Sci. Rep. Kanazawa Univ.
Vol. 64, pp. 15–39, 2020

ARTICLES

Stern's diatomic sequence and a series of tangent circles orthogonal to the unit circle

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(Received October 5, 2019 and accepted in revised form February 12, 2020)

Abstract We study a series of tangent circles orthogonal to the unit circle on the complex plane. In particular, we study the case that the number of tangent circles is three. By operating inversions to the circles, we have an infinite family of circles. We show that the inverse of the radius of a circle in the family is a linear sum of the inverses of the radii of beginning three circles, and then their coefficients are expressed by using Stern's diatomic sequence (Theorem 4.4). As a corollary, we obtain a formula to compute π (Corollary 4.5).

Keywords. Stern's diatomic sequence, Möbius transformation.
2010 Mathematics Subject Classification. 11Y35, 11Y55, 28A20.

1 Introduction

Stern's diatomic sequence is a sequence $\{a_m\}_{m=0}^{\infty}$ defined by

$$a_0 = 0,\ a_1 = 1,\ a_{2m} = a_m,\ a_{2m+1} = a_m + a_{m+1}\,(m \in \mathbb{Z}_{\geq 0}).$$

As far as we know, M. A. Stern firstly defined it in [10], after that several authors have studied it (e.g. [4, 7, 9]). In the present paper, we refine a_m as $[2^n : m]\,(m, n \in \mathbb{Z}_{\geq 0}, 0 \leq m \leq 2^n)$, which is called *Stern's diatomic integer* with *depth n* and *order m*. We arrange Stern's diatomic integers as vertices of a fixed infinite graph. The resulting one is called Stern's diatomic table (cf. Definition 4.1 and Fig. 4-1). Precisely, each Stern's diatomic integer $[2^n : m]$ is situated on the n-th line (the depth n) and the order m from the left in Stern's diatomic table.

In Section 2, we give a definition of tangent transformations and a series of tangent circles. In the complex plane, for three different points a_1, a_2 and a_3 on the unit circle, let C_1, C_2 and C_3 be circles with centers p_1, p_2 and p_3 that are in contact with one another at a_1, a_2 and a_3. These circles are orthogonal to the unit circle. We define three Möbius transformations: $F_i(z) = (z - p_i)/(\overline{p_i}z - 1)\,(i = 1, 2, 3)$, which are called *tangent transformations* with *centers* p_i. Then $F_i(C_i) = C_i\,(i = 1, 2, 3)$, and composite transformations $F_iF_i = F_i \circ F_i\,(i = 1, 2, 3)$ are the identity transformation. For any non-negative integer n, we define

$$T_n = \{F_{i_n} \cdots F_{i_2}F_{i_1} \,|\, i_1, i_2, \cdots, i_n \in \{1, 2, 3\}, i_k \neq i_{k+1} (1 \leq k \leq n - 1)\},$$

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where if $n = 0$, T_0 consists of the identity transformation only. We set

$$TC_n = \{C_{i_1 i_2 \cdots i_n} = F_{i_n} \cdots F_{i_3} F_{i_2}(C_{i_1}) \mid F_{i_n} \cdots F_{i_3} F_{i_2} \in T_{n-1}, i_1 \neq i_2\}, T = \coprod_{n=0}^{\infty} T_n, TC = \coprod_{n=1}^{\infty} TC_n.$$

Then T has a free group structure whose generators are F_1, F_2 and F_3, and the unit is the identity transformation.

All circles of TC_n are arranged on the unit circle without any gaps, each circle of TC_n is orthogonal to the unit circle, and any two circles of TC_n are either tangent or disjoint from each other. We set

$$TC_n^1 = \{C_{i_1 i_2 \cdots i_n} \in TC_n \mid i_n = 1\}, TC_n^2 = \{C_{i_1 i_2 \cdots i_n} \in TC_n \mid i_n = 2\}, TC_n^3 = \{C_{i_1 i_2 \cdots i_n} \in TC_n \mid i_n = 3\},$$

then all circles of TC_n^1 are inside C_1, all circles of TC_n^2 are inside C_2, all circles of TC_n^3 are inside C_3, and $TC_n = \coprod_{k=1}^{3} TC_n^k$. About this part, the readers refer to [5, 6, 8].

In Section 3, we clarify the relationships among the circles of TC.

In Section 4, we prove the main result Theorem 4.4, which shows us a relationship between geometric problems and Stern's diatomic sequence. We note that in this paper, for any circle, we denote the radius of its circle by the same symbol. Then Theorem 4.4 states that if $C_{(n:m)} \in TC_n^1$ is the m^{th} $(1 \leq m \leq 2^{n-1})$ circle counterclockwise from a_1, it holds

$$\frac{1}{C_{(n:m)}} = \frac{(n:m)_1}{C_1} + \frac{(n:m)_2}{C_2} + \frac{(n:m)_3}{C_3},$$

where

$$\begin{aligned}
(n:m)_1 &= [2^n : 2^{n-1} + (m-1)][2^n : 2^{n-1} + m], \\
(n:m)_2 &= [2^n : m-1][2^n : m], \\
(n:m)_3 &= [2^n : 2^{n-1} - (m-1)][2^n : 2^{n-1} - m].
\end{aligned}$$

As an application of Theorem 4.4, Corollary 4.5 states that π is given by (4:39) via Stern's diatomic integers.

Starting with Theorem 4.4, careful consideration to Stern's diatomic integers gives us a chance to study the Markov Conjecture (cf. [2]) which is one of the important Diophantine problems. In the forthcoming paper, by using a binary number presentation of Stern's diatomic integer, we will define the *assembly function* (cf. [11]), which is essentially equivalent to Conway's box function, and clarify importance of the assembly function and the relationship with the Markov Conjecture.

2 A series of tangent circles

For any complex number p, we define a Möbius transformation on the complex plane: $F(z) = (z-p)/(\overline{p}z-1)$. We summarize the properties of this function.

Lemma 2.1. *Let* $\Delta = 1(-1) - (-p)\overline{p} = |p|^2 - 1$.
(1) *If* $\Delta = 0$*, then* $F(z)$ *is a constant function.*
(2) *If* $\Delta \neq 0$*, then we have the following results:*
(i) $F(0) = p, F(p) = 0, F(F(z)) = z$.
(ii) *For any point* z *on the unit circle,* $F(z)$ *is on the unit circle.*
(iii) *If* $|p| > 1$*, for any point* z *inside the unit circle,* $F(z)$ *is outside the unit circle, and for any point*

z outside the unit circle, $F(z)$ is inside the unit circle.
If $|p| < 1$, for any point z inside the unit circle, $F(z)$ is inside the unit circle, and for any point outside the unit circle, $F(z)$ is outside the unit circle.
(iv) *$F(z)$ is a conformal mapping and a circle-to-circle correspondence.*

Proof. (1) Since $p\overline{p} = |p|^2 = 1$, $F(z) = \dfrac{z-p}{\overline{p}z-1} = \dfrac{p(z-p)}{p\overline{p}z-p} = \dfrac{p(z-p)}{z-p} = p$.
(2) (i) These formulas can be easily confirmed.
(ii) If $|z| = 1$, from $z\overline{z} = |z|^2 = 1$, $|F(z)| = \left|\dfrac{z-p}{\overline{p}z-1}\right| = \left|\dfrac{z-p}{\overline{p}z-z\overline{z}}\right| = \dfrac{|z-p|}{|z||\overline{z}-\overline{p}|} = \dfrac{1}{|z|} = 1$.
(iii) We note that

$$|F(z)|^2 - 1 = \frac{z-p}{\overline{p}z-1}\cdot\frac{\overline{z}-\overline{p}}{p\overline{z}-1} - 1 = \frac{(1-|p|^2)(|z|^2-1)}{|\overline{p}z-1|^2}.$$

If $|p| > 1$, for any complex number z with $|z| > 1$, we have $|F(z)| < 1$, and for any complex number z with $|z| < 1$, we have $|F(z)| > 1$. If $|p| < 1$, for any complex number z with $|z| > 1$, we have $|F(z)| > 1$, and for any complex number z with $|z| < 1$, we have $|F(z)| < 1$.
(iv) It is a well-known result in the complex function theory.

□

Definition 2.2. Suppose p is an intersection point of two straight lines tangent to the unit circle C at $a_1, a_2\,(a_1 \neq \pm a_2)$. Let C' be a circle with center p and radius $r = |a_1 - p| = |a_2 - p|$. We define a Möbius transformation such that $F(z) = (z-p)/(\overline{p}z-1)$, and call it *the tangent transformation* of C and p *the center* of F.

We summarize the properties of $F(z)$.

