Sci. Rep. Kanazawa Univ.
Vol. 68, pp. 1–9, 2025 **ARTICLES**

A generalized averaging of a function and characterization of Sobolev spaces on the sphere

Ikhsan MAULIDI* [a,b] and Hiroshi OHTSUKA [c]

a) Department of Mathematics, Universitas Syiah Kuala, Banda Aceh, Indonesia
b) Graduate School of Natural Science and Technology, Kanazawa University, Kanazawa, Japan
c) Faculty of Mathematics and Physics, Kanazawa University, Kanazawa, Japan

(Received November 7, 2024 and accepted in revised form February 5, 2025)

Abstract We present a new characterization of Sobolev spaces on the sphere. We follow the idea of Barceló et al. (2020) and develop the square function they used to characterize the Sobolev spaces. We discuss in detail the weight and the range of the averaging in the definition of the square function, and we find it possible to limit the domain of the averaging within a local coordinate of the sphere.

Keywords. Sobolev spaces, Spherical harmonics, Square function.

1 Introduction

Several characterizations of Sobolev spaces without using the notions of distributional derivatives have been introduced by many researchers. Among them, we are interested in the work by Alabern et al. [2] on $\mathbf{R}^d$ ($d \geq 1$) by using the following notion of the square function for $\alpha \in (0,2)$:

$$S_\alpha(f)^2(x) := \int_0^\infty \left| \frac{f_{B(x,t)} - f(x)}{t^\alpha} \right|^2 \frac{dt}{t}, \quad x \in \mathbf{R}^d.$$

Here f is a locally integrable function on $\mathbf{R}^d$ and $f_{B(x,t)}$ is the averaging of f on the open ball with center x and radius t, that is,

$$f_{B(x,t)} := \frac{1}{|B(x,t)|} \int_{B(x,t)} f(y)dy$$

and $|B(x,t)|$ is the Lebesgue measure of $B(x,t)$.

Theorem 1.1 ([2, Theorem 1 and 3])**.** *For $1 < p < \infty$ and $0 < \alpha < 2$, the following are equivalent:*

(1) $f \in W^{\alpha,p}\left(\mathbf{R}^d\right)$. (2) $f \in L^p\left(\mathbf{R}^d\right)$ and $S_\alpha(f) \in L^p\left(\mathbf{R}^d\right)$.

**Corresponding author* Email: ikhsanmaulidi@stu.kanazawa-u.ac.jp

Later Barceló et al. [3] got a similar result for the Sobolev spaces $H^{\alpha}\left(\mathbf{S}^{d-1}\right) := W^{\alpha,2}\left(\mathbf{S}^{d-1}\right)$ on the $d-1$ dimensional sphere $\mathbf{S}^{d-1}$ ($d \geq 2$), introducing the following square function:

$$S_{\alpha}(f)^2(\xi) := \int_0^{\pi} \left| \frac{f_{C(\xi,t)} - f(\xi)}{t^{\alpha}} \right|^2 \frac{dt}{t}, \quad \xi \in \mathbf{S}^{d-1}.$$

Here f is an integrable function on $\mathbf{S}^{d-1}$ and $f_{C(\xi,t)}$ is the averaging of f on the spherical cap $C(\xi,t)$ with center ξ and radius $t \in (0,\pi)$, that is,

$$f_{C(\xi,t)} := \frac{1}{|C(\xi,t)|} \int_{C(\xi,t)} f(\tau) d\sigma(\tau), \quad C(\xi,t) := \{\eta \in \mathbf{S}^{d-1} : \xi \cdot \eta \geq \cos t\}, \tag{1.1}$$

where $|C(\xi,t)|$ is measured by the uniform surface measure σ on $\mathbf{S}^{d-1}$ induced from the Lebesgue measure on $\mathbf{R}^d$.

