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An Elementary Proof of the Prime Ideal Theorem with Remainder Term

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§ 1. **Introduction.** Let K be an algebraic number field of degree n and $\mathfrak{o}$ be the integral domain consisting of all integers in K (the unit ideal of K). Small german characters denote integral ideals in K; $\mathfrak{p}$ represents a prime ideal in K, and x denotes a real number $x \geqq 1$. We now introduce some ideal-theoretic functions:

$$\theta_K(x) = \sum_{N\mathfrak{p} \leqq x} \log N\mathfrak{p}, \tag{1.1}$$

$$\psi_K(x) = \sum_{N\mathfrak{a} \leqq x} \Lambda_K(\mathfrak{a}), \tag{1.2}$$

where

$$\Lambda_K(\mathfrak{a}) = \log N\mathfrak{p}, \text{ if } \mathfrak{a} = \mathfrak{p}^E \ (E = 1.2. \cdots), \quad = 0, \text{ otherwise.} \tag{1.3}$$

From now on, E denotes a non-negative integer.

Using Selberg's elementary method[8] of the prime number theorem Shapiro obtained the prime ideal theorem[9], i.e.

$$\theta_K(x) - x = o(x). \tag{1.4}$$

It is easy to deduce that (1.4) is equivalent to

$$\psi_K(x) - x = R_K(x) = o(x). \tag{1.5}$$

Kuhn succeeded elementarily in dealing with the remainder term in the rational number field with a certain constant c ($c=1/10$)[4]

$$R_K(x) = O\left(\frac{x}{\log^c x}\right). \tag{1.6}$$

In this note we propose to extend Kuhn's method so as to yield a proof of the prime ideal theorem (1.6) in K with the remainder term. We shall first give an inversion formula (Lemma 2.3) and derive from this formula the following:

$$\sum_{N\mathfrak{a} \leqq x} \frac{\Lambda_K(\mathfrak{a})}{N\mathfrak{a}} = \log x + O(1) \tag{1.7}$$

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and

$$R_K(x)\ \log x + \sum_{N\mathfrak{a}\leqq x} R_K\left(\frac{x}{N\mathfrak{a}}\right) \Lambda_K(\mathfrak{a}) = O(x). \tag{1.8}$$

by means of the method of Iseki-Tatuzawa[3] and Shapiro[9]. It is easily seen that (1.8) is equivalent to the following result of Selberg in K:

$$\psi_K(x)\ \log x + \sum_{N\mathfrak{a}\leqq x} \Lambda_K(\mathfrak{a})\ \psi_K\left(\frac{x}{N\mathfrak{a}}\right) = 2x\ \log\ x + O(x). \tag{1.9}$$

Ayoub's proof of (1.9)[1] is based on the Iseki-Tatuzawa's ideas and the Tcheby-scheff-Poincaré's formula[6].

We assume the following important result of Weber[10],[7]:

$$B_K(x) = \sum_{N\mathfrak{a}\leqq x} = \alpha x + \lambda_K(x), \tag{1.10}$$

where $\lambda_K(x)=O(x^{\omega})$ with $\omega=1-1/n$ and

$$\alpha = \frac{2^{n_1+n_2}\ \pi^{n_2}\ Rh}{w\sqrt{|d|}}$$

Here n_1 and n_2 are the numbers of real and pairs of complex conjugate fields, w is the order of the group of roots of unity in K, d is the discriminant of K, R is regulator and h is the class number. If we like to avoid (1.10) we must follow on the theory of abstract prime number theory of Shapiro and Forman[2],[5]. We shall assume more weaker result (Lemma 3.2) than Shapiro's prime ideal theorem (1.4) and from this result and (1.8) derive the Kuhn's fundamental formula in an algebraic number field K. Using this fundamental formula and (1.7) in much the same way as Khun uses it in the rational number field we shall prove the result (1.6) with a certain constaut c.

If f and g are functions of certain variables and g is positve. then the notation $f=O(g)$ or $f \ll g$ means that there exists a positive constant c such that $|f| \leqq cg$ in the domain designated and c is a positive constant depending only on K.

§ 2. **Some Inversion Fofmulas and Their Applications.** We first introduce Möbius function $\mu_K(\mathfrak{a})$ defined as $\mu_K(\mathfrak{a})=1$ if $\mathfrak{a}=\mathfrak{o}$, $=(-1)^{\nu}$ if $\mathfrak{a}$ is squarefree and is the product of ν distict prime ideals, $=0$ otherwise. Then it follows easily that $\sum_{\mathfrak{t}|\mathfrak{a}} \mu_K(\mathfrak{t})=[1/N\mathfrak{a}]$. The familiar Möbius inversion formula is given by

Lemma 2.1. *If $f(x)$ is a given complex valued function defined for all $x>0$ and $g(x)$ is defined by*

$$g(x) = \sum_{N\mathfrak{a}\leqq x} f\left(\frac{x}{N\mathfrak{a}}\right),$$

then

$$f(x) = \sum_{N\mathfrak{a}\leqq x} \mu_K(\mathfrak{a})\ g\left(\frac{x}{N\mathfrak{a}}\right).$$

As a Corollary of this Lemma we have:

Lemma 2.2. *If F_K is a complex valued function defined for all ideals $\mathfrak{a}$ in K and*

$$G_K(\mathfrak{a}) = \sum_{\mathfrak{t}|\mathfrak{a}} F_K(\mathfrak{t})$$

then

$$F_K(\mathfrak{a}) = \sum_{\mathfrak{t}|\mathfrak{a}} \mu_K(\mathfrak{t})\, G_K\left(\frac{\mathfrak{a}}{\mathfrak{t}}\right)$$