Lemma 2.3. *Under the setting in Definition 2.2, we have the following:*
(1) *For any point z on the circle C', $F(z)$ is also on C', and $z, F(z), 0$ are on the same straight line. For any point z inside C', $F(z)$ is outside C', and for any point z outside C', $F(z)$ is inside C'.*
(2) *For any point z on the unit circle C, $F(z)$ is also on C, and $z, F(z), p$ are on the same straight line. In particular, a_1 and a_2 are fixed points of F. For any point z inside C, $F(z)$ is outside C, and for any point z outside C, $F(z)$ is inside C.*
(3) *For two points $z_1, z_2\,(z_1 \neq \pm z_2)$ on the unit circle C, we put $z_1' = F(z_1), z_2' = F(z_2)$, then z_1', z_2' are also on the unit circle C. Let C_1 be a circle orthogonal to C at z_1, z_2, and C_2 be a circle orthogonal to C at z_1', z_2'. Then C_1 corresponds to C_2 under F, and C_2 conversely corresponds to C_1 under F.*

Proof. (1) We note that $|p| > 1$. Since

$$|F(z)-p|^2 - (|p|^2-1) = \frac{(|p|^2-1)\{(|p|^2-1) - |z-p|^2\}}{|\overline{p}z-1|^2},$$

if $|z-p| = \sqrt{|p|^2-1}$, then $|F(z)-p| = \sqrt{|p|^2-1}$, if $|z-p| > \sqrt{|p|^2-1}$, then $|F(z)-p| < \sqrt{|p|^2-1}$, and if $|z-p| < \sqrt{|p|^2-1}$, then $|F(z)-p| > \sqrt{|p|^2-1}$. Hence we have that for any point z on the circle C', $F(z)$ is also on C', for any point z inside C', $F(z)$ is outside C', and for any point z outside C', $F(z)$ is inside C'. If z is a point on C', $F(z)$ is also a point on C'. Then

$$|p|^2 - 1 = |z-p|^2 = (z-p)(\overline{z}-\overline{p}) = z\overline{z} - \overline{p}z - p\overline{z} + |p|^2.$$

From $\overline{p}z - 1 = (z-p)\overline{z}$, we have

$$F(z) = \frac{z-p}{\overline{p}z-1} = \frac{z-p}{(z-p)\overline{z}} = \frac{1}{\overline{z}} = \frac{z}{|z|^2}. \tag{2:1}$$

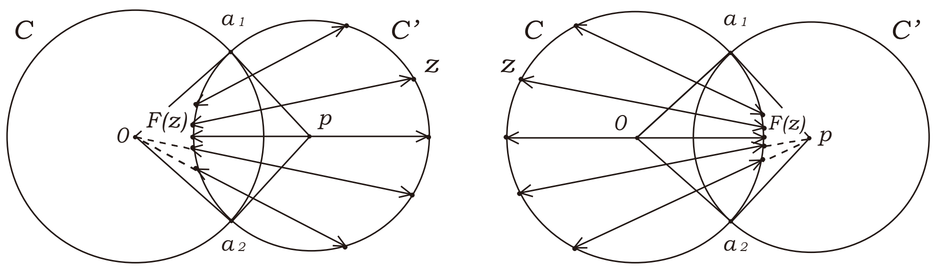

Fig 2-1 Fig 2-2

Therefore, the origin 0, $F(z), z$ are on the same straight line. If z is a fixed point, from (2:1), we have $|z| = 1$, which concludes that the fixed points on C' are only a_1, a_2.
(2) Since $|p| > 1$, from Lemma 2.1 (2) (ii), (iii), we have that for any point z on the unit circle C, $F(z)$ is also on C, for any point z inside C, $F(z)$ is outside C, and for any point z outside C, $F(z)$ is inside C. If z is a point on C, from $1 = |z|^2 = z\overline{z}$,

$$F(z) - p = \frac{z-p}{\overline{p}z-1} - p = \frac{(1-|p|^2)z}{\overline{p}z-1} = \frac{(1-|p|^2)z(p\overline{z}-1)}{|\overline{p}z-1|^2} = \frac{(|p|^2-1)(z-p)}{|\overline{p}z-1|^2}. \tag{2:2}$$

Hence $F(z), z, p$ are on the same straight line. Let z be a fixed point. Then we have $|F(z) - p| = |z - p|$. By (2:2), $|p - z| = |\overline{p} - \overline{z}| = |\overline{p}z - 1| = \sqrt{|p|^2 - 1}$. Therefore, we conclude that the fixed points of F are only a_1, a_2.
(3) By Lemma 2.1 (2) (i) and Lemma 2.3 (2), z'_1, z'_2 are on the unit circle C, and $z_1 = F(z'_1), z_2 = F(z'_2)$. Since F is a conformal mapping and a circle-to-circle correspondence, the circle C_1 corresponds to the circle C_2 under F, and the circle C_2 conversely corresponds to the circle C_1 under F. □

In the complex plane, for three different points a_1, a_2 and a_3 on the unit circle, where $\triangle a_1a_2a_3$ is an acute triangle, let C_1, C_2 and C_3 be circles with centers p_1, p_2 and p_3 that are in contact with each other at a_1, a_2 and a_3. These circles are orthogonal to the unit circle C. We define three Möbius transformations:

$$F_1(z) = \frac{z-p_1}{\overline{p_1}z-1}, \qquad F_2(z) = \frac{z-p_2}{\overline{p_2}z-1}, \qquad F_3(z) = \frac{z-p_3}{\overline{p_3}z-1},$$

which are called *tangent transformations* with *centers* $p_i\,(i = 1,2,3)$. Then we note that composite transformations $F_iF_i = F_i \circ F_i\,(i = 1,2,3)$ are the identity transformation, and $F_iC_i = F_i(C_i) = C_i\,(i = 1,2,3)$.

Definition 2.4. For any non-negative integer n, we define

$$T_n = \{F_{i_n} \cdots F_{i_2}F_{i_1} \,|\, i_1, i_2, \cdots, i_n \in \{1,2,3\}, i_k \neq i_{k+1} (1 \leq k \leq n-1)\},$$

and, for any positive integer n,

$$TC_n = \{C_{i_1i_2\cdots i_n} = F_{i_n} \cdots F_{i_3}F_{i_2}(C_{i_1}) \,|\, F_{i_n} \cdots F_{i_3}F_{i_2} \in T_{n-1},\, i_1 \neq i_2\},$$

where if $n = 0$, $F_{i_n} \cdots F_{i_2}F_{i_1}$ represents the identity transformation. $F_{i_n} \cdots F_{i_2}F_{i_1}$ is also called a *tangent transformation* and $C_{i_1i_2\cdots i_n} = F_{i_n} \cdots F_{i_3}F_{i_2}(C_{i_1})$ is called a *tangent circle* with *rank n*.

We set $T = \coprod_{n=0}^{\infty} T_n$, $TC = \coprod_{n=1}^{\infty} TC_n$, then T has a free group structure whose generators are F_1, F_2 and F_3, and the unit is the identity transformation. All tangent circles with rank n are arranged on the unit circle without any gaps, each tangent circle with rank n is orthogonal to the unit circle, and any two tangent circles with rank n are either tangent or disjoint from each other. Further we set $TC_n^1 = \{C_{i_1 i_2 \cdots i_n} \in TC_n \mid i_n = 1\}, TC_n^2 = \{C_{i_1 i_2 \cdots i_n} \in TC_n \mid i_n = 2\}$ and $TC_n^3 = \{C_{i_1 i_2 \cdots i_n} \in TC_n \mid i_n = 3\}$, then all tangent circles of TC_n^1 are inside C_1, all tangent circles of TC_n^2 are inside C_2, all tangent circles of TC_n^3 are inside C_3 and $TC_n = \coprod_{k=1}^{3} TC_n^k$. Finally, we set $IF = \{C_1, C_2, C_3, F_1, F_2, F_3\}$, and call it the *initial figure*.

From now on, for any circle H, we represent the radius of its circle by the same symbol H. In particular, we represent the radius of a tangent circle $C_{i_1 i_2 \cdots i_n}$ by the same symbol $C_{i_1 i_2 \cdots i_n}$.

3 The relationships among tangent circles

In this section, we study several relationships among the radii of tangent circles.

Lemma 3.1. *In Figure 3-1, let $a_1, a_2, a_3\,(a_1 \neq \pm a_3)$ be three points on the unit circle C, and C_1 be a circle orthogonal to C at a_1, a_3. Suppose c_1 is a circle orthogonal to C at a_1, a_2, and c_2 is a circle orthogonal to C at a_2, a_3. Then C_1, c_1, c_2 are touching at a_1, a_2, a_3. We set $l = |a_1 - a_3|$. Then we have*

$$(1)\ C_1 = \frac{c_1 + c_2}{1 - c_1 c_2}, \qquad (2)\ l = \frac{2C_1}{\sqrt{1 + (C_1)^2}}. \tag{3:1}$$

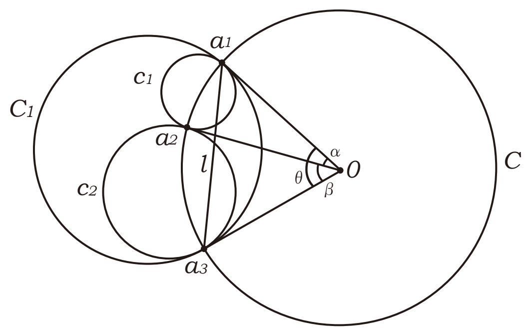

Fig 3-1

Proof. (1) We put $\theta = \angle a_1 0 a_3, \alpha = \angle a_1 0 a_2, \beta = \angle a_2 0 a_3$. Then from $\theta = \alpha + \beta$, we have $\theta/2 = \alpha/2 + \beta/2$. Hence,

$$C_1 = \tan\frac{\theta}{2} = \tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \frac{\tan\dfrac{\alpha}{2} + \tan\dfrac{\beta}{2}}{1 - \tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}} = \frac{c_1 + c_2}{1 - c_1 c_2}.$$