Theorem 1.2 ([3, Theorem 1.1]). *For $0 < \alpha < 2$, the following are equivalent:*

(1) $f \in H^{\alpha}\left(\mathbf{S}^{d-1}\right)$. (2) $f \in L^2\left(\mathbf{S}^{d-1}\right)$ and $S_{\alpha}(f) \in L^2\left(\mathbf{S}^{d-1}\right)$.

The purpose of this paper is to extend Theorem 1.2 by generalizing the concept of averaging. For a fixed $\xi \in \mathbf{S}^{d-1}$, we introduce the Euclidean angle $\theta \in [0,\pi]$ between $\tau \in \mathbf{S}^{d-1}$ and ξ. Then the uniform surface-area measure σ on $\mathbf{S}^{d-1}$ is expressed as

$$d\sigma = d\sigma(\theta,\tau') = \sin^{d-2}\theta d\theta d\sigma',$$

where $\tau' \in \mathbf{S}^{d-2}$ and σ' is the uniform surface-area measure of $\mathbf{S}^{d-2}$, cf [1, (1.17)].

Definition 1.3 (The generalized averaging). *Let $T \in (0,\pi]$ and*

$$\rho \in L^{\infty}(0,T) \text{ satisfies } \rho \geq 0 \text{ a.e. and } \int_0^T \rho(\theta) d\theta > 0. \tag{1.2}$$

Then, for an integrable function f on $\mathbf{S}^{d-1}$ and $t \in (0,T]$, we define

$$A_t^{\rho} f(\xi) := z_t^{-1} \int_{C(\xi,t)} f(\theta,\tau') \rho\left(\frac{T}{t}\theta\right) d\sigma(\theta,\tau')$$

as the generalized averaging of f around ξ with radius t, where z_t is chosen in order to satisfy

$$z_t^{-1} \int_{C(\xi,t)} \rho\left(\frac{T}{t}\theta\right) d\sigma(\theta,\tau') = 1,$$

that is,

$$z_t = \left|\mathbf{S}^{d-2}\right| \frac{t}{T} \int_0^T \sin^{d-2}\left(\frac{t\theta}{T}\right) \rho(\theta) d\theta.$$

We note that, thanks to the assumptions (1.2) on ρ, the generalized averaging A_t^{ρ} is well-defined for $f \in L^2\left(\mathbf{S}^{d-1}\right)$. We also note that the averaging $f_{C(\xi,t)}$ defined in (1.1) used in Barceló et al.[3] is nothing but the case when $T = \pi$ and $\rho(\theta) \equiv 1$ for $0 \leq \theta \leq \pi$.

Definition 1.4 (The generalized square function). *Under Definition 1.3 and for $0 < \alpha < 2$, we call the following $S_{\alpha}^{\rho,T}(f)(\xi)$ the generalized square function for f:*

$$S_{\alpha}^{\rho,T}(f)^2(\xi) := \int_0^T \left| \frac{A_t^{\rho} f(\xi) - f(\xi)}{t^{\alpha}} \right|^2 \frac{dt}{t}.$$

Theorem 1.5 (Main Theorem)**.** *For fixed* $T \in (0,\pi]$*,* $\rho \in L^{\infty}(0,T)$ *satisfying* (1.2)*, and* $0 < \alpha < 2$*, the following are equivalent:*

(1) $f \in H^{\alpha}\left(\mathbf{S}^{d-1}\right)$. (2) $f \in L^2\left(\mathbf{S}^{d-1}\right)$ and $S_{\alpha}^{\rho,T}(f) \in L^2\left(\mathbf{S}^{d-1}\right)$.

The motivation why we introduce the above generalization is two folds. One is to make more general averaging possible in the characterization of Sobolev spaces by an elementary argument via introducing ρ. Indeed, similar weight functions are introduced in [4, Cor 5.2] for $\mathbf{R}^d$ cases, for example. Another motivation is toward a similar characterization of the Sobolev spaces on compact manifolds. By introducing $T < \pi$, we are able to discuss this problem in a *local* coordinate of $\mathbf{S}^{d-1}$. One of the motivation of Barceló et al. is to extend the results of Alabern et al. [2] to the case on manifolds and they consider their results as a first step toward the purpose [3, p.2]. We think that our result is the next step to it.