A complex valued function $m_K(\mathfrak{a})$ (or $a(x)$) is called a completely multiplicative idealtheoretic (additive number-theoretic) function if for any pair of ideals $\mathfrak{a}_1$, $\mathfrak{a}_2$ (reals x_1, x_2)

$$m_K(\mathfrak{a}_1 \cdot \mathfrak{a}_2) = m_K(\mathfrak{a}_1)\, m_K(\mathfrak{a}_2) \qquad (a(x_1 x_2) = a(x_1) + a(x_2)).$$

It follows immediately that $a(1) = 0$ and if $m_K(\mathfrak{a})$ is not identically zero (only such multiplicative functions will be considered below), $m_K(\mathfrak{o}) = 1$.

Next, we define

$$(2.1) \qquad \Lambda_K^{(E)}(\mathfrak{a}) = \sum_{\mathfrak{t}|\mathfrak{a}} \mu_K(\mathfrak{t})\, a^E\left(\frac{N\mathfrak{a}}{N\mathfrak{t}}\right),$$

then, using Lemma 2.2 we have

$$(2.2) \qquad a^E(N\mathfrak{a}) = \sum_{\mathfrak{t}|\mathfrak{a}} \Lambda_K^{(E)}(\mathfrak{t}).$$

For $E=0$, we obtain

$$(2.3) \qquad \Lambda_K^{(0)}(\mathfrak{a}) = \sum_{\mathfrak{t}|\mathfrak{a}} \mu_K(\mathfrak{t}) = [1/N\mathfrak{a}]$$

$$(2.4) \qquad \sum_{\mathfrak{t}|\mathfrak{a}} \Lambda_K^{(0)}(\mathfrak{t}) = \sum_{\mathfrak{t}|\mathfrak{a}} [1/N\mathfrak{t}] = 1.$$

For $E=1$, we have

$$(2.5) \qquad \Lambda_K^{(1)}(\mathfrak{a}) = \sum_{\mathfrak{t}|\mathfrak{a}} \mu_K(\mathfrak{t})\, a\left(\frac{N\mathfrak{a}}{N\mathfrak{t}}\right)$$

$$(2.6) \qquad \sum_{\mathfrak{t}|\mathfrak{a}} \Lambda_K^{(1)}(\mathfrak{t}) = a(N\mathfrak{a}).$$

We obtain the following inversion formula:

Lemma 2.3. *Let $m_K(\mathfrak{a})$ ($a(x)$) be a complex valued comletely multiplicative idealtheoretic (number-theoretic) function. If $F(x)$ is a complex valued function defined for $x>0$ and $G^{(E)}(x)$ is defined by*

$$(2.7) \qquad G^{(E)}(x) = \sum_{N\mathfrak{t} \leq x} m_K(\mathfrak{t})\, F\left(\frac{x}{N\mathfrak{t}}\right) a^E(x),$$

then

$$H^{(E)}(x) = \sum_{N\mathfrak{t} \leq x} \mu_K(\mathfrak{t})\, m_K(\mathfrak{t})\, G^{(E)}\left(\frac{x}{N\mathfrak{t}}\right) = F(y)\, a^E(x)$$

and

$$= \sum_{N\mathfrak{t}\leq x} m_K(\mathfrak{t})\, F\left(\frac{x}{N\mathfrak{t}}\right)\left(a\left(\frac{x}{N\mathfrak{t}}\right) + \Lambda(\mathfrak{t})\right)^{(E)}.$$

Here $(\cdots\cdots)^{(E)}$ *means the symbolic power and* $(a(\cdots))^{(E)} = a^E(\cdots)$, $(\Lambda(\cdots))^{(E)} = \Lambda^{(E)}(\cdots)$.

proof. Using Lemma 2.1 we have

$$H^{(E)}(x) = \sum_{N\mathfrak{t}\leq x} \mu_K(\mathfrak{t})\, m_K(\mathfrak{t})\, G^{(E)}\left(\frac{x}{N\mathfrak{t}}\right) = F(x)\, a^E(x)$$

from (2.7)

$$= \sum_{N\mathfrak{t}\leq x} \mu_K(\mathfrak{t})\, m_K(\mathfrak{t}) \sum_{N\mathfrak{a}\leq \frac{x}{N\mathfrak{t}}} m_K(\mathfrak{a})\, F\left(\frac{x}{N\mathfrak{t}}/N\mathfrak{a}\right) a^E\left(\frac{x}{N\mathfrak{t}}\right)$$

$$= \sum_{N\mathfrak{a}\leq x} m_K(\mathfrak{a})\, F\left(\frac{x}{N\mathfrak{a}}\right) \sum_{\mathfrak{t}|\mathfrak{a}} \mu_K(\mathfrak{a})\, a^E\left(\frac{x}{N\mathfrak{t}}\right)$$

$$= \sum_{N\mathfrak{a}\leq x} m_K(\mathfrak{a})\, F\left(\frac{x}{N\mathfrak{a}}\right) \sum_{\mathfrak{t}|\mathfrak{a}} \mu_K(\mathfrak{t}) \left(a\left(\frac{x}{N\mathfrak{a}}\right) + a\left(\frac{N\mathfrak{a}}{N\mathfrak{t}}\right)\right)^E.$$

Combining the above with the definition (2.1) of $\Lambda_K^{(E)}(\mathfrak{a})$ we get

$$H^{(E)}(x) = \sum_{N\mathfrak{a}\leq x} m_K(\mathfrak{a})\, F\left(\frac{x}{N\mathfrak{a}}\right)\left(a\left(\frac{x}{N\mathfrak{a}}\right) + \Lambda(\mathfrak{a})\right)^{(E)},$$

and the Lemma follows.

