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The 8 puzzle with Neighbors Swap Motion is solvable

Prihardono ARIYANTO and Kenichi KAWAGOE*

Bukalapak, Metropolitan Tower Lantai 22 JI. R. A. Kartini Kav. 14 Cilandak Barat Kec. Cilandak, Kota Jakarta Selatan, Daerah Khusus Ibukota Jakarta 12430
Institute of Liberal Arts and Science, Kanazawa University
Kanazawa, 920-1192, Japan

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Abstract From the movements of “M_{13}” on $PG(2,3)$, we introduce a notion of Neighbors Swap Motions on the square. By applying Neighbors Swap Motions, the 8 puzzle turns out to be solvable.

Keywords. Neighbors Swap Motion, permutation.

1 Introduction

In [1], Conway constructed a groupoid of certain permutations of 13 letters on the projective plane $PG(2,3)$, called “M_{13}”. Inspired by this construction, Egner and Beth made a program to play M_{13}, which is one of the sliding puzzles [3]. Here we demonstrate M_{13}. The projective plane $PG(2,3)$ has 13 points and 13 lines. One line contains 4 points, and one point is an intersection of 4 lines. Any two points are on one line. In Figure 1, we give $PG(2,3)$ as a graph:

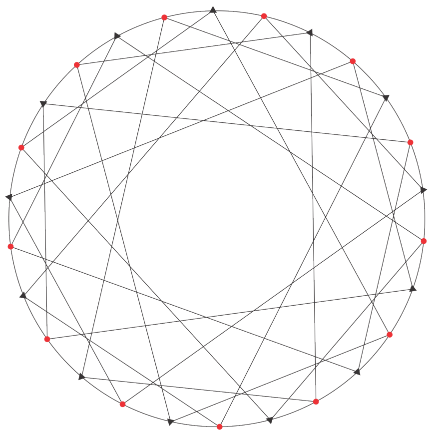


Figure 1: The projective plane $PG(2,3)$ as a graph

*Corresponding author Email: kawagoe@se.kanazawa-u.ac.jp

The red vertices of the graph stand for points of $PG(2,3)$. The black vertices of the graph stand for lines of $PG(2,3)$. Each edge stands for incidence relations. We put integers $1,\ldots,12$ on the red points. The last red point is free, called a hole. A movement of M_{13} is defined by

1. Select one red point except for the hole. Then there exists only one line which contains the selected point and the hole. Exchange the selected point and the hole.
2. The line contains exactly two points except for the selected point and the hole. Exchange the two points.

We demonstrate one of the movements of M_{13}. We start with the initial state of M_{13} shown in Figure 2. For example, we select the target point labeled with 8. Then there exists only one line containing the hole and the point labeled with 8, which lies between points labeled with 8 and 9 (Figure 3).

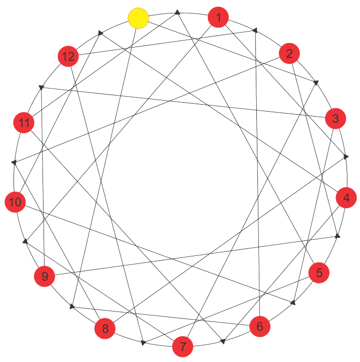


Figure 2: The initial state of M_{13}

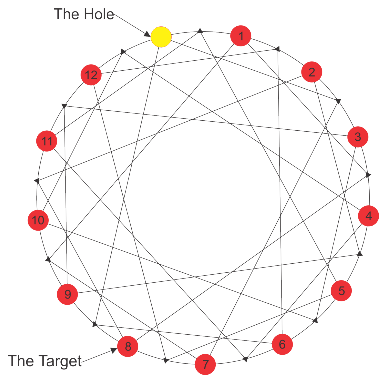


Figure 3: The selection of the target point

First, we exchange the point labeled with 8 and the hole (Figure 4). Next, we exchange the point labeled with 6 and 9 (Figure 5).

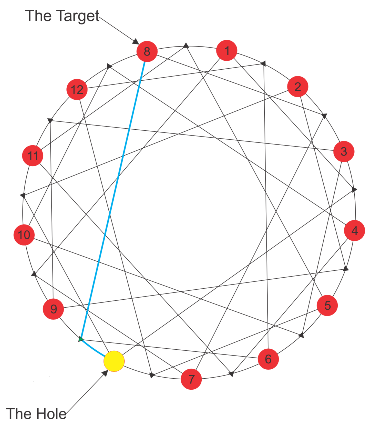


Figure 4: The first exchange

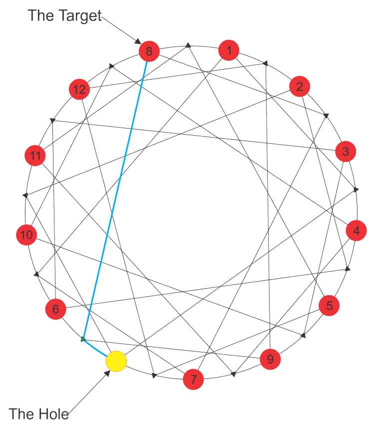


Figure 5: The second exchange

In M_{13}, for any labeling of the red points, it is known that we can transform the state into the initial state by a finite sequence of these movements.