We end the introduction with a remark that Alabern et al. [2] and Barceló et al. [3, p.2] considered a more general setting that includes higher-order derivatives: $W^{\alpha,p}\left(\mathbf{R}^d\right)$ and $H^{\alpha}\left(\mathbf{S}^{d-1}\right)$ for $1 < p < \infty$ and $\alpha \geq 2$. In this paper, however, we concentrate on the case $0 < \alpha < 2$ in order to simplify the explanation of the main ideas of our generalization. We will present a full generalization elsewhere in the near future.

The contents of this paper is as follows: in Section 2 we will summarize the settings of our paper, e.g., several facts on the spherical harmonics and the definition of Sobolev spaces using spherical harmonics. In Section 3, we will prove the main theorem (Theorem 1.5).

2 Overview of notions and notation

In this paper, we will follow [3] for notation in general. All functions take values in the set of complex numbers $\mathbf{C}$. The symbol $A \sim B$ denotes $A \lesssim B$ and $A \gtrsim B$, where $A \lesssim B$ means that there exists a constant $C > 0$ independent of the parameters attached to the quantities A and B such that $A \leq CB$. If we want to clarify the dependence of the constant C on some parameters, we write $\lesssim_{d,\rho,T}$ and $\sim_{d,\rho,T}$, etc. For $l = 0,1,2,...,$

$$\mathbb{H}_l^d := \operatorname{span}\{Y_l^j : 1 \leq j \leq \nu(l)\}$$

is the space of all spherical harmonics of degree l on $\mathbf{S}^{d-1}$, where $\{Y_l^j\}$ is an orthonormal basis in $L^2(\mathbf{S}^{d-1})$ and $\nu(l) = \dim_{\mathbf{C}}\mathbb{H}_l^d$, which is known to satisfy $\nu(l) = O(l^{d-2})$ for $l \gg 1$ [1, p.16 and p.19]. Here we recall the following lemma:

Lemma 2.1 ([3, Lemma 3.1], see also [1, Theorem 2.8])**.** *Let* $\xi \in \mathbf{S}^{d-1}$ *and* $L \in \mathbb{H}_l^d$ *such that* $L(R\eta) = L(\eta)$ *for all rotations* R *in* $\mathbf{R}^d$ *such that* $R(\xi) = \xi$*. Then*

$$L(\eta) = L(\xi)P_{l,d}(\eta\cdot\xi),$$

where $P_{l,d}$ *is the Legendre polynomial of degree* l *in* d *dimensions.*

The explicit formula for $P_{l,d}$ is known [1, (2.19)]:

$$P_{l,d}(s) = l!\Gamma\left(\frac{d-1}{2}\right)\sum_{k=0}^{[l/2]}(-1)^k\frac{(1-s^2)^k s^{l-2k}}{4^k k!(l-2k)!\Gamma(k+\frac{d-1}{2})}. \tag{2.1}$$

It holds that

$$P_{l,d}(1) = 1 \quad \text{for } l = 0,1,\cdots.$$

For later use, we recall the following estimates: for every $t \in [-1,1]$, it holds that

$$|P_{l,d}^{(k)}(t)| \leq P_{l,d}^{(k)}(1) \sim l^k(l+d-2)^k \sim l^{2k} \tag{2.2}$$

for $k \in \mathbf{N} \cup \{0\}$, see [3, (18), (44)] and [1, pp.58-59]. We also recall the so-called addition theorem [1, Theorem 2.9]

$$\sum_{j=1}^{\nu(l)} Y_l^j(\eta)\overline{Y_l^j(\tau)} = \frac{\nu(l)}{|\mathbf{S}^{d-1}|} P_{l,d}(\eta\cdot\tau).$$

As a consequence of this, the following reproducing formula holds for $Y_l \in \mathbb{H}_l^d$ [1, p.23]:

$$Y_l(\xi) = \frac{\nu(l)}{|\mathbf{S}^{d-1}|}\int_{\mathbf{S}^{d-1}} P_{l,d}(\xi\cdot\eta)Y_l(\eta)d\sigma(\eta). \tag{2.3}$$