Making use of this inversion formula with $m_K(\mathfrak{a}) = 1$ and $a(x) = \log x$, we shall now derive (1.7) and (1.8). Here we use the following well-known results without proofs:

$$(2.8)\quad \begin{cases} \sum_{N\mathfrak{a}\leq x} \dfrac{\log^E N\mathfrak{a}}{N\mathfrak{a}} = \dfrac{1}{E+1}\alpha \log^{E+1} x + \gamma_E + O(x^{\omega-1}\log^E x), \\ \sum_{N\mathfrak{a}\leq x} \log^E N\mathfrak{a} = \alpha\, x \log^E x - \alpha\, Ex + O(x^{\omega}\log^E x) \\ \sum_{N\mathfrak{a}\leq x} \dfrac{\log^E N\mathfrak{a}}{N\mathfrak{a}^{\omega}} = \alpha\, n \log x.\ x^{1-\omega} + O(\log^{E+1} x) \end{cases}$$

where $\omega = 1-1/n$ and γ_E is a constant depending only on K.

Case 1. For $F(x)=1$, $E=0$, we have

$$G^{(0)}(x) = B_K(x) = \alpha x + O(x^{\omega}), \quad H^{(0)}(x) = 1$$

and

$$(2.9)\qquad \sum_{N\mathfrak{a}\leq x} \frac{\mu_K(\mathfrak{a})}{N\mathfrak{a}} = O(1).$$

Case 2. For $F(x)=x$, $E=O$, using Of (2.8) we have

$$G^{(0)}(x) = \sum_{N\mathfrak{a}\leq x} \frac{x}{N\mathfrak{a}} = \alpha x \log x + \gamma_0 x + O(x^{\omega}), \quad H^{(0)}(\mathrm{x}) = x$$

and

$$(2.10) \qquad \sum_{N\mathfrak{a}\leqq x} \frac{\mu_K(\mathfrak{a})}{N\mathfrak{a}} \log\left(\frac{x}{N\mathfrak{a}}\right) = O(1).$$

Case 3. For $F(x)=1$, $E=1$, using of (1.10), we have

$$G^{(1)}(x) = \sum_{N\mathfrak{a}\leqq x} \log x = \alpha x \log x + O(x^{\omega}\log x), \quad H^{(1)}(x) = \log x + \psi_K(x)$$

and

$$(2.11) \qquad \psi_K(x) = O(x).$$

Case 4. For $F(x)=x$, $E=1$, using of (2.8) and (2.11), we have

$$G^{(1)}(x) = \sum_{N\mathfrak{a}\leqq x} \left(\frac{x}{N\mathfrak{a}}\right) \log x = \alpha x \log x + \gamma_0 x \log x + O(x^{\omega}\log x),$$

$$H^{(1)}(x) = x \log x + x \sum_{N\mathfrak{a}\leqq x} \frac{\Lambda_K(\mathfrak{a})}{N\mathfrak{a}}$$

and

$$(2.12) \qquad \alpha \sum_{N\mathfrak{a}\leqq x} \frac{\mu_K(\mathfrak{a})}{N\mathfrak{a}} \log^2\left(\frac{x}{N\mathfrak{a}}\right) = \log x + \sum_{N\mathfrak{a}\leqq x} \frac{\Lambda_K(\mathfrak{a})}{N\mathfrak{a}} + O(1)$$

Case 5. For $F(x)=x\log x$, $E=0$, using of (2.8) and (2.12), we have

$$G^{(0)}(x) = \sum_{N\mathfrak{a}\leqq x} \frac{x}{N\mathfrak{a}} \log\left(\frac{x}{N\mathfrak{a}}\right)$$

$$= \frac{1}{2} x \log^2 x + \gamma_0 x \log x - \gamma_1 x + 0(x^{\omega}\log x),$$

$$H^{(0)}(x) = x \log x$$

and

$$(1.7) \qquad \sum_{N\mathfrak{a}\leqq x} \frac{\Lambda_K(\mathfrak{a})}{N\mathfrak{a}} = \log x + O(1).$$

Case 6. For $F(x)=R_k+\gamma_0/\alpha+1$, $E=1$ using of (2.8) and (1.7) we have

$$G^{(1)}(x) = \sum_{N\mathfrak{a}\leqq x} R_K\left(\frac{x}{N\mathfrak{a}}\right) \log x + (1+\gamma_0/\alpha) \sum_{N\mathfrak{a}\leqq x} \log x = 0(x^{\omega}\log^2 x),$$

$$H^{(1)}(x) = R_K(x)\log x + \sum_{N\mathfrak{a}\leqq x} R_K\left(\frac{x}{N\mathfrak{a}}\right) \Lambda_K(\mathfrak{a}) + O(x)$$

and

$$(1.8) \qquad R_K(x)\log x + \sum_{N\mathfrak{a}\leqq x} R\left(\frac{x}{N\mathfrak{a}}\right) \Lambda_K(\mathfrak{a}) = O(x).$$

§ 3. Kuhn's Fundamental Theorem in an Algebraic Number Field.