(2) From $C_1 = \tan\dfrac{\theta}{2}$ and $l = 2\sin\dfrac{\theta}{2}$, we have the result by simple calculation. □

Theorem 3.2. *In Figure 3-2, let C_1, C_2, c be circles orthogonal to the unit circle C, where C_1, C_2 circumscribe each other at a point a_3, and C_1, c inscribe each other at the same point a_3. For $F(z) = (z-p)/(\overline{p}z-1)$ $(|p|>1)$, let C_1', C_2', c' be the images of C_1, C_2, c under F respectively. Then we have the following relationships among these radii.*

$$(1)\ \frac{C_1}{C_2}\cdot\frac{C_2+c}{C_1-c}=\frac{C_1'}{C_2'}\cdot\frac{C_2'+c'}{C_1'-c'}. \quad (2)\ c'=\frac{c(C_1+C_2)}{\left(\frac{C_1}{C_1'}\right)(C_2+c)+\left(\frac{C_2}{C_2'}\right)(C_1-c)}. \tag{3:2}$$

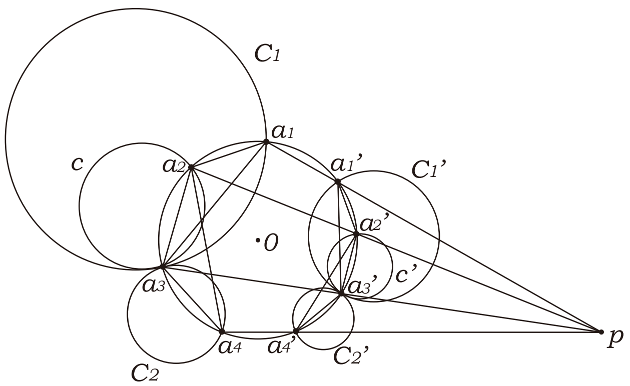

Fig 3-2

Proof. Let $a_1, a_2, a_4, a_1', a_2', a_4'$ be intersection points of C and $C_1, c, C_2, C_1', c', C_2'$ respectively, and we set $|a_4-a_3|=l_1$, $|a_3-a_1|=l_2$, $|a_4-a_2|=l_3$, $|a_2-a_1|=l_4$, $|a_4'-a_3'|=l_1'$, $|a_3'-a_1'|=l_2'$, $|a_4'-a_2'|=l_3'$, $|a_2'-a_1'|=l_4'$. Then since cross ratios are invariant under F, we have

$$\frac{(a_4-a_3)(a_2-a_1)}{(a_3-a_1)(a_4-a_2)}=\frac{(a_4'-a_3')(a_2'-a_1')}{(a_3'-a_1')(a_4'-a_2')},$$

which concludes that

$$\frac{l_1'}{l_1}\cdot\frac{l_4'}{l_4}=\frac{l_2'}{l_2}\cdot\frac{l_3'}{l_3}. \tag{3:3}$$

Suppose c_1 is the circle orthogonal to C at 2 points a_1, a_2, and c_2 is the circle orthogonal to C at 2 points a_2, a_4. Let c_1', c_2' be the images of c_1, c_2 under F respectively. We substitute (3:1) into the square of (3:3). Then we have

$$\frac{(C_2')^2}{(C_2)^2}\cdot\frac{1+(C_2)^2}{1+(C_2')^2}\cdot\frac{(c_1')^2}{(c_1)^2}\cdot\frac{1+(c_1)^2}{1+(c_1')^2}=\frac{(C_1')^2}{(C_1)^2}\cdot\frac{1+(C_1)^2}{1+(C_1')^2}\cdot\frac{(c_2')^2}{(c_2)^2}\cdot\frac{1+(c_2)^2}{1+(c_2')^2}.$$

By Lemma 3.1, we have $c_1=\dfrac{C_1-c}{1+C_1c}$, $c_2=\dfrac{C_2+c}{1-C_2c}$, $c_1'=\dfrac{C_1'-c'}{1+C_1'c'}$, $c_2'=\dfrac{C_2'+c'}{1-C_2'c'}$. Hence,

$$\frac{(C_2')^2}{(C_2)^2}\cdot\frac{1+(C_2)^2}{1+(C_2')^2}\cdot\frac{(C_1'-c')^2}{(C_1-c)^2}\cdot\frac{(1+C_1c)^2+(C_1-c)^2}{(1+C_1'c')^2+(C_1'-c')^2}$$

$$=\frac{(C_1')^2}{(C_1)^2}\cdot\frac{1+(C_1)^2}{1+(C_1')^2}\cdot\frac{(C_2'+c')^2}{(C_2+c)^2}\cdot\frac{(1-C_2c)^2+(C_2+c)^2}{(1-C_2'c')^2+(C_2'+c')^2}.$$

Therefore,

$$\frac{C_1}{C_2}\cdot\frac{C_2+c}{C_1-c}=\frac{C_1'}{C_2'}\cdot\frac{C_2'+c'}{C_1'-c'}\,. \tag{3:4}$$

By (3:4), we immediately obtain (3:2). □

For the radii of a series of tangent circles of TC, we rewrite Theorem 3.2.

Corollary 3.3. *Let $IF=\{C_1,C_2,C_3,F_1,F_2,F_3\}$ be the initial figure, where*

$$F_1(z)=\frac{z-p_1}{\overline{p_1}z-1},\qquad F_2(z)=\frac{z-p_2}{\overline{p_2}z-1},\qquad F_3(z)=\frac{z-p_3}{\overline{p_3}z-1}.$$

Suppose H_1,H_2 are the tangent circles with rank n, h_1,h_2 are the tangent circles with rank $n+1$, and these circles are touching each other at a point a on C. Let H_1',H_2',h_1',h_2' be the images of H_1,H_2,h_1,h_2 under F_1 respectively. Then,

$$h_1'=\frac{h_1(H_1+H_2)}{\left(\frac{H_1}{H_1'}\right)(H_2+h_1)+\left(\frac{H_2}{H_2'}\right)(H_1-h_1)},\qquad h_2'=\frac{h_2(H_1+H_2)}{\left(\frac{H_1}{H_1'}\right)(H_2-h_2)+\left(\frac{H_2}{H_2'}\right)(H_1+h_2)}.$$

Theorem 3.4. *Let $IF=\{C_1,C_2,C_3,F_1,F_2,F_3\}$ be the initial figure, where*

$$F_1(z)=\frac{z-p_1}{\overline{p_1}z-1},\qquad F_2(z)=\frac{z-p_2}{\overline{p_2}z-1},\qquad F_3(z)=\frac{z-p_3}{\overline{p_3}z-1}.$$

Suppose this IF has been already rotated around the origin such that p_1 locates on the positive real axis. Let a_1,a_2,a_3 be contact points between C_3 and C_1, C_1 and C_2, C_2 and C_3 respectively. For any point a on C, let a' be the image of a under F_1, and $\theta(0\le\theta<2\pi), r(\theta)(0\le\theta<2\pi), \alpha(0<\alpha<\pi/2)$ be the arguments of complex numbers a,a',a_2 respectively. Then, we have the following:

(1) *If $\theta=0$, then $r(\theta)=\pi$.*

(2) *If $\theta=\pi$, then $r(\theta)=0$.*

(3) *If $\theta\ne 0,\pi$, then* $\tan\frac{\theta}{2}\tan\frac{r(\theta)}{2}=\tan^2\frac{\alpha}{2}\,.$ (3:5)

Proof. (3) Under this situation, we denote $a=e^{i\theta}, a'=e^{ir(\theta)}, a_1=e^{-i\alpha}, a_2=e^{i\alpha}, p_1=1/\cos\alpha$. Since the argument of the complex number $z=(a'-p_1)/(a-p_1)$ equals to 0, z is a real number. Hence,

$$\frac{a'-p_1}{a-p_1}=\frac{\overline{a'}-p_1}{\overline{a}-p_1}.$$

$$e^{i(r(\theta)-\theta)}\cos^2\alpha-e^{ir(\theta)}\cos\alpha-e^{-i\theta}\cos\alpha=e^{i(\theta-r(\theta))}\cos^2\alpha-e^{i\theta}\cos\alpha-e^{-ir(\theta)}\cos\alpha.$$

From $0<\alpha<\pi/2$, we have $\cos\alpha\ne 0$. Hence

$$(e^{i(r(\theta)-\theta)}-e^{-i(r(\theta)-\theta)})\cos\alpha+(e^{i\theta}-e^{-i\theta})-(e^{ir(\theta)}-e^{-ir(\theta)})=0.$$

$$\sin\frac{r(\theta)-\theta}{2}\left(\cos\frac{r(\theta)-\theta}{2}\cos\alpha-\cos\frac{r(\theta)+\theta}{2}\right)=0.$$

