and let* $q=4t$ $(t>0)$, $p \neq q$. *If* ∂W_1, ∂W_2 *are tangentially homotopy equivalent, then* W_1, W_2 *are diffeomorphic under the following additional assumptions : For* $i_* : \pi_{q-1}(SO_p) \to \pi_{q-1}(SO_{p+1})$, *there exists a direct sum decomposition* $\mathrm{Im}\, i_* = F+T$ *by the torsion subgroup* T *and a free part* F *such that* $i_*^s : \pi_{q-1}(SO_{p+1}) \to \pi_{q-1}(SO) \cong Z$ *is monic on* F *and* $J : \pi_{q-1}(SO_{p+1}) \to \pi_{p+q}(S^{p+1})$ *is monic on* T.

Proof. Let $f : \partial W_1 \to \partial W_2$ be a tangential homotopy equivalence. By Theorem 2.2 and the remark, and by Lemma 2.6, we have $J \circ \alpha_1 = J \circ \alpha_2 \circ h$ and $i_*^s \circ \alpha_1 = i_*^s \circ \alpha_2 \circ h$ for the isomorphism $h=(i_2)_* \circ f_* \circ (i_1)_*^{-1} : H_1 \to H_2$. Since W_1, W_2 are of type 0, α_1, α_2 take values in $\mathrm{Im}\, i_*$. Let $x \in H_1$, $\alpha_1(x)=a+b$, $a \in F$, $b \in T$, and let $(\alpha_2 \circ h)(x)=a'+b'$, $a' \in F$, $b' \in T$. Then, $i_*^s(a)=i_*^s(a+b)=(i_*^s \circ \alpha_1)(x)=(i_*^s \circ \alpha_2 \circ h)(x)=i_*^s(a'+b')=i_*^s(a')$. Since i_*^s is monic on F, we have $a=a'$. Therefore, $0=J(\alpha_1(x)-(\alpha_2 \circ h)(x))=J(b-b')$. Since J is monic on T, we have $b=b'$. Thus, $\alpha_1(x)=(\alpha_2 \circ h)(x)$ for any $x \in H_1$ and therefore W_1, W_2 are diffeomorphic by Theorem 2.1.

3. J-homomorphism in the metastable range

As is seen in the previous sections, our main theorems depend heavily on the results on J-homomorphism. Let $J^{(i)} : \pi_n(SO_{n+2-i}) \to \pi_{2n+2-i}(S^{n+2-i})$ be the J-homomorphism. The following is seen in [25] .

PROPOSITION 3.1. (i) *If* $n \neq 4t-1$, *then* $J^{(i)}$, $i=1,2,3$, *are monic.*

(ii) *If* $n=4t-1$, *then* $i_*^s : \pi_n(SO_{n+2-i}) \to \pi_n(SO)$ *is monic for* $i=2$, 3, *and monic on* $i_*(\pi_n(SO_n))$ *if* $i=1$.

By studying further, we have the following results.

PROPOSITION 3.2. (i) *If* $n=8s$ *or* $8s+2$, *then* $J^{(i)}$, $i=4, 5$, are monic for $s>0$, *and* $J^{(6)}$ *is also monic for* $s>1$.

(ii) *If* $n=8s+1$, $8s+4$, *or* $8s+5$, *then* $J^{(i)}$, $i=4,5,6$, *are monic for* $s\geqq 0$.

(iii) *If* $n=8s+6$, *then* $J^{(i)}$, $i=4,5,6$, *are monic for* $s>0$.