Lemma 3.1.

$$(3.1) \qquad |R_K(x)\log x| \leqq 2\left|\sum_{N\mathfrak{a}^2\leqq x} (\Lambda_K(\mathfrak{a})-1) R_K\left(\frac{x}{N\mathfrak{a}}\right)\right| + O(x).$$

prof. A simple calculation yields

$$\sum_{N\mathfrak{a}\leqq x} \Lambda_K(\mathfrak{a})\,\psi_K\left(\frac{x}{N\mathfrak{a}}\right) = 2 \sum_{N\mathfrak{a}^2\leqq x} \Lambda_K(\mathfrak{a})\,\psi_K\left(\frac{x}{N\mathfrak{a}}\right) + O(x),$$

so that (1.8) may be rewritten as (3.1).

From now on we assume the following Shapiro's

Lemma 3.2. *We have a constant* $\alpha(x_1)>0$ *such that*

$$|R_K(x)| \leqq \alpha(x_1)\,x, \quad \alpha(x_1) \leqq 0.11 \quad (x \geqq x_1) \tag{3.2}$$

for x_1 *sufficiently large.*

Making use of Lemma 3.2 for η: $x_1 \leqq N \leqq \eta$, we obtain

$$|R_K(\eta)| \leqq \beta(N)\,\eta \text{ with } \beta(N) \leqq \alpha(x_1) \leqq 0.11 \tag{3.3}$$

for x_1 sufficiently large. Taking

$$N^2 \leqq L \leqq \sqrt{x}, \tag{3.4}$$

we get for all $\mathfrak{a}$: $N<N\mathfrak{a}\leqq L$ $\left(x_1 \leqq N \leqq N\mathfrak{a}, \frac{x}{N\mathfrak{a}}\right)$

$$\begin{cases} \left|R_K\left(\frac{x}{N\mathfrak{a}}\right)\right| \Big/ \frac{x}{N\mathfrak{a}} \leqq \beta(N), \\ |R_K(N\mathfrak{a})|\,/N\mathfrak{a} \leqq \beta(N). \quad \beta(N) \leqq \alpha(x_1). \end{cases} \tag{3.5}$$

Let for all x

$$\delta = 1/\sqrt{\log N}, \tag{3.6}$$

$$\eta_m = N(1+\delta)^m, \ \eta_m\eta_{-m} = x, \quad (0 \leqq m \leqq r), \tag{3.7}$$

where

$$r = [\log(L/N)\,/\log(1+\delta)]. \tag{3.8}$$

Next, we define for all m. $(0\leqq m\leqq r)$

$$\eta_m^{-1}\{R_K(\eta_{m+1}) - R_K(\eta_m)\} = \sigma_K(m), \tag{3.9}$$

$$\eta_m^{-1} R_K(\eta_{-m}) = \rho_K(m). \tag{3.10}$$

Then, from (3.3) we have

$$|R_K(\eta_m)|\,/\eta_m, \ |R_K(\eta_{-m})\,/\eta_{-m}| \leqq \beta(N). \tag{3.11}$$

Taking Δ : $0\leqq\Delta\leqq\delta$, we obtain from (1.9) and (2.11)

$$0 \leqq \psi_K(\eta+\Delta\eta) - \psi_K(\eta) \leqq 2\Delta\eta + O(\eta\log^{-1}\eta) \tag{3.12}$$

This formula may be written ln the form

$$\frac{1}{\eta}\,|R_K(\eta+\Delta\eta) - R_K(\eta)| \leqq \Delta + c/\log\eta \tag{3.13}$$

for sufficiently large constant $c=c(K)>0$.

Lemma 3.3. *For* m : $0\leqq m\leqq r$ *we have*

$$(3.14) \qquad |\rho_K(m)| \leqq \beta(N),$$

$$(3.15) \qquad |\rho_K(m+1) - \rho_K(m)| \leqq \tau = \delta(1+\beta(N)) + c\delta^2 \quad (= O(\delta)),$$

$$(3.16) \qquad |\sigma_K(m)| \leqq \varepsilon = \delta + c'\delta^2,$$

$$(3.17) \qquad \Big|\sum_{v \leqq m \leqq v+u} \sigma_K(m)\Big| \leqq 2\beta(N) + (u+1)\beta(N)\delta.$$

Proof. From (3.13) with $\eta = \eta_{-(m+1)}$, $\Delta = \delta$ and (3.14) we have (3.15). From (3.13) with $\eta = \eta_m$, $\Delta = \delta$ we have (3.16). Finally, using (3.11) we obtain

$$\Big|\sum_{v \leqq m \leqq v+u} \sigma_k(m)\Big| \leqq |\eta_v^{-1} R_K(\eta_v)| + \delta \Big|\sum_{v+1 \leqq m \leqq v+u} \eta_m^{-1} R_K(\eta - m)\Big|$$

$$+ |\eta_{v+u}^{-1} R_K(\eta_{v+u+1})| \leqq 2\beta(N) + (u+1)\beta(N)\delta.$$

Lemma 3.4.

$$(3.18) \qquad \frac{1}{x} \sum_{N \leqq N\mathfrak{a} \leqq L} (\Lambda_K(\mathfrak{a}) - 1) R_K\Big(\frac{x}{N\mathfrak{a}}\Big) = \sum_{o \leqq m \leqq r-1} \rho_K(m)\sigma_K(m) + O(\delta \log L).$$

Proof. From (3.13) with $\eta = \frac{x}{N\mathfrak{a}}$ and $o \leqq \Delta \leqq \delta$, we obtain

$$(3.19) \qquad R_K\Big(\frac{x}{N\mathfrak{a}}\Big) = R_K(\eta_{-m}) + O(\delta\eta_{-m}) = \eta_{-m}(\rho_K(m) + O(\delta))$$

for $\eta_{-(m+1)} \leqq x/N\mathfrak{a} < \eta_{-m}$.