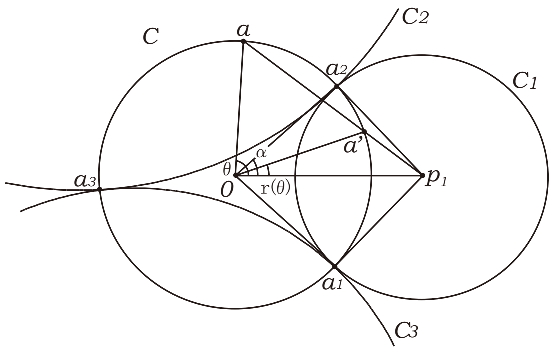

Fig 3-3

Hence

$$\sin\frac{r(\theta)-\theta}{2}=0\cdots\cdots\langle 1\rangle \quad \text{or} \quad \cos\frac{r(\theta)-\theta}{2}\cos\alpha-\cos\frac{r(\theta)+\theta}{2}=0\cdots\cdots\langle 2\rangle.$$

In the case $\langle 1\rangle$: Since $-\pi < (r(\theta)-\theta)/2 < \pi$, we have $r(\theta)=\theta$, which implies $\theta = r(\theta) = \alpha$ or $\theta = r(\theta) = 2\pi-\alpha$. In the both cases, we have (3:5).
In the case $\langle 2\rangle$:

$$\cos\alpha\left(\cos\frac{r(\theta)}{2}\cos\frac{\theta}{2}+\sin\frac{r(\theta)}{2}\sin\frac{\theta}{2}\right)-\left(\cos\frac{r(\theta)}{2}\cos\frac{\theta}{2}-\sin\frac{r(\theta)}{2}\sin\frac{\theta}{2}\right)=0.$$

$$(1+\cos\alpha)\sin\frac{r(\theta)}{2}\sin\frac{\theta}{2}=(1-\cos\alpha)\cos\frac{r(\theta)}{2}\cos\frac{\theta}{2}.$$

Since $0<\alpha<\pi/2$ and $\theta\neq 0,\pi$, we have $1+\cos\alpha\neq 0$, $\cos\frac{\theta}{2}\neq 0, \cos\frac{r(\theta)}{2}\neq 0$. Hence,

$$\tan\frac{\theta}{2}\tan\frac{r(\theta)}{2}=\frac{1-\cos\alpha}{1+\cos\alpha}=\tan^2\frac{\alpha}{2}.$$

(1), (2) It is clear that if $\theta=0$, then $r(\theta)=\pi$, and if $\theta=\pi$, then $r(\theta)=0$. □

4 The radii of tangent circles with rank n

Definition 4.1. For two non-negative integers m and n with $0\leq m\leq 2^n$, we define an integer $[2^n : m]$ by the following rules:

1. $[2^0:0]=0, [2^0:1]=1$,
2. $[2^{n+1}:2m]=[2^n:m] \qquad (0\leq m\leq 2^n)$,
3. $[2^{n+1}:2m+1]=[2^n:m]+[2^n:m+1] \qquad (0\leq m\leq 2^n-1)$.

We call $[2^n : m]$ *Stern's diatomic integer* (SDI, for short) for *depth n* and *order m*. SDIs are expressed in Fig. 4-1. We call this table of SDIs *Stern's diatomic table* (SDT, for short).

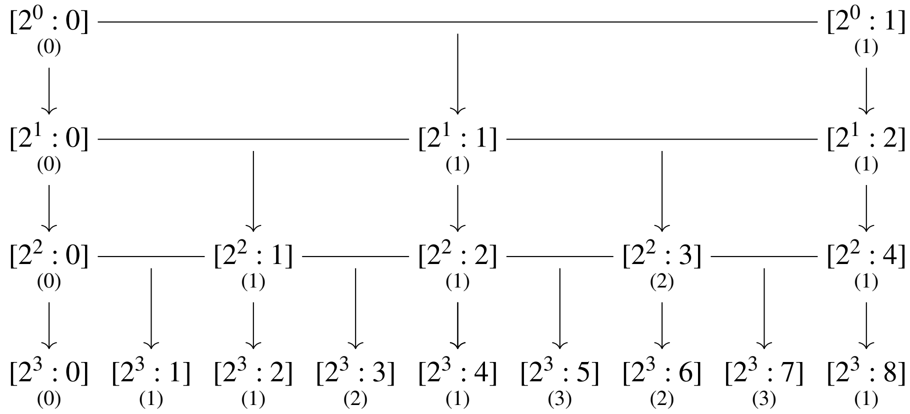

Fig. 4-1

From the definition, we can immediately obtain the following relations:

(1) $[2^{n+1} : m] = [2^n : m]$ $(0 \le m \le 2^n)$,

(2) $[2^n : 2m] = [2^n : m]$ $(0 \le m \le 2^{n-1})$,

(3) $[2^n : 2^{n-1} + m] = [2^n : 2^n - m]$ $(0 \le m \le 2^{n-1})$.

Lemma 4.2. *Let m and n be two integers with $n \ge 1, 1 \le m \le 2^n - 1$. If $m = 2^p(2k+1)$ $(p, k \in \mathbb{Z}_{\ge 0})$, then*

$$[2^n : m-1] + [2^n : m+1] = (2p+1)[2^n : m].$$

Proof. We prove this identity by induction on p. If $p = 0$, then $m = 2k+1$, and

$$\begin{aligned}[2^n : m-1] + [2^n : m+1] &= [2^n : 2k] + [2^n : 2k+2] = [2^{n-1} : k] + [2^{n-1} : k+1] \\ &= [2^n : 2k+1] = [2^n : m].\end{aligned}$$

For some integer $p \ge 0$, suppose $[2^n : m-1] + [2^n : m+1] = (2p+1)[2^n : m]$, where m is an arbitrary integer such that $m = 2^p(2k+1)$. For any integer $m = 2^{p+1}(2k+1)$,

$$\begin{aligned}[2^n : m-1] + [2^n : m+1] &= [2^n : 2^{p+1}(2k+1) - 1] + [2^n : 2^{p+1}(2k+1) + 1] \\ &= [2^{n-1} : 2^p(2k+1) - 1] + 2[2^{n-1} : 2^p(2k+1)] + [2^{n-1} : 2^p(2k+1) + 1] \\ &= (2p+1)[2^{n-1} : 2^p(2k+1)] + 2[2^{n-1} : 2^p(2k+1)] \\ &= (2p+1)[2^n : 2^{p+1}(2k+1)] + 2[2^n : 2^{p+1}(2k+1)] = (2p+3)[2^n : m],\end{aligned}$$

which completes the proof. □

Definition 4.3. By using SDIs, we define the following integers. For $m = 1, 2, \cdots, 2^{n-1}$,

$$(n : m)_1 = [2^n : 2^{n-1} + (m-1)][2^n : 2^{n-1} + m], \tag{4:1}$$

$$(n : m)_2 = [2^n : m-1][2^n : m], \tag{4:2}$$

$$(n : m)_3 = [2^n : 2^{n-1} - (m-1)][2^n : 2^{n-1} - m]. \tag{4:3}$$

Theorem 4.4. *Suppose $IF = \{C_1, C_2, C_3, F_1, F_2, F_3\}$ is the initial figure, where*

$$F_1(z) = \frac{z - p_1}{\overline{p_1}z - 1}, \qquad F_2(z) = \frac{z - p_2}{\overline{p_2}z - 1}, \qquad F_3(z) = \frac{z - p_3}{\overline{p_3}z - 1}.$$

Let a_1, a_2, a_3 be contact points on the unit circle C between C_3 and C_1, C_1 and C_2, C_2 and C_3 respectively, and $C_{(n:m)} \in TC_n^1$ be the m^{th} $(1 \le m \le 2^{n-1})$ tangent circle counterclockwise from a_1 with rank n in C_1. Then,

$$\frac{1}{C_{(n:m)}} = \frac{(n:m)_1}{C_1} + \frac{(n:m)_2}{C_2} + \frac{(n:m)_3}{C_3}. \tag{4:4}$$

We obtain the radii of tangent circles with rank n in C_2, C_3 by replacing C_1, C_2, C_3 of (4:4) with C_2, C_3, C_1 or C_3, C_1, C_2. To be precise, suppose a_2 is the contact point between C_1 and C_2, and a_3 is the contact point between C_2 and C_3. Let $H_{(n:m)} \in TC_n^2$ be the m^{th} $(1 \le m \le 2^{n-1})$ tangent circle counterclockwise from a_2 with rank n in C_2. Then,

$$\frac{1}{H_{(n:m)}} = \frac{(n:m)_1}{C_2} + \frac{(n:m)_2}{C_3} + \frac{(n:m)_3}{C_1}. \tag{4:5}$$

Let $K_{(n:m)} \in TC_n^3$ be the m^{th} $(1 \le m \le 2^{n-1})$ tangent circle counterclockwise from a_3 with rank n in C_3. Then,

$$\frac{1}{K_{(n:m)}} = \frac{(n:m)_1}{C_3} + \frac{(n:m)_2}{C_1} + \frac{(n:m)_3}{C_2}. \tag{4:6}$$

Proof. We prove this proposition by induction on rank n. If $n = 1$, then $m = 1$ and $C_{(1:1)} = C_1$. Meanwhile, by (4:1), (4:2) and (4:3), we have

$$(1:1)_1 = [2:1][2:2] = 1, \ (1:1)_2 = [2:0][2:1] = 0, \ (1:1)_3 = [2:1][2:0] = 0.$$

Hence we have

$$\frac{1}{C_{(1:1)}} = \frac{1}{C_1} = \frac{(1:1)_1}{C_1} + \frac{(1:1)_2}{C_2} + \frac{(1:1)_3}{C_3}.$$

Similarly, we can confirm that C_2, C_3 are expressed by (4:5), (4:6).