Since the spherical harmonic Y_l^j is an eigenfunction of the Laplace-Beltrami operator $-\Delta$ satisfying

$$-\Delta Y_l^j = l(l+d-2)Y_l^j, \tag{2.4}$$

it holds that

$$\|\nabla f\|^2_{L^2\left(\mathbf{S}^{d-1}\right)} = \sum_{l=0}^{\infty} l(l+d-2)(\sum_{j=1}^{\nu(l)} |\hat{f}_{lj}|^2)$$

for $f \in C^\infty(\mathbf{S}^{d-1})$ satisfying $f = \sum_{l=0}^{\infty}\sum_{j=1}^{\nu(l)} \hat{f}_{lj}Y_l^j$. Here $\hat{f}_{lj} := \int_{\mathbf{S}^{d-1}} f\overline{Y_j^l}d\sigma$ and ∇ is the gradient operator on $\mathbf{S}^{d-1}$, see [1, (3.3), (3.6), Proposition 3.3] for definitions. It is known that

$$\sum_{j=1}^{\nu(l)} |\hat{f}_{lj}|^2 = \|\sum_{j=1}^{\nu(l)} \hat{f}_{lj}Y_l^j\|^2_{L^2\left(\mathbf{S}^{d-1}\right)} = O(l^{d-2k-2})$$

if $f \in C^k(\mathbf{S}^{d-1})$([1, (3.26)]). Using the relation (2.4), we define

$$(-\Delta)^{\frac{\alpha}{2}} f := \sum_{l=0}^{\infty} \{l(l+d-2)\}^{\frac{\alpha}{2}} (\sum_{j=1}^{\nu(l)} \hat{f}_{lj}Y_l^j) \quad \text{in } L^2\left(\mathbf{S}^{d-1}\right),$$

that is,

$$\|(-\Delta)^{\frac{\alpha}{2}} f\|^2_{L^2\left(\mathbf{S}^{d-1}\right)} = \sum_{l=0}^{\infty} \{l(l+d-2)\}^{\alpha} (\sum_{j=1}^{\nu(l)} |\hat{f}_{lj}|^2) \tag{2.5}$$

for $f \in C^\infty(\mathbf{S}^{d-1})$. Especially, $\|(-\Delta)^{\frac{1}{2}} f\|_{L^2\left(\mathbf{S}^{d-1}\right)} = \|\nabla f\|_{L^2\left(\mathbf{S}^{d-1}\right)}$. Following [3], we define $H^\alpha(\mathbf{S}^{d-1})$ as the completion of $C^\infty(\mathbf{S}^{d-1})$ with respect to the norm

$$\|f\|^2_{H^\alpha(\mathbf{S}^{d-1})} := \sum_{l=0}^{\infty}\sum_{j=1}^{\nu(l)} \left(1 + l^{\frac{1}{2}}(l+d-2)^{\frac{1}{2}}\right)^{2\alpha} |\hat{f}_{lj}|^2$$
$$\left(= \left\| \left((I + (-\Delta)^{\frac{1}{2}}\right)^{\alpha} f \right\|^2_{L^2\left(\mathbf{S}^{d-1}\right)}\right),$$

see also [1, Definition 3.23].

3 Proof of Main Theorem (Theorem 1.5)

3.1 Preliminaries

We start from establishing the representation formula of $A_t^\rho f$ and $\|S_\alpha^{\rho,T}\|_{L^2\left(\mathbf{S}^{d-1}\right)}$ in terms of spherical harmonics.