From (3.9) and (3.13) with $\eta = \eta_m$, $\Delta = \delta$

$$\sum_{\eta_m < N\mathfrak{a} \leqq \eta_{m+1}} (\Lambda_K(\mathfrak{a}) - 1) = \eta_m \sigma_K(m)$$

and

$$\sum_{\eta_m < N\mathfrak{a} \leqq \eta_{m+1}} \Lambda_K(\mathfrak{a}) \leqq 2\delta\eta_m + O(\eta_m\delta^2).$$

Then we have

$$(3.20) \qquad \sum_{\eta_m < N\mathfrak{a} \leqq \eta_{m+1}} |\Lambda_K(\mathfrak{a}) - 1| = O(\delta\eta_m).$$

From (3.7), (3.19) and (3.20) we have

$$\sum_{\eta_m < N\mathfrak{a} \leqq \eta_{m+1}} (\Lambda_K(\mathfrak{a}) - 1) R_K\Big(\frac{x}{N\mathfrak{a}}\Big)$$

$$= \sum_{\eta_m < N\mathfrak{a} \leqq \eta_{m+1}} (\Lambda_K(\mathfrak{a}) - 1)(R_K(\eta_m) + O(\delta\eta_m))$$

From (2.8) we have directly

$$\sum_{\eta_r < N\mathfrak{a} \leqq L} \frac{1}{N\mathfrak{a}} = O(\log(1+\delta))$$

and

$$(3.21) \qquad \sum_{\eta_r < N\mathfrak{a} \leqq L} \frac{N\mathfrak{a}}{x} (\Lambda_K(\mathfrak{a}) - 1) \frac{1}{N\mathfrak{a}}$$

$$\ll \frac{1}{x} \log L \sum_{\eta_r < N\mathfrak{a} \leqq L} \frac{x}{N\mathfrak{a}} \ll \log L.$$

From (3.8), (3.19), (3.20) and (3.21) finally we obtain

$$\frac{1}{x}\Big|\sum_{N<N\mathfrak{a}\leqq L}(A_K(\mathfrak{a})-1)\,R_K\Big(\frac{x}{N\mathfrak{a}}\Big)\Big|\leqq\frac{1}{x}\Big|\sum_{\eta_0<N\mathfrak{a}\leqq\eta_r}\Big|+\frac{1}{x}\Big|\sum_{\eta_r<N\mathfrak{a}\leqq L}\Big|$$
$$\ll\frac{1}{x}\sum_{m=0}^{r-1}\Big\{x\,(\rho_K(m)\,\sigma_K(m)+O\,(\delta^2))\Big\}+O\,(\delta\,\log\,L)$$
$$\ll\sum_{m=0}^{r-1}\rho_k(m)\,\sigma_k(m)+O\,(\delta\,\log L).$$

For all m: $0\leqq m\leqq r$, we define

$$\chi_i=\sum_{\rho>0}\rho_K(m)\,\sigma_K(m)\quad\text{for } a_i\leqq m<a_i+k_i\quad(1\leqq i\leqq G)\tag{3.22}$$

$$\nu_j=\sum_{\rho>0}\rho_K(m)\,\sigma_K(m)\quad\text{for } b_j\leqq m<b_j+l_j\quad(1\leqq j\leqq H).\tag{3.23}$$

Obviously we have

$$\Big|\sum_{m=0}^{r-1}\rho_K(m)\,\sigma_K(m)\Big|\leqq\sum_{i\leqq G}|\chi_i|+\sum_{j\leqq H}|\nu_j|,\tag{3.24}$$

$$\sum_{1\leqq i\leqq G}k_i+\sum_{1\leqq g\leqq H}l_j\leqq r.\tag{3.25}$$

Lemma 3.5. *For* $k_i\tau\leqq 4\beta$ $(2\leqq i\leqq G-1)$ *or* $k_i\tau>4\beta$

$$\frac{1}{k_i\delta}|\chi_i|\leqq\frac{3}{4}\,\beta(N)\,(1+\beta(N))+O\,(\delta).\tag{3.26}$$

For $l_j\tau\leqq 4\beta$ $(2\leqq j\leqq H-1)$ *or* $l_j\tau>4\beta$

$$\frac{1}{l_j\delta}|\nu_j|\leqq\frac{3}{4}\,\beta(N)\,(1+\beta\,(N))+O\,(\delta),\tag{3.27}$$

where $\tau=\delta\,(1+\beta(N))+c\,\delta^2$ *and* $\beta=\beta(N)$.