Proof. For the cases $n=8s$, $8s+1$, the results are already known by Lemma 2.2, Remark 1, and Lemma 3.2 of [8]. We consider the following diagram which is commutative up to sign :

$$(*)\qquad \begin{array}{ccccccc}
\pi_n(SO_{n-4}) & \xrightarrow{i^{(6)}} & \pi_n(SO_{n-3}) & \xrightarrow{i^{(5)}} & \pi_n(SO_{n-2}) & \xrightarrow{i^{(4)}} & \pi_n(SO_{n-1}) \\
\downarrow J^{(6)} & & \downarrow J^{(5)} & & \downarrow J^{(4)} & & \downarrow J^{(3)} \\
\pi_{2n-4}(S^{n-4}) & \xrightarrow{E^{(6)}} & \pi_{2n-3}(S^{n-3}) & \xrightarrow{E^{(5)}} & \pi_{2n-2}(S^{n-2}) & \xrightarrow{E^{(4)}} & \pi_{2n-1}(S^{n-1}),
\end{array}$$

where the upper homomorphisms are induced from inclusions of the rotation groups and the lowers are suspension homomorphisms. Since $\pi_{n+1}(S^{n-3})=0$ $(n>8)$, $\pi_{n+1}(S^{n-4})=0$ $(n>10)$, and $\pi_n(S^{n-4})=0$ $(n>9)$, $i^{(5)}$ is monic for $n>8$ and $i^{(6)}$ is an isomorphism for $n>10$. We note also that $E^{(5)}$, $E^{(6)}$ are monic then. Hence, if $J^{(4)}$ is monic, then $J^{(5)}$, $J^{(6)}$ are monic for $n>8$, $n>10$ respectively.

In the following diagram $(n>6)$ which is commutative up to sign

$$(**)\qquad \begin{array}{ccccc}
 & \overset{\partial^{(4)}}{\nearrow} & \pi_n(SO_{n-2}) & \xrightarrow{i^{(4)}} & \pi_n(SO_{n-1}) \\
Z_8+Z_8=\pi_{n+1}(S^{n-2}) & & \downarrow J^{(4)} & & \downarrow J^{(3)} \\
\alpha_1(n-2),\ \nu_{n-2} & \underset{P^{(4)}}{\searrow} & \pi_{2n-2}(S^{n-2}) & \xrightarrow{E^{(4)}} & \pi_{2n-1}(S^{n-1}),
\end{array}$$

$J^{(3)}$ is monic if $n\neq 4t-1$ by Proposition 3.1. Here, $P^{(4)}=[\ \ ,\iota_{n-2}]$, $\partial^{(4)}$ is the boundary homomorphism, and $\alpha_1(n-2)$, ν_{n-2} denote the corresponding generators. If $n=8s+2$ or $8s+5$ $(s>0)$, then there exist the isomorphisms $\pi_n(SO_{n-k})\cong\pi_{n+1}(V_{m,m-n+k})$, $k=1,2$, by [12], where m is sufficiently large. Therefore, $i^{(4)}$ is precisely known from the results on homotopy groups of Stiefel manifolds of [18]. In fact, if $n=8s+2$ $(s>0)$, $i^{(4)}: \pi_{8s+2}(SO_{8s})=Z_{24}+Z_8\longrightarrow Z_8=\pi_{8s+2}(SO_{8s+1})$ maps Z_8 isomorphically onto Z_8 and Z_{24} to zero. If $n=8s+5$ $(s>0)$, $i^{(4)}: \pi_{8s+5}(SO_{8s+3})=Z_2+Z_2\longrightarrow Z_2+Z_2=\pi_{8s+5}(SO_{8s+4})$ maps as $i^{(4)}(1,0)=(0,0)$ and $i^{(4)}(0,1)=(1,0)$. Note that $i^{(4)}: \pi_5(SO_3)=Z_2\longrightarrow Z_2+Z_2=\pi_5(SO_4)$ maps as $i^{(4)}(1)=(1,0)$ since $SO_4=SO_3\times S^3$. In addition, if $n=8s+4$ $(s>0)$, $i^{(4)}: \pi_{8s+4}(SO_{8s+2})=Z_{12}\longrightarrow Z_2=\pi_{8s+4}(SO_{8s+3})$ is trivial since there exists the following exact sequence induced from the canonical fibering :

$$\begin{array}{ccccccc} \pi_{8s+4}(SO_{8s+2}) & \xrightarrow{i^{(4)}} & \pi_{8s+4}(SO_{8s+3}) & \longrightarrow & \pi_{8s+4}(S^{8s+2}) & \longrightarrow & \pi_{8s+3}(SO_{8s+2}). \\ \| & & \| & & \| & & \| \\ Z_{12} & & Z_2 & & Z_2 & & Z \end{array}$$