Lemma 3.1. *For every ρ satisfying* (1.2)*, $T \in (0,\pi)$, and $t \in (0,T)$, the numbers*

$$m_{l,t}^\rho := \frac{|\mathbf{S}^{d-2}|}{z_t} \cdot \frac{t}{T} \int_0^T P_{l,d}\left(\cos \frac{t}{T}\theta\right) \sin^{d-2}\left(\frac{t}{T}\theta\right) \rho(\theta) d\theta.$$

form a bounded sequence $\{m_{l,t}^\rho\}_{l=0}^\infty \subset \mathbf{R}$ satisfying the following representation for every $f = \sum_{l=0}^\infty \sum_{j=1}^{\nu(l)} \hat{f}_{lj} Y_l^j \in L^2\left(\mathbf{S}^{d-1}\right)$:

$$A_t^\rho f = \sum_{l=0}^\infty m_{l,t}^\rho \sum_{j=1}^{\nu(l)} \hat{f}_{lj} Y_l^j \quad in\ L^2\left(\mathbf{S}^{d-1}\right).$$

Proof. It is obvious that $m_{l,t}^\rho$ is bounded by some constant independent of $l = 0,1,2,\cdots$ from (2.2). Therefore it is enough to see that $A_t^\rho Y_l = m_{l,t}^\rho Y_l$ for any $Y_l \in \mathbb{H}_l^d$. Since we have (2.3) for $\xi \in \mathbf{S}^{d-1}$, it holds that

$$A_t^\rho Y_l(\xi) = \int_{\mathbf{S}^{d-1}} L(\eta) Y_l(\eta) d\sigma(\eta),$$

where

$$L(\eta) := \frac{1}{z_t} \cdot \frac{\nu(l)}{|\mathbf{S}^{d-1}|} \int_{C(\xi,t)} P_{l,d}(\tau \cdot \eta) \rho\left(\frac{T}{t}\theta\right) d\sigma(\tau) \in \mathbb{H}_l^d.$$

Then, for a rotation R in $\mathbf{R}^d$ satisfying $R(\xi) = \xi$, it holds that

$$\begin{aligned} L(R\eta) &= \frac{1}{z_t} \cdot \frac{\nu(l)}{|\mathbf{S}^{d-1}|} \int_{C(\xi,t)} P_{l,d}(R^{-1}\tau \cdot \eta) \rho\left(\frac{T}{t}\theta\right) d\sigma(\tau) \\ &= \frac{1}{z_t} \cdot \frac{\nu(l)}{|\mathbf{S}^{d-1}|} \int_{C(R\xi,t)} P_{l,d}(\tau \cdot \eta) \rho\left(\frac{T}{t}\theta\right) d\sigma(\tau) = L(\eta). \end{aligned}$$

Therefore, applying Lemma 2.1, we get

$$A_t^\rho Y_l(\xi) = L(\xi) \int_{\mathbf{S}^{d-1}} P_{l,d}(\eta \cdot \xi) Y_l(\eta) d\sigma(\eta) = m_{l,t}^\rho Y_l(\xi),$$

where

$$\begin{aligned} m_{l,t}^\rho &= z_t^{-1} \int_{C(\xi,t)} P_{l,d}(\tau \cdot \xi) \rho\left(\frac{T}{t}\theta\right) d\sigma(\tau) \\ &= \frac{|\mathbf{S}^{d-2}|}{z_t} \cdot \frac{t}{T} \int_0^T P_{l,d}\left(\cos \frac{t}{T}\theta\right) \sin^{d-2}\left(\frac{t}{T}\theta\right) \rho(\theta) d\theta. \end{aligned}$$

□

Lemma 3.2. *For $f = \sum_{l=0}^\infty \sum_{j=1}^{\nu(l)} \hat{f}_{lj} Y_l^j \in L^2\left(\mathbf{S}^{d-1}\right)$, it holds that*

$$\|S_\alpha^{\rho,T}(f)\|_{L^2(\mathbf{S}^{d-1})}^2 = \sum_{l=1}^\infty I_\alpha^{\rho,T}(l) \left(\sum_{j=1}^{\nu(l)} |\hat{f}_{lj}|^2\right),$$

where

$$I_\alpha^{\rho,T}(l) := \int_0^T |M_{l,t}^\rho|^2 \frac{dt}{t^{2\alpha+1}},$$
$$M_{l,t}^\rho := \frac{|\mathbf{S}^{d-2}|}{z_t} \cdot \frac{t}{T} \int_0^T \left\{P_{l,d}(1) - P_{l,d}\left(\cos \frac{t}{T}\theta\right)\right\} \sin^{d-2}\left(\frac{t}{T}\theta\right) \rho(\theta) d\theta.$$