Proof. From (3.15)

$$-\tau+\rho_K(m-1)\leqq\rho_K\,(m)\leqq\tau+\rho_K(m-1),$$

hence from the obvious inequalities

$$\rho_K(a_i-1)\leqq 0,\quad \rho_K(a_i+k_i)\leqq 0,$$

we have

$$\rho_K(m)\leqq(m-a_i+1)\,\tau,\tag{3.28}$$

$$\rho_K(m)\leqq(a_i+k_i-m)\,\tau\qquad\text{for } a_i\leqq m\leqq a_i+k_i-1.\tag{3.29}$$

Case (a): $k_i\tau\leqq 2\beta(N)$ $(i\neq 1,\ G)$: From (3.16), (3.28), (3.29) and $\varepsilon/\delta=1+O(\delta)$, $\tau=O(\delta)$,

$$\omega_i=\chi_i/k_i\delta=\frac{1}{k_i\delta}\sum_{\rho>0}\rho_K\,(m)\,\sigma_K\,(m)\qquad(a_i\leqq m<a_i+k_i)$$

$$=\frac{1}{k_i\delta}\left(\sum_{\substack{\rho>0,\ (3.28)\\ a_i\leqq m_i\leqq a+[k_i/2]-1}}\rho_K\,(m)\,\sigma_K\,(m)+\sum_{\substack{\rho>0,\ (3.29)\\ a_i+[k_i/2]\leqq m\|\wedge a_i+k_i-1}}\rho_K(m)\,\sigma_K(m)\right)$$

$$=\frac{1}{k_i\delta}\varepsilon\Big(\sum_{a_i\leqq m\leqq a_i+[k_i/2]-1}(m-a_i+1)\tau+\sum_{a_i+[k_i/2]\leqq m\leqq a_i+k_i-1}(a_i+k_i-m)\tau\Big)$$

$$\leqq\frac{1}{2}\beta(N)+O(\delta) \tag{3.30}$$

Case (b): $2\beta(N)<k_i\tau\leqq 4\beta(N)(i\neq 1,G)$: From (3.14), (3.16), (3.28) and (3.29)

$$\omega_i=\frac{1}{k_i\delta}\Big(\sum_{\substack{\rho>0,\ (3.28)\\ a_i\leqq m\leqq a_i+[\beta/\tau]-1}}\rho_K(m)\sigma_K(m)+\sum_{\substack{\rho>0,\ (3.14)\\ a_i+[\beta/\tau]\leqq m\leqq a_i+k_i-[\beta/\tau]-1}}\rho_K(m)\sigma_K(m)+\sum_{\substack{\rho>0,\ (3.29)\\ a_i+k_i-[\beta/\tau]\leqq m\leqq a_i+k_i-1}}\rho_K(u)\sigma_K(m)\Big)$$

$$\leqq\frac{\varepsilon}{k_i\delta}\Big(\tau\sum_{a_i\leqq m\leqq_i\alpha+[\beta/\tau]-1}(m-a_i+1)+\beta\sum_{a_i+[\beta/\tau]\leqq m\leqq a_i+k_i-[\beta/\tau]-1}1+\tau\sum_{a_i+k_i-[\beta/\tau]\leqq m\leqq a_i+k_i-1}(a_i+k_i-m)\Big)$$

$$\leqq 0.75\beta(N)+O(\delta) \tag{3.31}$$

Case(c): $k_i\tau>4\beta$: From (3.14), (3.16), and (3.17)

$$\omega_i=\frac{1}{k_i\delta}\Big(\sum_{\substack{\rho>0,\ (3.14)\\ \sigma(m_i')>0\\ 1\leqq i\leqq k_i'}}\rho_K(m_i')\sigma_K(m_i')+\sum_{\substack{\rho>0\\ \sigma(m_i'')<0\\ 1\leqq i\leqq k_i''}}\rho_K(m_i'')\sigma_K(m_i'')+\sum_{\substack{\rho>0\\ \sigma(m_i''')=0\\ 1\leqq i\leqq k_i'''}}\rho_K(m_i''')\sigma_K(m_i''')\Big)$$

$$=\frac{1}{k_i\delta}\Big(\beta(N)\sum_{\substack{\sigma(m_i')>0\\ 1\leqq i\leqq k_i}}\sigma_K(m'^i)\Big)\leqq\frac{1}{k_i\delta}\beta(N)\Big(\beta(N)+\frac{1}{2}\beta(N)k_i\delta+\frac{1}{2}k_i\varepsilon\Big)$$

$$\leqq 0.75\beta(N)(1+\beta(N))+O(\delta), \tag{3.32}$$

where $k_i'+k_i''+k_i'''=k_i$.

From (3.30), (3.31) and (3.32) we have (3.26). By the same method we can prove (3.27) for ν_j.

Kuhn's fundamental theorem. *For $x_1\leqq N$ and $L\leqq\sqrt{x}$, we have*

$$\frac{1}{x}\sum_{x_1\leqq N<N\mathfrak{a}\leqq L\leqq\sqrt{x}}\Big|(\Lambda_K(\mathfrak{a})-1)R_K\Big(\frac{x}{N\mathfrak{a}}\Big)\Big|$$
$$\leqq 0.75\beta(N)(1+\beta(N)\log(L/N)+c_1\log L/\sqrt{\log N}.$$

where c_1 is a positive constant depending only on K.

Proof. From Lemma 3.5 and (3.8): $r=[\log(L/N)/\log(1+\delta)]$ we have

$$\frac{1}{x}\sum_{N<N\mathfrak{a}\leqq L}\Big|(\Lambda_K(\mathfrak{a})-1)R_K\Big(\frac{x}{N\mathfrak{a}}\Big)\Big|$$
$$\leqq\{\sum_{i\leqq G}k_i\delta+\sum_{j\leqq H}l_j\delta\}\{0.75\beta(N)(1+\beta(N))+O(\delta)\}+(\delta\log L)$$
$$\leqq 0.75\beta(N)(1+\beta(N))\log(L/N)+O(\log L/\sqrt{\log N})$$

which was to be proved.