If $n = 2$, then $m = 1$ or 2, and $C_{(2:1)} = C_{31}, C_{(2:2)} = C_{21}$. First we show the radius of C_{21}.

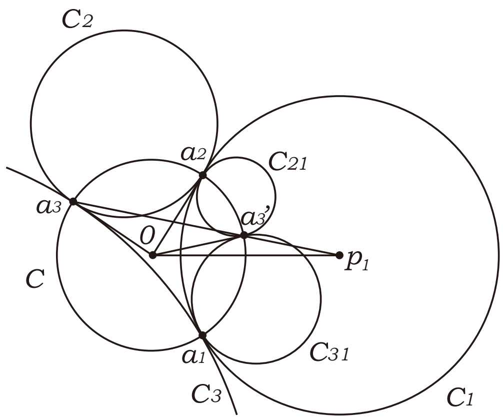

Fig 4-2

In Figure 4-2, we put $\angle p_1 0 a_3 = \theta$, $\angle p_1 0 a_3' = r(\theta)$ and $\angle p_1 0 a_2 = \alpha$. Then,

$$C_{21} = \tan\frac{\alpha - r(\theta)}{2} = \frac{\tan\frac{\alpha}{2} - \tan\frac{r(\theta)}{2}}{1 + \tan\frac{\alpha}{2}\tan\frac{r(\theta)}{2}} = \frac{\tan\frac{\theta}{2}\tan\frac{\alpha}{2} - \tan\frac{\theta}{2}\tan\frac{r(\theta)}{2}}{\tan\frac{\theta}{2} + \tan\frac{\alpha}{2}\tan\frac{\theta}{2}\tan\frac{r(\theta)}{2}}.$$

By Theorem 3.4, we have $C_{21} = \dfrac{\tan\frac{\theta}{2}\tan\frac{\alpha}{2} - \tan^2\frac{\alpha}{2}}{\tan\frac{\theta}{2} + \tan^3\frac{\alpha}{2}} \cdots\cdots\cdots \langle 1 \rangle$.

Note $C_1 = \tan\alpha$, $C_2 = \tan\dfrac{\theta-\alpha}{2} = \dfrac{\tan\frac{\theta}{2} - \tan\frac{\alpha}{2}}{1 + \tan\frac{\theta}{2}\tan\frac{\alpha}{2}}$, and substitute $\tan\dfrac{\theta}{2} = \dfrac{C_2 + \tan\frac{\alpha}{2}}{1 - C_2\tan\frac{\alpha}{2}}$ into $\langle 1 \rangle$. Then,

$$C_{21} = \frac{C_2\tan\frac{\alpha}{2} + C_2\tan^3\frac{\alpha}{2}}{C_2 + \tan\frac{\alpha}{2} + \tan^3\frac{\alpha}{2} - C_2\tan^4\frac{\alpha}{2}} = \frac{C_2\tan\frac{\alpha}{2}}{C_2\left(1 - \tan^2\frac{\alpha}{2}\right) + \tan\frac{\alpha}{2}}$$

$$= \frac{2C_2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2C_2\left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\right) + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{C_2\tan\alpha}{2C_2 + \tan\alpha} = \frac{C_1C_2}{2C_2 + C_1}.$$

Similarly, we have the radius of C_{31} as follows:

$$C_{31} = \tan\frac{\alpha + r(\theta)}{2} = \frac{\tan\frac{\alpha}{2} + \tan\frac{r(\theta)}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{r(\theta)}{2}} = \frac{\tan\frac{\theta}{2}\tan\frac{\alpha}{2} + \tan\frac{\theta}{2}\tan\frac{r(\theta)}{2}}{\tan\frac{\theta}{2} - \tan\frac{\alpha}{2}\tan\frac{\theta}{2}\tan\frac{r(\theta)}{2}}.$$

By Theorem 3.4, we have $C_{31} = \dfrac{\tan\frac{\theta}{2}\tan\frac{\alpha}{2} + \tan^2\frac{\alpha}{2}}{\tan\frac{\theta}{2} - \tan^3\frac{\alpha}{2}} \cdots\cdots \langle 2 \rangle$.

Note $C_3 = \tan\dfrac{(2\pi-\alpha)-\theta}{2} = \tan\left(\pi - \dfrac{\alpha+\theta}{2}\right) = -\tan\dfrac{\alpha+\theta}{2} = -\dfrac{\tan\frac{\alpha}{2} + \tan\frac{\theta}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{\theta}{2}}$, and substitute $\tan\dfrac{\theta}{2} = \dfrac{C_3 + \tan\frac{\alpha}{2}}{C_3\tan\frac{\alpha}{2} - 1}$ into $\langle 2 \rangle$. Then we have

$$C_{31} = \frac{C_3\tan\frac{\alpha}{2} + C_3\tan^3\frac{\alpha}{2}}{C_3 + \tan\frac{\alpha}{2} + \tan^3\frac{\alpha}{2} - C_3\tan^4\frac{\alpha}{2}} = \frac{C_3\tan\frac{\alpha}{2}}{C_3\left(1-\tan^2\frac{\alpha}{2}\right) + \tan\frac{\alpha}{2}}$$

$$= \frac{2C_3\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2C_3\left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\right) + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{C_3\sin\alpha}{2C_3\cos\alpha + \sin\alpha} = \frac{C_1C_3}{2C_3 + C_1}.$$

From the above mentioned, we have

$$\frac{1}{C_{31}} = \frac{2}{C_1} + \frac{1}{C_3}, \qquad \frac{1}{C_{21}} = \frac{2}{C_1} + \frac{1}{C_2}. \tag{4:7}$$

Meanwhile, if $n = 2$ and $m = 1$,

$$(2:1)_1 = [4:2][4:3] = 2,\ (2:1)_2 = [4:0][4:1] = 0,\ (2:1)_3 = [4:2][4:1] = 1.$$

If $n = 2$ and $m = 2$,

$$(2:2)_1 = [4:3][4:4] = 2,\ (2:2)_2 = [4:1][4:2] = 1,\ (2:2)_3 = [4:1][4:0] = 0.$$

Hence we have

$$\frac{1}{C_{(2:1)}} = \frac{1}{C_{31}} = \frac{(2:1)_1}{C_1} + \frac{(2:1)_2}{C_2} + \frac{(2:1)_3}{C_3}, \quad \frac{1}{C_{(2:2)}} = \frac{1}{C_{21}} = \frac{(2:2)_1}{C_1} + \frac{(2:2)_2}{C_2} + \frac{(2:2)_3}{C_3}.$$

Therefore all the radii of tangent circles with rank 2 in C_1 are given by (4:4), which immediately concludes that all the radii of tangent circles with rank 2 in C_2, C_3 are given by (4:5), (4:6).

Suppose all the radii of tangent circles with rank $n, n+1$ in C_1 are given by (4:4). To be precise, for any order $m (1 \le m \le 2^{n-1})$, the radius of the m^{th} tangent circle from a_1 with rank n in C_1 is given by

$$\frac{1}{C_{(n:m)}} = \frac{(n:m)_1}{C_1} + \frac{(n:m)_2}{C_2} + \frac{(n:m)_3}{C_3}, \tag{4:8}$$

and for any order $m (1 \le m \le 2^n)$, the radius of the m^{th} tangent circle from a_1 with rank $n+1$ in C_1 is given by

$$\frac{1}{C_{(n+1:m)}} = \frac{(n+1:m)_1}{C_1} + \frac{(n+1:m)_2}{C_2} + \frac{(n+1:m)_3}{C_3}. \tag{4:9}$$

Then, we can conclude that for any order $m (1 \le m \le 2^{n-1})$, the radius of the m^{th} tangent circle from a_2 with rank n in C_2 is given by

$$\frac{1}{H_{(n:m)}} = \frac{(n:m)_1}{C_2} + \frac{(n:m)_2}{C_3} + \frac{(n:m)_3}{C_1}, \tag{4:10}$$

and the radius of the m^{th} tangent circle from a_3 with rank n in C_3 is given by

$$\frac{1}{K_{(n:m)}} = \frac{(n:m)_1}{C_3} + \frac{(n:m)_2}{C_1} + \frac{(n:m)_3}{C_2}. \tag{4:11}$$