Proof of Lemma 3.2. From Lemma 3.1, it holds that

$$A_t^\rho f(\xi) - f(\xi) = \sum_{l=1}^{\infty} \sum_{j=1}^{\nu(l)} (m_{l,t}^\rho - 1) \hat{f}_{lj} Y_l^j(\xi) = \sum_{l=1}^{\infty} \sum_{j=1}^{\nu(l)} (-M_{l,t}^\rho) \hat{f}_{lj} Y_l^j(\xi)$$

since $m_{0,t}^\rho = 1$. Then we have the conclusion from Fubini's theorem and the orthogonality of $\{Y_l^j\}$ in $L^2\left(\mathbf{S}^{d-1}\right)$. □

Theorem 1.5 follows from (2.5) and the following fact:

$$I_\alpha^{\rho,T}(l) = \int_0^T |M_{l,t}^\rho|^2 \frac{dt}{t^{2\alpha+1}} \sim_{d,\rho,T} l^{2\alpha} \sim_{d,\rho,T} \{l(l+d-2)\}^\alpha.$$

The proof of this fact is divided into the following Lemma 3.6 and Lemma 3.8, which give the upper and lower bound, respectively. We start from the following elementary lemma:

Lemma 3.3. *For every $\rho \in L^\infty(0,T)$ satisfying* (1.2)*, there exists $[t_\rho, T_\rho] \subset [0,T] \cap (0,\pi)$ such that*

$$\int_{t_\rho}^{T_\rho} \rho(\theta) d\theta > 0.$$

Proof. Suppose that the conclusion does not hold. Then $\int_{1/N}^{T-1/N} \rho(\theta) d\theta = 0$ for every sufficiently large $N \in \mathbf{N}$, which leads to $\int_0^T \rho(\theta) d\theta = 0$ because $\rho \in L^\infty(0,T)$. This contradicts the assumption $\int_0^T \rho(\theta) d\theta > 0$. □

Next, we prepare the following lemma concerning the general behavior of z_t as $t \to 0$.

Lemma 3.4. *For every $\rho \in L^\infty(0,T)$ satisfying* (1.2)*, it holds that*

$$z_t \sim_{d,\rho,T} t^{d-1} \tag{3.1}$$

uniformly for every $t \in [0,T]$.

Proof. The upper estimate follows from the fact that $\sin\theta \le \theta$ for $\theta \ge 0$. The lower estimate follows from Lemma 3.3. Indeed, since $\sin\theta \ge \frac{\sin T_\rho}{T_\rho}\theta$ for $\theta \in [0, T_\rho]$, it follows that

$$z_t \ge \left|\mathbf{S}^{d-2}\right| \left(\frac{\sin T_\rho}{T_\rho}\right)^{d-2} \cdot \left(\frac{t}{T}\right)^{d-1} \int_{t_\rho}^{T_\rho} \theta^{d-2} \rho(\theta) d\theta \ge C_2(\rho) t^{d-1}$$

for every $t \in [0,T]$. □

3.2 Upper bound

First we prepare a rough estimate for $M^{\rho}_{l,t}$.

Lemma 3.5. *For every $l \in \mathbf{N}$ and $t \in (0,T]$, it holds*

$$0 \le M^{\rho}_{l,t} \le 2. \tag{3.2}$$

Proof. From (2.2), we get

$$0 \le P_{l,d}(1) - P_{l,d}\left(\cos\frac{t}{T}\theta\right) \le 2.$$

Then the conclusion follows. □

Lemma 3.6. *Suppose that $\rho \in L^{\infty}(0,T)$ satisfies (1.2). Then for $l \in \mathbf{N}$, it holds that*

$$I^{\rho,T}_{\alpha}(l) \lesssim_{d,\rho,T} l^{2\alpha}.$$

Proof. In order to get the conclusion, the estimate (3.2) is not enough around $t=0$. So we study the fine behavior of $M^{\rho}_{l,t}$ around $t=0$.