§ 4. **Proof of the Prime Ideal Theorem.** The details of this section are similar as in[4] but we shall include them for completeness. From (1.7) we obtain for any $x \geqq x_1$

$$(4.1) \qquad \left| \sum_{N\mathfrak{a} \leqq x} \frac{\Lambda_K(\mathfrak{a})}{N\mathfrak{a}} - \log x \right| \leqq c_2$$

where c_2 is a positive constant depending only on K. Now we define

$$(4.2) \qquad \Sigma(N,L) = \frac{1}{x} \sum_{\substack{N < N\mathfrak{a} \leqq L \\ (N^2 \leqq L \leqq \sqrt{x})}} (\Lambda_K(\mathfrak{a}) - 1) R_K\left(\frac{x}{N\mathfrak{a}}\right)$$

Then we obtain the following

Lemma 4.1. *For* $x_1 \leqq N$ *and* $L \leqq \sqrt{x}$ *we have*

$$(4.3) \qquad |\Sigma(N,L)| \leqq \frac{3}{4} \beta(N)(1+\beta(N)) \log(L/N) + c_3\sqrt{\log L},$$

where c_3 *is a positive constant depending only on* K.

Proof. It t is an integer with $N^{2^t} \leqq L < N^{2^{t+1}}$, we have

$$|\Sigma(N,L)| \leqq |\Sigma(N^{2^{t-j-1}}, N^{2^{t-j}})| + |\Sigma(N^{2^{t-1}}, L)|$$

Using kuhn's fundamental theorem we obtain

$$|\Sigma(N,L)| \leqq \frac{3}{4}(N)(1+\beta(N))\log(L/N)$$
$$+ c_1\left(\frac{\log L}{\sqrt{\log N^{2^{t-1}}}} + (\sqrt{2}+1+\frac{1}{\sqrt{2}}+\cdots\cdots)\sqrt{\log N^{2^{t-1}}}\right)$$
$$\leqq \frac{3}{4}\beta(N)(1+\beta(N))\log(L/N) + c_3\sqrt{\log L}.$$

Lemma 4.2. *If* $\sqrt{x} \geqq \zeta \geqq x_2$, $\varepsilon_1 = 10^{-3}$ *for sufficiently large* x_2, *then we have*

$$(4.4) \qquad \alpha_2 = \frac{3}{4}\alpha_1(x_1)(1-\alpha_1(x_1))(1+\varepsilon_1) < 0.092,$$

$$(4.5) \qquad \varepsilon_1\alpha_2 \log\zeta > c_3\sqrt{\log\zeta},$$

$$(4.6) \qquad |\Sigma(0,\zeta)| < \alpha_2 \log\zeta - \frac{1}{2}c$$

$$(4.7) \qquad \zeta^{-2}|R_K(\zeta^2)| < \alpha_2$$

Proof. (4.4) is easy. Taking $x_1 \leqq \sqrt{x}$ and $\alpha_1(x_1) = 0.11$ with (3.2), we get from (2.8) and (4.1)

$$(4.8) \qquad |\Sigma(0,x_1)| \leqq \alpha_1(x_1)\left(\sum_{N\mathfrak{a} \leqq x_1} \frac{\Lambda_K(x)}{N\mathfrak{a}} + \sum_{N\mathfrak{a} \leqq x_1} \frac{1}{N\mathfrak{a}}\right)$$
$$\leqq \alpha_1(x_1)((1+\alpha_1(x))\log_1 + c_4)$$

where c_4 is a positive constant depending only on K. Then for all $x \geqq \zeta^2$ and all $\zeta \geqq x_1{}^2$ we obtain from (4.3) and (4.8)

$$| \Sigma (0,\zeta) | \leqq | \Sigma (0,x_1) | + | \Sigma (x_1,\zeta) |$$

$$\leqq \alpha_1(x_1) ((1 + \alpha_1(x_1) \log x + c_4)$$

$$+ \frac{3}{4} \alpha_1(x_1) (1 + \alpha_1(x_1)) \log \frac{\zeta}{x_1} + c_3 \sqrt{\log \zeta}$$

$$\leqq \frac{3}{4} \alpha_1(x_1) (1 + \alpha_1(x_1)) \log \zeta + c_3 \sqrt{\log \zeta} + c_5 (x_1) \tag{4.9}$$

where

$$c_5(x_1) = \alpha_1(x_1) (c- \frac{3}{4} \alpha_1(x_1) (1 + \alpha_1(x_1))) \log x_1 + \alpha_1(x_1) c_4. \tag{4.10}$$

Applying Lemma 3.1 we have

$$\frac{1}{x} | R_K(x) | \log x \leqq 2 | \Sigma (0, \sqrt{x}) | + c_6. \tag{4.11}$$

we have for all $\zeta \geqq x_2$ the following inequality

$$\frac{3}{4} \varepsilon_1 \alpha_1(x_1) (1 + \alpha_1(x_1)) \log \zeta \geqq c_3 \sqrt{\log \zeta} + c_5(x_1) + \frac{1}{2} c_6, \tag{4.12}$$

where x_2 is sufficiently large. Then we obtain (4.5). From (4.9), (4.10) and (4.12) we obtain (4.6). Finally from (4.11) we have (4.7).