On the other hand, by [20] and [13], we know that

$$o(\operatorname{Im} P^{(4)})=\begin{cases} 24 & \text{if } n=8s+2 \quad (s>0), \\ 12 & \text{if } n=8s+4 \quad (s>0), \\ 2 & \text{if } n=8s+5 \quad (s>0), \end{cases}$$

where $o(G)$ denotes the order of a group G. Then, it is easily seen that $J^{(4)}$ is monic for $s>0$, from the diagram (* *). Only for $n=8s+4$ $(s>0)$, we can see it also by the following diagram which is commutative up to sign:

$$\begin{array}{ccccc} 0=\pi_n(SO_{n-3}) & \xrightarrow{i_*} & \pi_n(SO_{n-2}) & \xrightarrow{\pi_*} & \pi_n(S^{n-3}) \\ & & \downarrow J^{(4)} & \cong & \downarrow E^{n-2} \\ & & \pi_{2n-2}(S^{n-2}) & \xrightarrow{H} & \pi_{2n-2}(S^{2n-5}), \end{array}$$

where π_* is induced from the projection and H is the Hopf homomorphism. For the case $s=0$ in (ii), the assertion holds trivially except $J^{(4)}: \pi_5(SO_3)\to\pi_8(S^3)$. However, it must be monic since $i^{(4)}: \pi_5(SO_3)\to\pi_5(SO_4)$ is monic and $J^{(3)}: \pi_5(SO_4)\to\pi_9(S^4)$ by Proposition 3.1.

Let $n=8s+6$ $(s>0)$. The homotopy groups of rotation groups in this case are known by [1] if $s>1$. If $s=1$, those are known by [10], that is, $\pi_{14}(SO_{10})\cong Z_8$, $\pi_{14}(SO_{11})\cong Z_8$, and $\pi_{14}(SO_{12})\cong Z_4+Z_{24}$. We have the following diagram which is commutative up to sign:

$$\begin{array}{ccccccc} & & \underbrace{\begin{matrix} (s=1) & (s>1) \\ Z_4+Z_{24} & Z_4+Z_{48} \end{matrix}} & & (s>0)\ Z_8 & & \\ & & \| & & \| & & \\ & \overset{\partial^{(4)}}{\nearrow} & \pi_{8s+6}(SO_{8s+4}) & \xrightarrow[\text{epi}]{i^{(4)}} & \pi_{8s+6}(SO_{8s+5}) & \overset{\pi_*=0}{\searrow} & \\ Z_8+Z_8=\pi_{8s+7}(S^{8s+4}) & \text{mono } (s>1) & \downarrow J^{(4)} & \text{mono} & \downarrow J^{(3)} & & \pi_{8s+6}(S^{8s+4})=Z_2, \\ & \underset{P^{(4)}}{\searrow} & \pi_{16s+10}(S^{8s+4}) & \xrightarrow{E^{(4)}} & \pi_{16s+11}(S^{8s+5}) & \underset{H}{\nearrow} & \end{array}$$

where π_* is induced from the projection and H is the Hopf homomorphism. Since $\pi_*=0$

by (iv) of Lemma 2.2 of [5] , $i^{(4)}$ is epic for $s\geqq 0$ and therefore $\partial^{(4)}$ is monic for $s>1$. Since $J^{(3)}$ is monic by Proposition 3.1, we have the exact sequence

$$Z_3+Z_8=\pi_{8s+7}(S^{8s+4})\xrightarrow{P^{(4)}}\mathrm{Im}\ J^{(4)}\xrightarrow{E^{(4)}}\mathrm{Im}\ J^{(3)}=Z_8\longrightarrow 0.$$