2 The 8 puzzle with Neighbors Swap Motion

M_{13} seems to be some kind of sliding puzzle on the circle. As for the sliding puzzle, the 15 puzzle is known as one of the most famous sliding puzzles, which is demonstrated on the square 4×4. Furthermore, we know that it has two types, solvable or not solvable. Solvable means that a labeling of 15 letters can be transformed into the standard labeling of $1, \ldots, 15$ from top left to bottom right. It is well known that the labeling of 15 letters is an even permutation if and only if it is solvable. Similarly, the 8 puzzle is known as a sliding puzzle on the square 3×3, which has also two types, solvable or not solvable. Here, we consider another kind of permutations on the square 3×3. We call 9 small squares on the 3×3 square a *cell*, and we label 9 cells with $1, \ldots, 8$ and the hole.

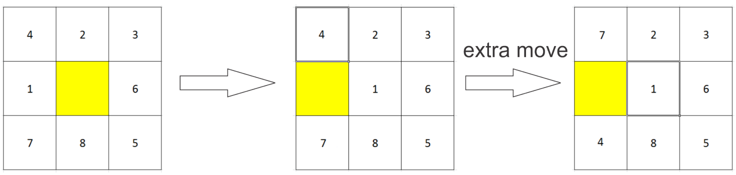
Definition 2.1. *Select a target cell that has a side in common with the hole. Then we define a movement by exchanging the cell and hole, and furthermore*

1. *if the number of the cell each of which has a side in common with the target cell is 2, the extra movement is defined by swapping between the two cells.*
2. *if the number of the cell each of which has a side in common with the target cell is 3, the extra movement is defined by swapping between the two cells, where the two swapped cells are in the same row or in the same column with the target cell.*

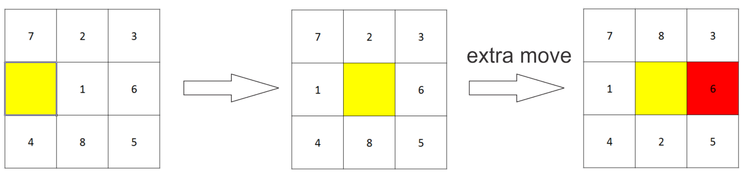
We call this movement a Neighbors Swap Motion.

We give examples of the Neighbors Swap Motions in the following figures. The yellow cell stands for the hole.

Example 2.1. 1. *Select the target cell labeled with* 1. *Then cells labeled with* 4 *and* 7 *have a side in common with the hole. Hence we exchange between the cell labeled with* 1 *and the hole, and between two cells labeled with* 4 *and* 7.



2. *Select the target cell labeled with* 1. *Then cells labeled with* 2, 6, 8 *have a side in common with the hole. Since the cell labeled with* 6 *(the red cell) is in the same row with the target and the hole, we exchange between the cell labeled with* 1 *and the hole, and between two cells labeled with* 2 *and* 8.



There are 4 kinds of movements, and we define them as follows:

Definition 2.2. 1. *$u :=$ move the hole up (the target cell down)*

2. *$d :=$ move the hole down (the target cell up)*

3. *l := move the hole left (the target cell right)*
4. *r := move the hole right (the target cell left)*

Theorem 2.1. *Every two cells that have a side in common are exchangeable by a finite sequence of Neighbors Swap Motions.*

Proof. We consider a graph G_8 represented by vertices that are states of permutations of the $1,2,3,4,5,6,7,8$ on the 3×3 square, where we put the hole 0 at the lower right. The number of vertices of G_8 is $8! = 40320$. Edges are defined if one state is obtained from the other state by a Neighbors Swap Motion. We may assume that the hole is at the lower right because we consider all permutations of $1, \dots, 8$. With a breadth-first search for G_8, we can show that the graph G_8 is connected. In fact, we find transpositions of two cells that have a side in a common. All such transpositions are given in the following table.

	l	l	r	r	l	r								
	u	u	d	u	l	r	l	l	r	d	r	d		
	u	u	d	u	l	d	u	d	r	d				
	u	u	l	l	d	u	d	d	u	r	r	l	r	d
	u	d	u	u	d	d								
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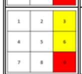
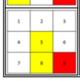
	u	u	d	d	u	d								
	u	u	d	d	u	l	l	u	d	d	u	r	r	d
	l	r	l	l	r	r								

Figure 6: Transpositions by Neighbors Swap Motions

□

From this theorem, we have the following corollary.

Corollary 2.1. *The* 8*-puzzle with Neighbors Swap Motion is solvable.*

3 Comments

Since the Neighbors Swap Motions behave differently at the corner and the side and the inside, permutations induced by the Neighbors Swap Motions are very complicated. Hence, we first use computer calculations to determine whether G_8 is connected or not. In fact, by finding all transpositions, we can show that G_8 is connected. For the 4×4 square, the corresponding graph is so big that we cannot determine whether the graph is connected or not.

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