Using the mean value theorem and (2.2) for $k=1$, we have

$$M^{\rho}_{l,t} \lesssim_d l^2 \frac{|\mathbf{S}^{d-2}|}{z_t} \cdot \frac{t}{T} \int_0^T \left\{1-\cos\left(\frac{t}{T}\theta\right)\right\} \sin^{d-2}\left(\frac{t}{T}\theta\right)\rho(\theta)d\theta.$$

Here we note that

$$\left\{1-\cos\left(\frac{t}{T}\theta\right)\right\}\sin^{d-2}\left(\frac{t}{T}\theta\right) = \frac{\sin^d\left(\frac{t}{T}\theta\right)}{1+\cos\left(\frac{t}{T}\theta\right)} \le \left(\frac{t}{T}\theta\right)^d.$$

Then, by using (3.1), we get

$$0 \le M^{\rho}_{l,t} \lesssim_{d,\rho,T} l^2 t^2 \quad \text{for } 0 < t \le T.$$

Take $a \in (0,T)$ and set $t_l := \frac{a}{l}$ for $l \in \mathbf{N}$. Then $al \le T$ holds for every $l \in \mathbf{N}$. Therefore we get

$$\begin{aligned} I^{\rho,T}_{\alpha}(l) = \int_0^T |M^{\rho}_{l,t}|^2 \frac{dt}{t^{2\alpha+1}} &\le \int_0^{t_l} |M^{\rho}_{l,t}|^2 \frac{dt}{t^{2\alpha+1}} + \int_{t_l}^{\pi} |M^{\rho}_{l,t}|^2 \frac{dt}{t^{2\alpha+1}} \\ &\lesssim_{d,\rho,T} l^4 t_l^{4-2\alpha} + t_l^{-2\alpha} \sim_{d,\rho,T} l^{2\alpha}. \end{aligned}$$

□

3.3 Lower bound

We start from preparing another estimates of $P_{l,d}(s)$.

Lemma 3.7. *Set*

$$k_{l,d} := \begin{cases} \frac{1}{2} & (l=1), \\ \frac{d+1}{(l+d-1)(l-1)} \ (\le 1) & (l \ge 2). \end{cases}$$

Then, for every $\varepsilon \in (0,1)$, it holds that

$$P'_{l,d}(s) \ge (1-\varepsilon)P'_{l,d}(1) > 0 \quad \text{for } 1-\varepsilon k_{l,d} \le s \le 1.$$

Proof. The conclusion is obvious for $l = 1$ since $P_{1,d}(s) = s$. Suppose $l \geq 2$. The mean value theorem and (2.2) for $k = 2$ give us

$$P'_{l,d}(s) \geq P'_{l,d}(1) - P^{(2)}_{l,d}(1)(1-s).$$

From (2.1), we have

$$\frac{P'_{l,d}(1)}{P^{(2)}_{l,d}(1)} = \frac{d+1}{(l+d-1)(l-1)} = k_{l,d} \leq 1$$

for $l \geq 2$. Therefore, for

$$s \geq 1 - \varepsilon \frac{P'_{l,d}(1)}{P^{(2)}_{l,d}(1)} = 1 - \varepsilon k_{l,d},$$

it holds that

$$P'_{l,d}(s) \geq P'_{l,d}(1) - P^{(2)}_{l,d}(1)(1-s) \geq (1-\varepsilon)P'_{l.d}(1) > 0.$$