Furthermore, we can conclude that for any order $m\,(1 \le m \le 2^n)$, the radius of the m^{th} tangent circle from a_2 with rank $n+1$ in C_2 is given by

$$\frac{1}{H_{(n+1:m)}} = \frac{(n+1:m)_1}{C_2} + \frac{(n+1:m)_2}{C_3} + \frac{(n+1:m)_3}{C_1}, \tag{4:12}$$

and the radius of the m^{th} tangent circle from a_3 with rank $n+1$ in C_3 is given by

$$\frac{1}{K_{(n+1:m)}} = \frac{(n+1:m)_1}{C_3} + \frac{(n+1:m)_2}{C_1} + \frac{(n+1:m)_3}{C_2}. \tag{4:13}$$

Under the assumptions (4:8), (4:9), (4:10), (4:11), (4:12), (4:13), we will prove that for any order $m\,(1 \le m \le 2^{n+1})$, the radius of the m^{th} tangent circle from a_1 with rank $n+2$ in C_1 is given by

$$\frac{1}{C_{(n+2:m)}} = \frac{(n+2:m)_1}{C_1} + \frac{(n+2:m)_2}{C_2} + \frac{(n+2:m)_3}{C_3}.$$

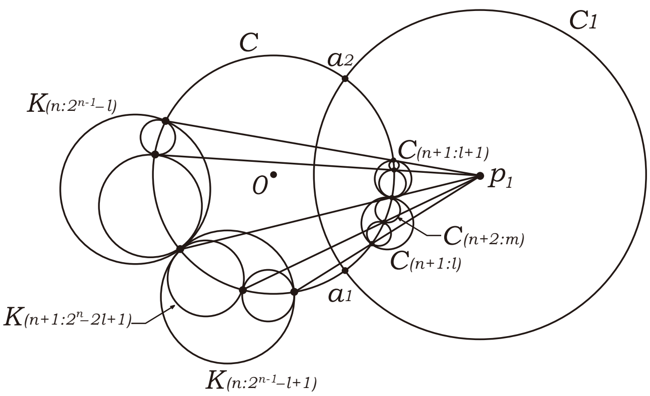

Fig 4-3

(1) Let m be an even integer such that $2 \le m \le 2^n - 2$, and we put $m = 2l\,(l = 1, 2, \cdots, 2^{n-1} - 1)$. In Figure 4-3, $C_{(n+2:m)}$ inscribes $C_{(n+1:l)}$ and circumscribes $C_{(n+1:l+1)}$, where $C_{(n+1:l)}$ is the l^{th} tangent circle from a_1 with rank $n+1$ in C_1 and $C_{(n+1:l+1)}$ is the $(l+1)^{th}$ tangent circle from a_1 with rank $n+1$ in C_1. Then, the inverse image of $C_{(n+1:l)}$ under F_1 is $K_{(n:2^{n-1}-l+1)}$, which is the $(2^{n-1} - l + 1)^{th}$ tangent circle from a_3 with rank n in C_3, the inverse image of $C_{(n+1:l+1)}$ under F_1 is $K_{(n:2^{n-1}-l)}$, which is the $(2^{n-1} - l)^{th}$ tangent circle from a_3 with rank n in C_3, and the inverse image of $C_{(n+2:m)}$ under F_1 is $K_{(n+1:2^n-2l+1)}$, which is the $(2^n - 2l + 1)^{th}$ tangent circle from a_3 with rank $n+1$ in C_3.

From the assumption of induction,

$$\frac{1}{C_{(n+1:l)}}=\frac{(n+1:l)_1}{C_1}+\frac{(n+1:l)_2}{C_2}+\frac{(n+1:l)_3}{C_3}, \tag{4:14}$$

$$\frac{1}{C_{(n+1:l+1)}}=\frac{(n+1:l+1)_1}{C_1}+\frac{(n+1:l+1)_2}{C_2}+\frac{(n+1:l+1)_3}{C_3}, \tag{4:15}$$

$$\frac{1}{K_{(n:2^{n-1}-l+1)}}=\frac{(n:2^{n-1}-l+1)_1}{C_3}+\frac{(n:2^{n-1}-l+1)_2}{C_1}+\frac{(n:2^{n-1}-l+1)_3}{C_2}, \tag{4:16}$$

$$\frac{1}{K_{(n:2^{n-1}-l)}}=\frac{(n:2^{n-1}-l)_1}{C_3}+\frac{(n:2^{n-1}-l)_2}{C_1}+\frac{(n:2^{n-1}-l)_3}{C_2}, \tag{4:17}$$

$$\frac{1}{K_{(n+1:2^n-2l+1)}}=\frac{(n+1:2^n-2l+1)_1}{C_3}+\frac{(n+1:2^n-2l+1)_2}{C_1}+\frac{(n+1:2^n-2l+1)_3}{C_2}.$$

Hence, by Corollary 3.3, we can calculate $C_{(n+2:m)}$ as follows:

$$\begin{aligned}
&C_{(n+2:m)}\\
&=\frac{K_{(n+1:2^n-2l+1)}\left\{K_{(n:2^{n-1}-l+1)}+K_{(n:2^{n-1}-l)}\right\}}{\left\{\frac{K_{(n:2^{n-1}-l+1)}}{C_{(n+1:l)}}\right\}\left\{K_{(n:2^{n-1}-l)}+K_{(n+1:2^n-2l+1)}\right\}+\left\{\frac{K_{(n:2^{n-1}-l)}}{C_{(n+1:l+1)}}\right\}\left\{K_{(n:2^{n-1}-l+1)}-K_{(n+1:2^n-2l+1)}\right\}}\\
&=\frac{\left\{\frac{1}{K_{(n:2^{n-1}-l+1)}}+\frac{1}{K_{(n:2^{n-1}-l)}}\right\}}{\frac{1}{C_{(n+1:l)}}\left\{\frac{1}{K_{(n+1:2^n-2l+1)}}+\frac{1}{K_{(n:2^{n-1}-l)}}\right\}+\frac{1}{C_{(n+1:l+1)}}\left\{\frac{1}{K_{(n+1:2^n-2l+1)}}-\frac{1}{K_{(n:2^{n-1}-l+1)}}\right\}}.
\end{aligned} \tag{4:18}$$

By (4:16) and (4:17),

$$\begin{aligned}
\frac{1}{K_{(n:2^{n-1}-l+1)}}+\frac{1}{K_{(n:2^{n-1}-l)}}&=\frac{(n:2^{n-1}-l+1)_1+(n:2^{n-1}-l)_1}{C_3}\\
&\quad+\frac{(n:2^{n-1}-l+1)_2+(n:2^{n-1}-l)_2}{C_1}+\frac{(n:2^{n-1}-l+1)_3+(n:2^{n-1}-l)_3}{C_2}.
\end{aligned}$$

In this identity, we set $l=2^p(2l'+1)$ $(p,l'\in\mathbb{Z}_{\geq 0})$. Then by Lemma 4.2,

$$\begin{aligned}
&(n:2^{n-1}-l+1)_1+(n:2^{n-1}-l)_1\\
&=[2^n:2^n-l][2^n:2^n-l+1]+[2^n:2^n-l-1][2^n:2^n-l]\\
&=[2^n:2^n-l]([2^n:2^n-l-1]+[2^n:2^n-l+1])\\
&=[2^n:2^n-l](2p+1)[2^n:2^n-l]=(2p+1)[2^n:2^n-l]^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(n:2^{n-1}-l+1)_2+(n:2^{n-1}-l)_2&=(2p+1)[2^n:2^{n-1}-l]^2,\\
(n:2^{n-1}-l+1)_3+(n:2^{n-1}-l)_3&=(2p+1)[2^n:l]^2.
\end{aligned}$$

Hence we have

$$\frac{1}{K_{(n:2^{n-1}-l+1)}}+\frac{1}{K_{(n:2^{n-1}-l)}}=(2p+1)\left(\frac{[2^n:2^n-l]^2}{C_3}+\frac{[2^n:2^{n-1}-l]^2}{C_1}+\frac{[2^n:l]^2}{C_2}\right). \tag{4:19}$$

Meanwhile,

$$\frac{1}{K_{(n+1:2^n-2l+1)}}-\frac{1}{K_{(n:2^{n-1}-l+1)}}=\frac{(n+1:2^n-2l+1)_1-(n:2^{n-1}-l+1)_1}{C_3}$$
$$+\frac{(n+1:2^n-2l+1)_2-(n:2^{n-1}-l+1)_2}{C_1}+\frac{(n+1:2^n-2l+1)_3-(n:2^{n-1}-l+1)_3}{C_2}.$$

In this identity,

$$\begin{aligned}&(n+1:2^n-2l+1)_1-(n:2^{n-1}-l+1)_1\\&=[2^{n+1}:2^{n+1}-2l][2^{n+1}:2^{n+1}-2l+1]-[2^n:2^n-l][2^n:2^n-l+1]\\&=[2^n:2^n-l]([2^{n+1}:2^{n+1}-2l+1]-[2^n:2^n-l+1])\\&=[2^n:2^n-l]([2^n:2^n-l]+[2^n:2^n-l+1]-[2^n:2^n-l+1])=[2^n:2^n-l]^2.\end{aligned}$$

Similarly, we have

$$\begin{aligned}(n+1:2^n-2l+1)_2-(n:2^{n-1}-l+1)_2&=[2^n:2^{n-1}-l]^2,\\(n+1:2^n-2l+1)_3-(n:2^{n-1}-l+1)_3&=[2^n:l]^2.\end{aligned}$$