Since $H[\alpha_1(8s+4),\iota_{8s+4}]=\pm\alpha_1$, $H[\nu_{8s+4},\iota_{8s+4}]=\pm 2\nu$ stably by (5.32) of [24] , we know that $o([\alpha_1(8s+4),\iota_{8s+4}])=3$ and $o([\nu_{8s+4},\iota_{8s+4}])=4$ or 8. Here, $o(\gamma)$ denotes the order of a group element γ. If $s=1$, then we have $o(\mathrm{Im}\ J^{(4)})\leqq 96$. Therefore, the above sequence shows that $o([\nu_{12},\iota_{12}])=4$ and $o(\mathrm{Im}\ J^{(4)})=96$. Hence, $J^{(4)}$ is monic if $s=1$. Let $s>1$. Since $o([\nu_{8s+4},\iota_{8s+4}])=8$ by Table 2 of [16] , the above sequence shows that $o(\mathrm{Im}\ J^{(4)})=192$ and therefore $J^{(4)}$ is monic. Thus, $J^{(4)}$ is monic for $n=8s+6$ $(s>0)$.

Since $J^{(5)}$, $J^{(6)}$ follow $J^{(4)}$, this completes the proof.

PROPOSITION 3.3. *If $n=4t-1$ $(t\geqq 3)$, there exist the direct sum decompositions $\pi_n(SO_{n+2-i})\cong Z_2+Z$, $i=4,5,6$, for which we have*

(i) *$i_*^s:\pi_n(SO_{n+2-i})\rightarrow\pi_n(SO)$ is monic on the free part for $i=4,5,6$.*

(ii) *If $n=8s+3$ $(s>0)$, then $J^{(i)}$, $i=4,5,6$, are monic on the respective torsion subgroups.*

(iii) *If $n=8s+7$ $(s>0)$, then, on the respective torsion subgroups, $J^{(i)}$, $i=4,5,6$, are monic if $s\neq 2^l-1$ and are all trivial if $s=2^l-1$.*

Proof. (i) will come to be clear in proving (ii) and (iii) by (ii) of Proposition 3.1.

(ii) By [12] , we have the exact sequence

$$0\longrightarrow\pi_{8s+4}(V_{m,m-8s-i})\longrightarrow\pi_{8s+3}(SO_{8s+i})\longrightarrow\pi_{8s+3}(SO_m)=Z\longrightarrow 0,$$

for $s\geqq 2$, $i\geqq -2$ or $s=1$, $i\geqq 0$, where m is sufficiently large. Hence, by [18] we know the following correspondence of the noted generators :

$$\begin{array}{ccccccc}
\pi_{8s+3}(SO_{8s-1}) & \xrightarrow{i^{(6)}} & \pi_{8s+3}(SO_{8s}) & \xrightarrow{i^{(5)}} & \pi_{8s+3}(SO_{8s+1}) & \xrightarrow{i^{(4)}} & \pi_{8s+3}(SO_{8s+2}),\\
\| & & \| & & \| & & \|\\
Z_2\ \ z_1 & & Z_2\ \ y_1 & & Z_2\ \ x_1 & & \\
+ & & + & & + & & \\
Z\ \ z_2 & & Z\ \ y_2 & & Z\ \ x_2 & & Z\ \ w
\end{array}$$

where each generator is mapped horizontally to a generator and the one with no corresponding generator is mapped to zero. If $s>1$, we take z_2 so that $i_*^s(z_2)=\tau$, the generator of $\pi_{8s+3}(SO)\cong Z$. Then, the unique element z_1 and z_2 will determine the other generators. If $s=1$, we take y_2 so that $i_*^s(y_2)=\tau$. Since $i^{(6)}$ is an isomorphism even if $s=1$, we choose z_1, z_2 so that $i^{(6)}(z_1)=y_1$ which is the unique non-trivial torsion element of $\pi_{11}(SO_8)$ and $i^{(6)}(z_2)=y_2$. The other generators are determined canonically.

By [20] or [13] , it is known that $[\nu_{n-2},\iota_{n-2}]\neq 0$ if $n=8s+3$ $(s>0)$. Hence, by the diagram (**) $J^{(4)}$ is monic on the torsion subgroup, and therefore $J^{(5)}$, $J^{(6)}$ by the diagram(*).