Lemma 4.3. For sufficiently large x; $\zeta \geqq x_2$ we have Lemma 4.2. Furthermore, when we put for all $k \geqq 2$ and $\kappa \geqq 2$

$$x_{k+1} \geqq x_k{}^{2^\kappa} > x_{k+1} - 1 \tag{4.13}$$

$$\alpha_{k+1} = (1 + \varepsilon_1) \alpha_k \varphi(\kappa, k), \ \varphi(\kappa, k) = \kappa^{-1} (1 + \frac{3}{4} (\kappa - 1) (1 + \alpha_k)) \tag{4.14}$$

then for all ζ: $\sqrt{x} \geqq \zeta \geqq x_{k+1}$ *we have the following inequalities*

$$| \Sigma (0,\zeta) | < \alpha_{k+1} \log \zeta - \frac{1}{2} c_6 \tag{4.15}$$

$$\frac{1}{\zeta^2} | R_k(\zeta^2) | < \alpha_{k+1}. \tag{4.16}$$

Proof. Obviously we have

$$0.75 < \varphi(\kappa,k) < 0.91, \tag{4.17}$$

$$\varphi(\kappa, k) = 0.75 (1 + \alpha_k) + \frac{1}{\kappa} (1 - 0.75 (1 + \alpha_k)). \tag{4.18}$$

From (4.17), (4.18) and (4.5) we have

$$\varepsilon_1 \alpha_k \varphi (\kappa, k) \log \zeta < c_1 \sqrt{\log \zeta}. \tag{4.19}$$

From (4.19) and Lemma 4.1 we have for ζ : $\zeta \geqq x_{k+1} \geqq x_k{}^{2\kappa}$

$$| \Sigma (0,\zeta) | \leqq | \Sigma (0,x_k{}^2) | + | \Sigma (x_k{}^2, \zeta) |$$

$$\leqq \alpha_k \log x_k{}^2 - \frac{1}{2} c_4 + | \Sigma (x_k{}^2, \zeta) |$$

$$\leqq \alpha_k \varphi (\kappa, k) \log \zeta + c_1 \sqrt{\log \zeta} - \frac{1}{2} c_4$$

$$< \alpha_{k+1} \log \zeta - \frac{1}{2} c_4.$$

Next, applying (4.11) we have (4.16).

From (4.4) and (4.14) we have

$$\alpha_\kappa < 0.1\,(0.92)^{k-1} \qquad (k \geqq 2). \tag{4.20}$$

Let I be an integer taken so large that $\alpha_I < 10^{-3}$ and let

$$Q(\kappa) = 1.001 \frac{1}{\kappa}(1 + 0.75(\kappa - 1)\,1.001),$$

then $Q(\kappa) > (1+\varepsilon_1)\varphi(\kappa.\ I-m)$, where $0.75 < \varphi < 0.91$. From (4.14) we obtain

$$\alpha_{I+m} < 10^{-3} Q^m(\kappa). \tag{4.21}$$

Now, we introduce for all non-negative integers m such that

$$y_{I+m} = \left[(x_2 + 1)^{(2\kappa)^{m+I-2}} \right] \tag{4.22}$$

Then, from (4.13) and (4.22), for x: $x_{I+m} < y_{I+m} \leqq \sqrt{x}$ we have

$$\frac{1}{x} \mid R_K(x) \mid < 10^{-3} Q^m(\kappa). \tag{4.23}$$

Thus, for all sufficiently large x: $y^2_{I+m} \leqq x < y^2_{I+m+1}$ we obtain fro (4.22)

$$(2\kappa)^m = O(\log x). \tag{4.24}$$

From (4.23) and (4.24) we have

$$\frac{1}{x} \mid R_K(x) \mid = O(Q^m(\kappa)) = O((2\kappa)^{-mc}) = O(\log^{-c} x),$$

where $c = c(\kappa) = -\log Q(\kappa) \log^{-1}(2\kappa)$ $(\kappa \geqq 2)$. Furthermore we can take c here not depending on κ as in[4] and the proof if the prime ideal theorem in the form of (1.6) is completed.

References

[1] Ayoub, R.G.: On Selberg's lemma for algebraic fields, Canadian J. Math., 7(1955), 138–143.

[2] Forman, W. and Shapiro H. N.: Abstract Prime Number Theorem. Comm. Pure Appl. Math., 2 (1949), 309–323.

[3] Iseki, K. and Tatuzawa T.: On Selberg's elementary proof of the prime number theorem. Proc. Jap. Acad., 27(1951), 340–342.

[4] Kuhn, P.: Eine Verbesserung des Restgliedes beim elementaren Beweis des Primzahlsatzes. Math. Scand., 3(1955), 75–89.

[5] Montaner, P.L. and Shapiro H.N.: On equivalent formulations of certain abstract prime number theorems. Dissertation, New York Univ. 1951.

[6] Landau, E.: Über die zu einem algebraischen Zahlkörper gehörige Zetafunktion und die Ausdehnung der Tschebyschefschen Primzahltheorie auf das Problem der Vertheilung der Primideale. J. reine angew. Math., 125(1903), 64–188.

[7] Rieger, C.: Über die Anzahl der Ideale in einer Idealklasse mod $\mathfrak{f}$ eines algebraischen Zahlkörpers. Math. Ann., 135(1958), 444–466.

[8] Selberg, A.: An elementary proof of the prime number theorem. Ann. of Math., 50 (1949), 305–313.

[9] Shapiro, H.N.: An elementary proof of the prime ideal theorem. Comm. Pure Appl. Math., 2(1949), 309–323.

[10] Weber, H.: Lehrbuch der Algebra, Bd. 2, Braunschweig, 1899.