Hence we have

$$\frac{1}{K_{(n+1:2^n-2l+1)}}-\frac{1}{K_{(n:2^{n-1}-l+1)}}=\frac{[2^n:2^n-l]^2}{C_3}+\frac{[2^n:2^{n-1}-l]^2}{C_1}+\frac{[2^n:l]^2}{C_2}. \tag{4:20}$$

Furthermore,

$$\frac{1}{K_{(n+1:2^n-2l+1)}}+\frac{1}{K_{(n:2^{n-1}-l)}}=\frac{(n+1:2^n-2l+1)_1+(n:2^{n-1}-l)_1}{C_3}$$
$$+\frac{(n+1:2^n-2l+1)_2+(n:2^{n-1}-l)_2}{C_1}+\frac{(n+1:2^n-2l+1)_3+(n:2^{n-1}-l)_3}{C_2}.$$

In this identity,

$$\begin{aligned}(n+1:2^n-2l+1)_1+(n:2^{n-1}-l)_1&=2(p+1)[2^n:2^n-l]^2,\\(n+1:2^n-2l+1)_2+(n:2^{n-1}-l)_2&=2(p+1)[2^n:2^{n-1}-l]^2,\\(n+1:2^n-2l+1)_3+(n:2^{n-1}-l)_3&=2(p+1)[2^n:l]^2.\end{aligned}$$

Hence we have

$$\frac{1}{K_{(n+1:2^n-2l+1)}}+\frac{1}{K_{(n:2^{n-1}-l)}}=2(p+1)\left(\frac{[2^n:2^n-l]^2}{C_3}+\frac{[2^n:2^{n-1}-l]^2}{C_1}+\frac{[2^n:l]^2}{C_2}\right). \tag{4:21}$$

We substitute (4:19), (4:20) and (4:21) into (4:18), then we have

$$\frac{1}{C_{(n+2:m)}}=\frac{2(p+1)}{2p+1}\frac{1}{C_{(n+1:l)}}+\frac{1}{2p+1}\frac{1}{C_{(n+1:l+1)}}.$$

We substitute (4:14), (4:15) into this identity. Then,

$$\frac{1}{C_{(n+2:m)}} = \frac{\frac{2p+2}{2p+1}(n+1:l)_1 + \frac{1}{2p+1}(n+1:l+1)_1}{C_1}$$
$$+ \frac{\frac{2p+2}{2p+1}(n+1:l)_2 + \frac{1}{2p+1}(n+1:l+1)_2}{C_2} + \frac{\frac{2p+2}{2p+1}(n+1:l)_3 + \frac{1}{2p+1}(n+1:l+1)_3}{C_3}.$$

In this identity,

$$\frac{2p+2}{2p+1}(n+1:l)_1 + \frac{1}{2p+1}(n+1:l+1)_1$$
$$= \frac{[2^{n+1}:2^n+l]}{2p+1}\Big\{[2^{n+1}:2^n+l+1] + (2p+2)[2^{n+1}:2^n+l-1]\Big\}$$
$$= \frac{[2^{n+1}:2^n+l]}{2p+1}\Big\{[2^{n+1}:2^n+l-1] + [2^{n+1}:2^n+l+1] + (2p+1)[2^{n+1}:2^n+l-1]\Big\}\cdots\langle 3\rangle.$$

Then, there exists some integer k_2 such that $2^n + l = 2^p(2k_2+1)$. By Lemma 4.2,

$$\langle 3\rangle = \frac{[2^{n+1}:2^n+l]}{2p+1}\Big\{(2p+1)[2^{n+1}:2^n+l] + (2p+1)[2^{n+1}:2^n+l-1]\Big\}$$
$$= [2^{n+1}:2^n+l]\Big([2^{n+1}:2^n+l-1] + [2^{n+1}:2^n+l]\Big)$$
$$= [2^{n+1}:2^n+l][2^{n+2}:2^{n+1}+2l-1]$$
$$= [2^{n+2}:2^{n+1}+2l][2^{n+2}:2^{n+1}+2l-1]$$
$$= [2^{n+2}:2^{n+1}+m][2^{n+2}:2^{n+1}+(m-1)] = (n+2:m)_1. \tag{4:22}$$

Similarly,

$$\frac{2p+2}{2p+1}(n+1:l)_2 + \frac{1}{2p+1}(n+1:l+1)_2$$
$$= \frac{[2^{n+1}:l]}{2p+1}\Big\{(2p+2)[2^{n+1}:l-1] + [2^{n+1}:l+1]\Big\}$$
$$= \frac{[2^{n+1}:l]}{2p+1}\Big\{(2p+1)[2^{n+1}:l-1] + [2^{n+1}:l-1] + [2^{n+1}:l+1]\Big\}\cdots\cdots\langle 4\rangle,$$

where $l = 2^p(2l'+1)$. By Lemma 4.2,

$$\langle 4\rangle = \frac{[2^{n+1}:l]}{2p+1}\Big\{(2p+1)[2^{n+1}:l-1] + (2p+1)[2^{n+1}:l]\Big\}$$
$$= [2^{n+1}:l]\Big([2^{n+1}:l-1] + [2^{n+1}:l]\Big) = [2^{n+2}:2l][2^{n+2}:2l-1]$$
$$= [2^{n+2}:m-1][2^{n+2}:m] = (n+2:m)_2. \tag{4:23}$$

Furthermore,

$$\begin{aligned}
&\frac{2p+2}{2p+1}(n+1:l)_3+\frac{1}{2p+1}(n+1:l+1)_3\\
&=\frac{[2^{n+1}:2^n-l]}{2p+1}\Big\{[2^{n+1}:2^n-l-1]+(2p+2)[2^{n+1}:2^n-l+1]\Big\}\\
&=\frac{[2^{n+1}:2^n-l]}{2p+1}\Big\{[2^{n+1}:2^n-l-1]+[2^{n+1}:2^n-l+1]+(2p+1)[2^{n+1}:2^n-l+1]\Big\}\cdots\langle 5\rangle.
\end{aligned}$$

Then, there exists some integer k_3 such that $2^n-l=2^p(2k_3+1)$. By Lemma 4.2,

$$\begin{aligned}
\langle 5\rangle &=\frac{[2^{n+1}:2^n-l]}{2p+1}\Big\{(2p+1)[2^{n+1}:2^n-l+1]+(2p+1)[2^{n+1}:2^n-l]\Big\}\\
&=[2^{n+1}:2^n-l]\Big([2^{n+1}:2^n-l]+[2^{n+1}:2^n-l+1]\Big)\\
&=[2^{n+2}:2^{n+1}-2l][2^{n+2}:2^{n+1}-2l+1]\\
&=[2^{n+2}:2^{n+1}-(m-1)][2^{n+2}:2^{n+1}-m]=(n+2:m)_3. \qquad (4{:}24)
\end{aligned}$$

Therefore, by (4:22), (4:23) and (4:24), we have

$$\frac{1}{C_{(n+2:m)}}=\frac{(n+2:m)_1}{C_1}+\frac{(n+2:m)_2}{C_2}+\frac{(n+2:m)_3}{C_3}.$$

(2) Let m be an odd integer such that $3\le m\le 2^n-1$, and we put $m=2l-1\,(l=2,3,\cdots,2^n)$. In Figure 4-4, $C_{(n+2:m)}$ inscribes $C_{(n+1:l)}$ and circumscribes $C_{(n+1:l-1)}$, where $C_{(n+1:l)}$ is the l^{th} tangent circle from a_1 with rank $n+1$ in C_1 and $C_{(n+1:l-1)}$ is the $(l-1)^{th}$ tangent circle from a_1 with rank $n+1$ in C_1. Then, the inverse image of $C_{(n+1:l)}$ under F_1 is $K_{(n:2^{n-1}-l+1)}$, which is the $(2^{n-1}-l+1)^{th}$ tangent circle from a_3 with rank n in C_3, the inverse image of $C_{(n+1:l-1)}$ under F_1 is $K_{(n:2^{n-1}-l+2)}$, which is the $(2^{n-1}-l+2)^{th}$ tangent circle from a_3 with rank n in C_3, and the inverse image of $C_{(n+2:m)}$ under F_1 is $K_{(n+1:2^n-2l+2)}$, which is the $(2^n-2l+2)^{th}$ tangent circle from a_3 with rank $n+1$ in C_3. From the assumption of induction,

$$\frac{1}{C_{(n+1:l)}}=\frac{(n+1:l)_1}{C_1}+\frac{(n+1:l)_2}{C_2}+\frac{(n+1:l)_3}{C_3}, \qquad (4{:}25)$$

$$\frac{1}{C_{(n+1:l-1)}}=\frac{(n+1:l-1)_1}{C_1}+\frac{(n+1:l-1)_2}{C_2}+\frac{(n+1:l-1)_3}{C_3}, \qquad (4{:}26)$$

$$\frac{1}{K_{(n:2^{n-1}-l+1)}}=\frac{(n:2^{n-1}-l+1)_1}{C_3}+\frac{(n:2^{n-1}-l+1)_2}{C_1}+\frac{(n:2^{n-1}-l+1)_3}{C_2}, \qquad (4{:}27)$$

$$\frac{1}{K_{(n:2^{n-1}-l+2)}}=\frac{(n:2^{n-1}-l+2)_1}{C_3}+\frac{(n:2^{n-1}-l+2)_2}{C_1}+\frac{(n:2^{n-1}-l+2)_3}{C_2}, \qquad (4{:}28)$$

$$\frac{1}{K_{(n+1:2^n-2l+2)}}=\frac{(n+1:2^n-2l+2)_1}{C_3}+\frac{(n+1:2^n-2l+2)_2}{C_1}+\frac{(n+1:2^n-2l+2)_3}{C_2}.$$