(iii) By [12] , we have the exact sequence

$$0 \longrightarrow \pi_{8s+8}(V_{m,m-8s-i}) \longrightarrow \pi_{8s+7}(SO_{8s+i}) \longrightarrow cZ(\subset Z = \pi_{8s+7}(SO_m)) \longrightarrow 0\,,$$

for $s \geqq 1$, $i \geqq 3$. Here, m is sufficiently large, $c=2$ if $s=1$, $i=3,4$, and $c=1$ otherwise. Hence, by [18] we know the following correspondence of the noted generators :

$$\begin{array}{ccccccc}
\pi_{8s+7}(SO_{8s+3}) & \xrightarrow{i^{(6)}} & \pi_{8s+7}(SO_{8s+4}) & \xrightarrow{i^{(5)}} & \pi_{8s+7}(SO_{8s+5}) & \xrightarrow{i^{(4)}} & \pi_{8s+7}(SO_{8s+6}), \\
\| & & \| & & \| & & \| \\
Z_2 \quad z_1 & & Z_2 \quad y_1 & & Z_2 \quad x_1 & & \\
+ & & + & & + & & \\
Z \quad z_2 & & Z \quad y_2 & \xrightarrow{\times 2(s=1)} & Z \quad x_2 & & Z \quad w
\end{array}$$

where each generator is mapped in a way similar to the above except the indicated one. If $s>1$, we take z_2 so that $i_*^s(z_2)=\tau$, the generator of $\pi_{8s+7}(SO) \cong Z$. Then, the unique element z_1 and z_2 will determine the other generators. If $s=1$, we can take z_2, x_2 so that $i_*^s(z_2)=2\tau$, $i_*^s(x_2)=\tau$. Then, with the unique element z_1, they will determine the other generators. Here, $i^{(5)}(y_2)=2x_2$ or x_1+2x_2. If $i^{(5)}(y_2)=x_1+2x_2$, we replace z_2, y_2 by $z_2'=z_1+z_2$, $y_2'=y_1+y_2$ respectively. Then, we have $i^{(5)}(y_2')=2x_2$. Thus, the above correspondence are obtained.

On the other hand, $[\upsilon_{8s+5}, \iota_{8s+5}] \neq 0$ if $s \neq 2^l-1$ by [20] or [13] and $[\upsilon_{8s+5}, \iota_{8s+5}]=0$ if $s=2^l-1$ by [15] and [17] . Hence, we have the result using the diagrams (*), (**) similarly, where we must note that $E^{(5)}$, $E^{(6)}$ are monic.

This completes the proof.

4. Proof of main theorems

Let M be a closed smooth m-manifold which is simply connected and statisfies the hypotheses (H_1), (H_2) in § 1. Then, by a set of surgeries of the generators of $H_p(M)$, M can be modified to a homotopy m-sphere Σ (cf. Theorem 6.3 of [3]). Therefore, by constructing conversely, $M \# (-\Sigma)$ is the boundary of a handlebody $W \in \mathscr{H}(m+1, r, q)$ and hence $M = \partial W \# \Sigma$, where $q=m-p$ and $r=$ rank $H_p(M)$. We can show that W is of type 0. In fact, if $p=q-1$, it follows from the homology exact sequence of $(W, \partial W)$

$$0 \longrightarrow H_q(\partial W) \longrightarrow H_q(W) \xrightarrow{j_*} H_q(W, \partial W) \longrightarrow H_{q-1}(\partial W) \longrightarrow 0.$$

