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On the Remark of Laasonen's Theorem

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The purpose of this paper is to investigate the absolute convergence of the following expansion by series of the *fuchsoid* function given by *Laasonen*.[1)]

§ 1. Let G_n be a *Fuchsian* group of genus zero formed by n parabolic, linear generating transformations;

$$(1) \qquad T_\nu : \frac{1}{T_\nu(z)-\zeta_\nu} = \frac{1}{z-\zeta_\nu} - \frac{1}{\eta_\nu} \qquad \left\{ \begin{matrix} |\eta_\nu| = 1, \ \eta_\nu \text{ real} \\ \nu = 1, 2, \cdots, n \end{matrix} \right\}$$

and yet be properly discontinuous on the principal circle H ($|z|=1$) except the singular points of G_n. If we consider the *Poincaré*'s theta series of -2 dimension with respect to G_n

$$(2) \qquad \Theta_n(z) = \sum_{G_n} \frac{1}{(s_\nu(z))^2} \cdot \frac{ds_\nu(z)}{dz},$$

it is absolutely convergent by the well-known *Burnside-Ritter*'s theorem.

Next let us consider the function

$$(3) \qquad f_n(z) = -\int_{a_n}^{z} \Theta_n(z)\,dz + C_n = -\sum_{G_n}\int_{a_n}^{z} \frac{1}{(s_\nu(z)^2)} d\,s_\nu(z) + C_n$$

$$= \sum_{G_n} \left[\frac{1}{s_\nu(z)} - \frac{1}{s(a_n)} \right] + C_n,$$

where a_n is any point in the fundamental region B_n with respect to G_n. B_n is sur-rounded by the boundaries which consist of the arcs of H and n pairs of circular arcs which are orthogonal to H and tangent at the parabolic vertices. (Fig 1) If we choose C_n so that the *Laurant* series of $f_n(z)$ may have not constant term at the origin, where $f_n(z)$ has a simple pole, (3) becomes

$$(4) \qquad f_n(z) = \frac{1}{z} + \sum_{G_n}{}' \left[\frac{1}{s_\nu(z)} - \frac{1}{s_\nu(O)} \right],$$

where $\sum_{G_n}'$ denotes the summation extended over the all transformations except S_o. (4) defines a meromorphic function which has poles at the equivalent points to the origin, in the outside of singular points of G_n.

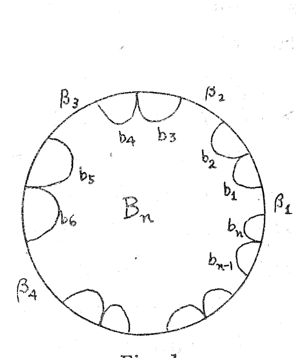

Fig. 1

Then $f_n(z)$ has the following properties;

(i) *$f_n(z)$ is an automorphic function with respect to G_n, that is,*

$$f_n(T_k(z)) \equiv f_n(z) \qquad (\forall T_k \varepsilon G_n)$$

(ii) *Since $f_n(z)$ is regular as the function of local parameter t at the parabolic vertices, it is a simple automorphic function.*

(iii) *Since $f_n(z)$ has only one pole at the origin in B_n, it takes any value only once in B_n and therefore it is the principal function of G_n.*

(iv) *The principal domain of $f_n(z)$, that is the image of B_n, coincides the outside of a circle.*

(v) *The principal function $f_n(z)$ of G_n converges to the principal function $f(z)$ of the fuchsoid group G which is approximated by the above Fuchsian subgroups $\{G_n\}$. The principal domain of $f(z)$ is the outside of the circle with radius ≥ 0.*

§ 2. Under what condition is the series (4) of $f_n(z)$ absolutely convergent for $n \to \infty$? To resolve this problem we need the following lemma and *Myrberg*'s consequences.[2)]

Lemma. Between $f_n(z)$ and the Blaschke-Product

$$(5) \qquad \Pi_n(z) = \Pi_{G^n} \frac{|a_\nu| - z e^{-i\alpha_\nu}}{1 - \bar{a}_\nu z} \qquad \left\{ \begin{array}{l} a_\nu = |a_\nu| e^{i\alpha_\nu} = s_\nu(0) \\ \bar{a}_\nu = |a_\nu| e^{-i\alpha_\nu} \end{array} \right\},$$

there exists the relation

$$(6) \qquad f_n(z) = b_n + \frac{a_n}{\Pi_n(z)} .$$

Proof. $\Pi_n(z)$ is the simple automorphic function which maps B_n onto the unit circle. Since $\Pi_n(z)$ and $f_n(z)$ have the same B_n and belong to the same G_n, there exists a rational relation between $\Pi_n(z)$ and $f_n(z)$. And since the boundaries of this principal domains are circles, the relation must be linear, hence we obtain (6). q. E. D.

Now the principal domain T_n onto which $\Pi_n(z)$ maps B_n is obtained by removing n points from $|\Pi_n| < 1$. These points in T_n cluster to the singular endpoint set[3)] on $|\Pi_n| = 1$ for $n \to \infty$, which corresponds to that on $|z| = 1$, that is, the cluster set of parabolic vertices.