□

Lemma 3.8. *Suppose that* $\rho \in L^\infty(0,T)$ *satisfies* (1.2). *Then*

$$I^{\rho,T}_\alpha(l) \gtrsim_{d,\rho,T} l^{2\alpha}.$$

Proof. We use Lemma 3.7 with $\varepsilon = \frac{1}{2}$ and take $a(l) \geq 0$ that satisfies

$$\cos a(l) = 1 - \frac{k_{l,d}}{2}.$$

Since $k_{l,d} \leq 1$, we may assume that $a(l) \leq \pi/3$. Moreover we may assume

$$a(l) \sim \sqrt{k_{l,d}} \sim \{l(l+d-2)\}^{-\frac{1}{2}} \sim_d l^{-1}.$$

Therefore we get

$$M^\rho_{l,t} \gtrsim_d l^2 \cdot \frac{|\mathbf{S}^{d-2}|}{z_t} \cdot \frac{t}{T} \int_0^{\min(T, a(l)T/t)} \left\{1 - \cos\left(\frac{t}{T}\theta\right)\right\} \sin^{d-2}\left(\frac{t}{T}\theta\right) \rho(\theta) d\theta$$

from the mean value theorem, Lemma 3.7 with $\varepsilon = \frac{1}{2}$, and (2.2). Moreover, for t satisfying

$$\min(T, a(l)T/t) \geq T_\rho \quad \Leftrightarrow \quad t \leq \frac{T}{T_\rho} a(l), \tag{3.3}$$

where $T_\rho > 0$ is in Lemma 3.3, it holds that

$$\begin{aligned} &\int_0^{\min(T,a(l)T/t)} \left\{1 - \cos\left(\frac{t}{T}\theta\right)\right\} \sin^{d-2}\left(\frac{t}{T}\theta\right) \rho(\theta) d\theta \\ &\geq \frac{1}{2} \int_{t_\rho}^{T_\rho} \sin^d\left(\frac{t}{T}\theta\right) \rho(\theta) d\theta \gtrsim_{d,\rho,T} t^d \int_{t_\rho}^{T_\rho} \theta^d \rho(\theta) d\theta \gtrsim_{d,\rho,T} t^d. \end{aligned}$$

Using (3.1), we get

$$M^\rho_{l,t} \gtrsim_{d,T,\rho} l^2 t^2$$

for t in the range (3.3). Therefore it follows that

$$\begin{aligned} I^{\rho,T}_\alpha(l) &= \int_0^T \left|M^\rho_{l,t}\right|^2 \frac{dt}{t^{2\alpha+1}} \geq \int_0^{\frac{T}{T_\rho}a(l)} \left|M^\rho_{l,t}\right|^2 \frac{dt}{t^{2\alpha+1}} \\ &\gtrsim_{d,\rho,T} l^4 \int_0^{\frac{T}{T_\rho}a(l)} t^{3-2\alpha} dt = l^4 \left\{\frac{T}{T_\rho} a(l)\right\}^{4-2\alpha} \gtrsim_{d,\rho,T} l^{2\alpha}. \end{aligned}$$

□

Acknowledgement

We thank Prof. Masato Kimura (Kanazawa Univ.) for his kind suggestions for the presentation. We also thank the anonymous referees for improving the presentation. This work was supported by Japan Society for the Promotion of Science Grant-in-Aid for Scientific Research Number 24K06794.

References

[1] Kendall Atkinson and Weimin Han, Spherical harmonics and approximations on the unit sphere: an introduction, Lecture Notes in Math., 2044, 2012, Springer, Heidelberg, 2012, x+244 pp.

[2] Roc Alabern, Joan Mateu, Joan Verdera, A new characterization of Sobolev spaces on $\mathbf{R}^n$, Math. Ann. **354** (2012) no.2, 589–626.

[3] Juan Antonio Barceló, Teresa Luque, Salvador Pérez-Esteva, Characterization of Sobolev spaces on the sphere, J. Math. Anal. Appl. **491** (2020) no.1, 124240, 23 pp.

[4] Shuichi Sato, Littlewood-Paley equivalence and homogeneous Fourier multipliers, Integral Equations Operator Theory **87** (2017) no.1, 15–44.