Because, since $H_q(W, \partial W) \cong H^q(W) \cong \mathrm{Hom}(H_q(W), Z)$ and $H_{q-1}(\partial W) \cong H_p(M)$, we have rank $H_q(W, \partial W) =$ rank $H_{q-1}(\partial W) = r$. So, j_* must be trivial. Since j_* can be represented by the matrix representation of the intersection number pairing which coincides with ϕ, ϕ must be trivial. Let $p<q-1$ and $2p>q$. By Lemma 1.1 of [7] , $\partial W - *$ has the homotopy type of $X = (\vee_{i=1}^r S_i^p) \cup_{\{\lambda_j\}} (\cup_{j=1}^r D_j^q)$, where $\lambda_j = \Sigma_{i=1}^r \lambda_{ij}$ as an element of $\pi_{q-1}(\vee_{i=1}^r S_i^p) = \Sigma_{i=1}^r \pi_{q-1}(S_i^p)$ and λ_{ij}, $i, j=1,2,\cdots,r$, are linking elements if $i \neq j$ and self-linking elements if $i=j$. From the hypothesis (H_1), $\partial W - *$ has also the homotopy type of $X' = (\vee_{i=1}^r S_i^p) \vee (\vee_{j=1}^r S_j^q)$. Hence, every λ_j must be trivial. In fact, there exists a homotopy equivalence $f: X \to X'$ and it may be assumed that $f(\vee_{i=1}^{r} S_i^p) \subset \vee_{i=1}^{r} S_i^p$. Let $\bar{f}: (X, \vee_{i=1}^{r} S_i^p) \to (X', \vee_{i=1}^{r} S_i^p)$ be the map defined by f. Then, we have the following commutative diagram :

$$
\begin{array}{ccc}
H_q(X) & \xrightarrow[\cong]{f_*} & H_q(X') \\
\downarrow \cong & & \downarrow \cong \\
H_q(X, \vee_{i=1}^{r} S_i^p) & \xrightarrow{\bar{f}_*} & H_q(X', \vee_{i=1}^{r} S_i^p) \\
\uparrow \cong & & \uparrow \cong \\
\pi_q(X, \vee_{i=1}^{r} S_i^p) & \xrightarrow{\bar{f}_*} & \pi_q(X', \vee_{i=1}^{r} S_i^p) \\
\partial \downarrow & & \downarrow \partial' \\
\pi_{q-1}(\vee_{i=1}^{r} S_i^p) & \xrightarrow{(f \mid \vee_{i=1}^{r} S_i^p)_*} & \pi_{q-1}(\vee_{i=1}^{r} S_i^p),
\end{array}
$$

where $(X, \vee_{i=1}^{r} S_i^p)$, $(X', \vee_{i=1}^{r} S_i^p)$ are $(q-1)$-connected and p, $q>1$ from assumption. Since f_* is an isomorphism, the two $\bar{f}_*$ are isomorphisms. Therefore, $(f \mid \vee_{i=1}^{r} S_i^p)_*$ is also an isomorphism by applying the five lemma to the homotopy exact sequences of the pairs $(X, \vee_{i=1}^{r} S_i^p)$, $(X', \vee_{i=1}^{r} S_i^p)$. Then, by the diagram, each λ_j must be trivial since ∂' is trivial. Let $\{e_1, e_2, \cdots, e_r\}$ be the basis of $H_q(W)$ determined by the handles. Then $\phi(e_i, e_j) = E\lambda_{ij}$ $(i \neq j)$ and $\phi(e_i, e_i) = E\pi_* \alpha(e_i) = E\lambda_{ii}$ by Lemma 7 and Lemma 3 of [23]. Hence $\phi = 0$, and therefore, W is of type 0.

Proof of Theorems 1,2. M_1, M_2 are respectively the boundaries $(\mathrm{mod}\,\Theta_m)$ of handlebodies W_1, W_2 of $\mathscr{H}(m+1, r, q)$ which are of type 0. In the cases (A), (B), $\cdots$, and (F), $J: \pi_{q-1}(SO_{p+1}) \to \pi_{p+q}(S^{p+1})$ corresponds to $J^{(1)}$, $J^{(2)}$, $\cdots$, and $J^{(6)}$, respectively. Let $n \neq 4t-1$. Then, $J^{(i)}$, $i=1, 2, 3$, are monic from Proposition 3.1, and $J^{(i)}$, $i=4,5,6$, are also monic from Proposition 3.2. Hence, by Corollary 2.3, M_1, M_2 are diffeomorphic $\mathrm{mod}\,\Theta_m$. Let $n=4t-1$. Then, by (ii) of Proposition 3.1 and Theorem 2.7, M_1, M_2 are diffeomorphic mod Θ_m for the cases (A), (B), and (C). By Proposition 3.3 and Theorem 2.8, M_1, M_2 are also diffeomorphic mod Θ_m for the cases (D), (E), and (F), under the hypothesis $s \neq 2^l - 1$ when $n=8s+7$, that is, $n \neq 2^l - 1 (l \geqq 4)$. We note that summing a homotopy sphere to a manifold does not affect its tangent bundle (cf. Theorem 2 of [19]).