Myrberg[4)] proved that the linear measure of this singular endpoint set depends upon the cuts which make T_n simply-connected and the distribution of these inner points.

If each of these n points in T_n is connected with the peripherie by a straight line along radius so that any two of them are not on the same radius, then we obtain a figure called *Radial Star* S_n. (Fig 2)

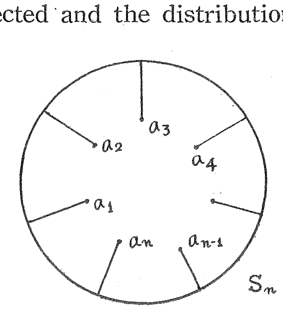

Fig. 2

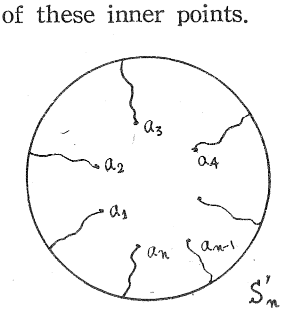

Fig. 3

If the images of n cuts are deformed into circular arcs orthogonal to H without moving the endpoints on H, the image of B_n is called *Normal Radial Star* S'_n. (Fig 3)

In Fig 1 denote by (b_n) the set of sides and (β_n) the set of circular arcs of H complementary to (b_n) in the boundary of B_n. Now we consider the harmonic measure $\omega_n(z) = \omega(z, (\beta_n), B_n)$, which is harmonic in B_n, 1 on (β_n) and 0 on (b_n).[5)] Of course $0 \leq \omega_n(z) \leq 1$, and the sequence $\{\omega_n(z)\}$ is monotone decreasing and converges uniformly to a harmonic function $\omega(z, (\beta), B)$ in a wide sense by *Harnack*'s theorem, where B is the fundamental region of the *fuchsoid* group $G = \lim\limits_{n\to\infty} G_n$. The limit function $\omega(z, (\beta), B)$ is called the harmonic measure of the fundamental region B, and $\omega(\Pi) = \omega(\Pi^{-1}(z))$ the harmonic measure of the *Stars* $S = \lim\limits_{n\to\infty} S_n$ and $S' = \lim\limits_{n\to\infty} S'_n$.

By using this harmonic measure, *Myrberg* got the following consequences.

Suppose that S is the *Radial Star* of the inner boundary points (a) whose coordinates are $r_\nu e^{i\theta_\nu}$ $(\nu = 1, 2, \cdots\cdots)$.

(i)° *If the series*

$$\sum_{\nu=1}^{\infty} (1 - r_\nu) \tag{7}$$

is finite, the harmonic measure of S *is positive.*

(ii)° *Let q_n be the monotone increasing function of n, that is, $q_n \to \infty$ for $n \to \infty$ and (a) is distributed so regular as*

$$a_n^{(k)} = \left(1 - \frac{1}{q_n}\right) e^{\frac{2\pi i k}{p_n}} \qquad (0 \leq k < p_n),$$

where P_n *is the distributing function of* (a).

If

$$\lim_{n\to\infty} \frac{p_n}{q_n} = \infty, \tag{8}$$

the harmonic measure of any Radial Star is zero.

(iii)° *If*

$$\sum_{n=1}^{\infty} \frac{p_n}{q_n} < \infty, \tag{9}$$

the Radial Star S has positive harmonic measure. (9) *follows from* (i)°, *if we put* $r_n = (1 - \frac{1}{q_n})$.

(iv)° *The linear measure of* (β) *in the radial fundamental region is zero under the condition* (8) *and positive under the condition* (9).

§ 3. Now let us prove the absolute convergence of the series (4) for $n \to \infty$, which is the expansion of $f_n(z)$, by the lemma and *Myrberg*'s consequences (i)°—(iv°).

From the property of the group G, two cases occur with respect to the behavior of (4) for $n \to \infty$.

(A) The case where G has no limit circle, that is, G is properly discontinuous on H. In this case, since the linear measure of (β) is obviously positive, we obtain

$$\sum_{\nu=1}^{\infty} (1 - |z_\nu|) < \infty \tag{10}$$

by the well-known *Burnside*'s theorem,[6] where $z_\nu = S_\nu(0)$ $(S_\nu \in G)$. It leads to the convergence of the *Blaschke-Product* $\Pi_G |S_\nu(z)|$ at once. From the condition (10) and the form

$$(11)\qquad \left|\frac{ds_\nu(z)}{dz}\right| = \frac{1}{|\gamma_\nu z+\delta_\nu|^2} = \frac{1-|s_\nu(z)|^2}{1-|z|^2},$$

where

$$S_\nu(z) = \frac{\alpha_\nu z+\beta_\nu}{\gamma_\nu z+\delta_\nu} \qquad (\alpha_\nu\delta_\nu-\beta_\nu\gamma_\nu=1)$$

is any transformation of G, we obtain the absolute convergence of the *Poincaré*'s theta saries of -2 dimension with respect to G.