Proof of Theorem 3. Let $n = 2^l - 1 (l \geqq 4)$. Then, each torsion subgroup of $\pi_n(SO_{n+2-i})$, $i=4, 5, 6$, which is isomorphic to Z_2, is incuded in Ker $J^{(i)}$ by (iii) of Proposition 3.3. Let $\mathscr{B}^{(i)}$ be the S^{n+1-i}-bundle over S^{n+1} which corresponds to such an element b_i of order 2 of Ker $J^{(i)}$, $i=4, 5, 6$. Then, $\mathscr{B}^{(i)}$, $i=4,5,6$, admit cross-sections. This is clear since $b_4 = i^{(5)}(b_5)$, $b_5 = i^{(6)}(b_6)$, and we have the exact sequence

$$\pi_n(SO_{n-5}) \longrightarrow \pi_n(SO_{n-4}) \longrightarrow \pi_n(S^{n-5})=0 \quad (n \geqq 12)$$

induced from the canonical fibering. Every b_i is sent to zero by i_*^s since $\pi_{8s+7}(SO) \cong Z$. Let $\bar{\mathscr{B}}^{(i)}$ be the $(n+2-i)$-disk bundle over S^{n+1} associated to $\mathscr{B}^{(i)}, i=4, 5, 6$, and let $B^{(i)}$, $\bar{B}^{(i)}$ be the total spaces of $\mathscr{B}^{(i)}, \bar{\mathscr{B}}^{(i)}$, respectively. Then, it is seen easily that $\bar{B}^{(i)}$, $i=4, 5, 6$, are parallelizable and therefore $B^{(i)}$, $i=4, 5, 6$, are π-manifolds. Thus, we have a pair of π-manifolds $B^{(i)}$, $S^{n+1-i} \times S^{n+1}$ for each $i=4$, 5, and 6. They are homotopy equivalent by Theorem 2.2 since $J^{(i)}(b_i)=0$, and hence tangentially homotopy equivalent since they are π-manifolds. However, they are not diffeomorphic even modulo homotopy m-spheres. Because, $\bar{B}^{(i)}$, $D^{n+2-i} \times S^{n+1}$ are considered as handlebodies of type 0 of $\mathscr{H}(m+1, 1, n+1)$ with the boundaries $B^{(i)}$, $S^{n+1-i} \times S^{n+1}$, respectively. Let α, α' be the homomorphisms belonging to the invariant systems of $\bar{B}^{(i)}$, $D^{n+2-i} \times S^{n+1}$ respectively. Then, $\alpha \neq 0$ and $\alpha'=0$, and hence they are not diffeomorphic by Theorem 2.1. Furthermore, Theorem 2.4 shows that $B^{(i)}$, $S^{n+2-i} \times S^{n+1}$ can not be diffeomorphic mod Θ_m. Generally, the following pair of M_1, M_2 forms such an example as above for $i=4, 5, 6$ and for any $r \geqq 1$:

$$M_1 = B^{(i)} \# \overbrace{S^{n+1-i} \times S^{n+1} \# \cdots \# S^{n+1-i} \times S^{n+1}}^{r-1},$$

$$M_2 = \overbrace{S^{n+1-i} \times S^{n+1} \# \cdots \# S^{n+1-i} \times S^{n+1}}^{r}.$$

The proof of the latter half of Theorem 3 is given as follows. Let $M = \partial W (\bmod \Theta_m)$, $M' = \partial W' (\bmod \Theta_m)$, where W, $W' \in \mathscr{H}(m+1, 1, n+1)$ and they are of type 0. Let $(H\,;\,\alpha)$, $(H'\,;\,\alpha')$ be the invariant systems of W, W' respectively. Assume that M, M' are tangentially homotopy equivalent but they are not diffeomorphic modΘ_m, where $n=2^l-1(l \geqq 4)$. Then, by Lemma 2.6, we have an isomorphism $h: H \rightarrow H'$ which satisfies $(i_*^s \circ \alpha)(e) = (i_*^s \circ \alpha')(h(e))$. Here, e is the basis element of $H(\operatorname{rank} H=1)$. Let $\alpha(e)=b+a$, $\alpha'(h(e))=b'+a'$, where b, b' are torsion elements, a, a' are elements of the free part, and $\alpha(e)$, $\alpha'(h(e))$ belong to $\pi_n(SO_{n+2-i}) \cong Z_2+Z$, $i=4, 5, 6$. By the above equality, we know that $i_*^s(a) = i_*^s(\alpha(e)) = i_*^s(\alpha'(h(e))) = i_*^s(a')$ and hence $a=a'$ since i_*^s is monic on the free part by Proposition 3.3. b, b' belong to the torsion subgroup which is isomorphic to Z_2, and therefore $b'=b$ or $b'=b+b_i$. If $b'=b$, then $\alpha(e)=\alpha'(h(e))$ and therefore W, W' are diffeomorphic by Theorem 2.1. So, $M=M'$ mod Θ_m and this is a contradiction. Hence, $b'=b+b_i$ and we have $\alpha'(h(e))=\alpha(e)+b_i$. Let $M''=\partial W''(\bmod \Theta_m)$, where W'' belongs to $\mathscr{H}(m+1, 1, n+1)$ and is of type 0, and let $(H''\,;\,\alpha'')$ be the invariant system of W''. If (M'', M) is such a pair as (M', M), then, similarly as the above, there exists an isomorphism $g: H \rightarrow H''$ such that $\alpha''(g(e))=\alpha(e)+b_i$. Therefore, $\alpha'(h(e))=\alpha''(g(e))$ for the basis $\{e\}$ of H. Hence, $\alpha' \circ h = \alpha'' \circ g$, that is, $\alpha' = \alpha'' \circ (g \circ h^{-1})$. Thus, W', W'' are diffeomorphic by Theorem 2.1, and therefore M', M'' are diffeomorphic mod Θ_m. This completes the proof.

Proof of Corollary 4. In the proof of the latter half of Theorem 3, we note that if M', M'' are just the boundaries of W', W'' respectively, then M', M'' are diffeomorphic. We have known that $S^{n-3} \times \Sigma^{n+1}$ and $B^{(4)}$ are respectively tangentially homotopy equivalent to

$S^{n-3}\times S^{n+1}$ but neither of them is diffeomorphic to $S^{n-3}\times S^{n+1} \bmod \Theta_{2n-2}$. We note that $B^{(4)}=\partial\bar{B}^{(4)}$, $S^{n-3}\times\Sigma^{n+1}=\partial(D^{n-2}\times\Sigma^{n+1})$, and $D^{n-2}\times\Sigma^{n+1}$ can be considered as a handlebody of $\mathscr{H}(2n-1, 1, n+1)$. In fact, represent the homology class $(*\times\Sigma^{n+1})$ by an imbedded $(n+1)$-sphere in $D^{n-2}\times\Sigma^{n+1}(n\geqq 8)$ and take the tubular neighbourhood N with boundary. Then, it is clear from the h-cobordism theorem that $D^{n-2}\times\Sigma^{n+1}$ is diffeomorphic to N, which can be considered as a handlebody of $\mathscr{H}(2n-1, 1, n+1)$. Clearly, $\bar{B}^{(4)}$ and N are of type 0. Thus, $S^{n-3}\times\Sigma^{n+1}$ is just diffeomophic to $B^{(4)}$. Similar arguments also hold for $B^{(5)}$, $S^{n-4}\times\Sigma^{n+1}$ and for $B^{(6)}$, $S^{n-5}\times\Sigma^{n+1}$. This completes the proof.

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