Therefore

$$(12)\qquad f(z)= \frac{1}{z} + \sum_G{}' \left[\frac{1}{s_\nu(z)} - \frac{1}{s_\nu(0)}\right],$$

which can be obtained from $f_n(z)$ by $n\to\infty$, is also absolutely convergent. Since the center $b_n=f_n(\infty)$ of the boundary circle of D_n converges to b and the linear measure of the image of (β) is also positive, D_n converges to the principal domain D of $f(z)$, that is, the outside of the limiting boundary circle with radius $|a|>0$ and center b. Then by the lemma the relation

$$(13)\qquad f(z)=b+\frac{a}{\Pi(z)}$$

exists between $f(z)$ and $\Pi(z)$ for $n\to\infty$.

(B) The case where G has limit circle.

By *Myrberg*'s consequences the linear measure of the singular endpoint set of the *Star* of $\Pi(z)$ depends upon the distribution of the infinte inner isolated boundary points (a) which are images of the parabolic vertices.

(1_B) If the distribution of (a) satisfies the condition (9), or, when the distribution is so irregular that the set (a) may not satisfy the condition (9), if it satisfies the condition (7), we obtain (10) just the same as (A). (10) is the necessary and sufficient condition for the existence of the *Green* function on D and also for the absolute convergence of the theta series of -2 dimension of G[7]. In this case the *Green* function with a logarithmic pole at the origin is given by $-\log \Pi_G |S_\nu(z)|$[8]

Therefore the series (4) is absolutely convergent for $n\to\infty$, and the relation (13) exists between $f(z)$ and $\Pi(z)$, and the principal domain of $f(z)$ consists of the outside of the circle with radius $|a|>0$ and the center b.

The case (1_B) contains (A) as the special case. Because (A) may be considered as the case where in (1_B) the order of p_n (*regular*) or the grade of distribution of (a) (*irregular*) is so small in comparison with that of q_n or that of approach of (a) to the peripherie that the series in (i)° is convegant at once.

(2_B) When the condition (8) or $\sum_{\nu=1}^{\infty}(1-r_\nu)=\infty$ for irregular distribution is satisfied with respect to the set (a), we need the following consideration.

Though the linear measure of (β) is zero under (8) by (ii)°, the condition (10) may occur for some group. But the series $\sum_{\nu=1}^{\infty}(1-|z_\nu|)$ is divergent in this case.

Because the boundary of D is either the limiting circle with radius >0 of the boundary circles of $\{D_n\}$ or the limiting point of the infinite isolated points from the constitution of G_n, and if the boundary is former, the linear measure of (β) is positive and it is inconsistent with the fact that the measure of (β) is zero. Since the boundary of D must be the only one limit point and hence the *Green* function on D cann't exist, and the series $\sum_{\nu=1}^{\infty}(1-|z_\nu|)$ is divergent. Therefore the *Poincaré*'s theta series of -2 dimension and the series (4) cann't be absolutely convergent for $n\to\infty$, and the relation (13) does not exist between $f(z)$ and $\Pi(z)$.

From the above consequences (A) and (B), we obtain the following theorem.

Theorem. Let G be the fuchsoid group which is approximated by the Fuchsian subgroup and p_n, q_n be the function of the set $(a)=(r_\nu e^{i\theta_\nu})$ in the principal domain onto $\Pi(z)$ maps B.

Then if the condition

(9) $$\sum_{n=1}^{\infty}\frac{p_n}{q_n}<\infty \qquad \textit{(regular)},$$

or the condition

$$\sum_{\nu=1}^{\infty}(1-r_\nu)<\infty \qquad \textit{(irregular)}$$

is satisfied, the principal function $f(z)$ with respect to G has the expansion

(12) $$f(z)=\frac{1}{z}+\sum_G{}'\left[\frac{1}{s_\nu(z)}-\frac{1}{s_\nu(0)}\right],$$

which is absolutely convergent and between f(z) *and Blaschke-Product $\Pi(z)$ the relation*

(13) $$f(z)=b+\frac{a}{\Pi(z)}$$

exists.

If the condition

(8) $$\lim_{n\to\infty}\frac{p_n}{q_n}=\infty \qquad \textit{(regular)},$$

or the condition

$$\sum_{\nu=1}^{\infty}(1-r_\nu)=\infty \qquad \textit{(irregular)}$$

is satisfied, (12) *is not absolutely convergent, and the relation* (13) *does not exist.*

Remark. In the investigation of the absolute convergence of the series (12), to use the principal domain of $\Pi(z)$, which is the well-known, general, and standard automorphic function with respect to G, is more convenient than that of $f(z)$. For the radius and the center of the boundary circle of $f_n(z)$ vary and converge to those of the limiting boundary circle of $f(z)$.

In the *fuchsoid* group, the relation which exists in the *Fuchsian* subgroup, as the lemma in the section 2 shows, does not always lead to the same relation for $n\to\infty$.

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