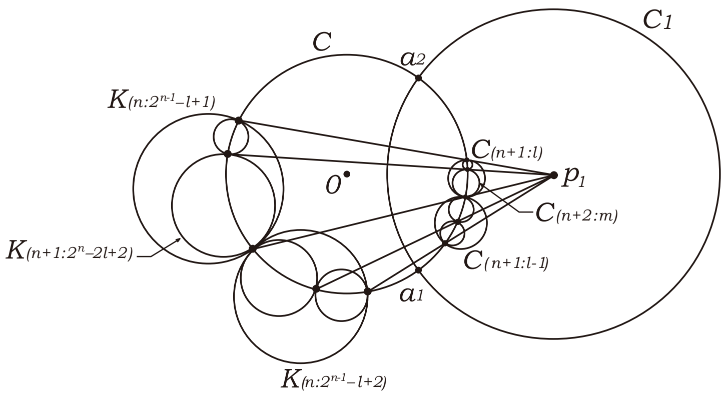

Fig 4-4

Hence, also in this case, by Corollary 3.3, we can calculate $C_{(n+2:m)}$ as follows:

$$
\begin{aligned}
&C_{(n+2:m)}\\
&= \frac{K_{(n+1:2^n-2l+2)}\left\{K_{(n:2^{n-1}-l+1)}+K_{(n:2^{n-1}-l+2)}\right\}}{\left\{\dfrac{K_{(n:2^{n-1}-l+1)}}{C_{(n+1:l)}}\right\}\left\{K_{(n:2^{n-1}-l+2)}+K_{(n+1:2^n-2l+2)}\right\}+\left\{\dfrac{K_{(n:2^{n-1}-l+2)}}{C_{(n+1:l-1)}}\right\}\left\{K_{(n:2^{n-1}-l+1)}-K_{(n+1:2^n-2l+2)}\right\}}\\
&= \frac{\left\{\dfrac{1}{K_{(n:2^{n-1}-l+1)}}+\dfrac{1}{K_{(n:2^{n-1}-l+2)}}\right\}}{\dfrac{1}{C_{(n+1:l)}}\left\{\dfrac{1}{K_{(n+1:2^n-2l+2)}}+\dfrac{1}{K_{(n:2^{n-1}-l+2)}}\right\}+\dfrac{1}{C_{(n+1:l-1)}}\left\{\dfrac{1}{K_{(n+1:2^n-2l+2)}}-\dfrac{1}{K_{(n:2^{n-1}-l+1)}}\right\}}. \qquad (4{:}29)
\end{aligned}
$$

By (4:27) and (4:28),

$$
\begin{aligned}
\frac{1}{K_{(n:2^{n-1}-l+1)}}+\frac{1}{K_{(n:2^{n-1}-l+2)}} &= \frac{(n:2^{n-1}-l+1)_1+(n:2^{n-1}-l+2)_1}{C_3}\\
&+\frac{(n:2^{n-1}-l+1)_2+(n:2^{n-1}-l+2)_2}{C_1}+\frac{(n:2^{n-1}-l+1)_3+(n:2^{n-1}-l+2)_3}{C_2}.
\end{aligned}
$$

In this identity,

$$
\begin{aligned}
&(n:2^{n-1}-l+1)_1+(n:2^{n-1}-l+2)_1\\
&=[2^n:2^n-l][2^n:2^n-l+1]+[2^n:2^n-l+1][2^n:2^n-l+2]\\
&=[2^n:2^n-l+1]\Big([2^n:2^n-l]+[2^n:2^n-l+2]\Big).
\end{aligned}
$$

We put $l-1=2^p(2l'+1)$ $(p,l'\in\mathbb{Z}_{\geq 0})$. Then, there exists some integer k_1 such that $2^n-l+1=2^p(2k_1+1)$. By Lemma 4.2,

$$(n:2^{n-1}-l+1)_1+(n:2^{n-1}-l+2)_1=(2p+1)[2^n:2^n-l+1]^2.$$

Similarly, we have

$$
\begin{aligned}
(n:2^{n-1}-l+1)_2+(n:2^{n-1}-l+2)_2&=(2p+1)[2^n:2^{n-1}-(l-1)]^2,\\
(n:2^{n-1}-l+1)_3+(n:2^{n-1}-l+2)_3&=(2p+1)[2^n:l-1]^2.
\end{aligned}
$$

Hence we have

$$\frac{1}{K_{(n:2^{n-1}-l+1)}}+\frac{1}{K_{(n:2^{n-1}-l+2)}}$$
$$=(2p+1)\left(\frac{[2^n:2^n-l+1]^2}{C_3}+\frac{[2^n:2^{n-1}-(l-1)]^2}{C_1}+\frac{[2^n:l-1]^2}{C_2}\right). \quad (4:30)$$

Meanwhile, in the same way as (4:20), (4:21), we have the following results:

$$\frac{1}{K_{(n+1:2^n-2l+2)}}-\frac{1}{K_{(n:2^{n-1}-l+1)}}=\frac{[2^n:2^n-l+1]^2}{C_3}+\frac{[2^n:2^{n-1}-(l-1)]^2}{C_1}+\frac{[2^n:l-1]^2}{C_2}. \quad (4:31)$$

$$\frac{1}{K_{(n+1:2^n-2l+2)}}+\frac{1}{K_{(n:2^{n-1}-l+2)}}$$
$$=2(p+1)\left(\frac{[2^n:2^n-l+1]^2}{C_3}+\frac{[2^n:2^{n-1}-(l-1)]^2}{C_1}+\frac{[2^n:l-1]^2}{C_2}\right). \quad (4:32)$$

We substitute (4:30), (4:31), (4:32) into (4:29), then we have

$$\frac{1}{C_{(n+2:m)}}=\frac{2(p+1)}{2p+1}\frac{1}{C_{(n+1:l)}}+\frac{1}{2p+1}\frac{1}{C_{(n+1:l-1)}}.$$

We substitute (4:25), (4:26) into this identity. Then we have

$$\frac{1}{C_{(n+2:m)}}=\frac{\frac{2p+2}{2p+1}(n+1:l)_1+\frac{1}{2p+1}(n+1:l-1)_1}{C_1}$$
$$+\frac{\frac{2p+2}{2p+1}(n+1:l)_2+\frac{1}{2p+1}(n+1:l-1)_2}{C_2}+\frac{\frac{2p+2}{2p+1}(n+1:l)_3+\frac{1}{2p+1}(n+1:l-1)_3}{C_3}.$$

In this identity,

$$\frac{2p+2}{2p+1}(n+1:l)_1+\frac{1}{2p+1}(n+1:l-1)_1$$
$$=\frac{[2^{n+1}:2^n+l-1]}{2p+1}\left\{[2^{n+1}:2^n+l-2]+(2p+2)[2^{n+1}:2^n+l]\right\}$$
$$=\frac{[2^{n+1}:2^n+l-1]}{2p+1}\left\{[2^{n+1}:2^n+l]+[2^{n+1}:2^n+l-2]+(2p+1)[2^{n+1}:2^n+l]\right\}\cdots\langle 6\rangle.$$

Then, there exists some integer k_2 such that $2^n+l-1=2^p(2k_2+1)$. By Lemma 4.2,

$$\langle 6\rangle=\frac{[2^{n+1}:2^n+l-1]}{2p+1}\left\{(2p+1)[2^{n+1}:2^n+l-1]+(2p+1)[2^{n+1}:2^n+l]\right\}$$
$$=[2^{n+1}:2^n+l-1]\left([2^{n+1}:2^n+l-1]+[2^{n+1}:2^n+l]\right)$$
$$=[2^{n+2}:2^{n+1}+2l-2][2^{n+2}:2^{n+1}+2l-1]$$
$$=[2^{n+2}:2^{n+1}+(m-1)][2^{n+2}:2^{n+1}+m]=(n+2:m)_1. \quad (4:33)$$

Similarly,

$$\frac{2p+2}{2p+1}(n+1:l)_2+\frac{1}{2p+1}(n+1:l-1)_2=(n+2:m)_2. \tag{4:34}$$

Furthermore,

$$\frac{2p+2}{2p+1}(n+1:l)_3+\frac{1}{2p+1}(n+1:l-1)_3=(n+2:m)_3. \tag{4:35}$$

Hence, also in this case, by (4:33), (4:34) and (4:35), we have

$$\frac{1}{C_{(n+2:m)}}=\frac{(n+2:m)_1}{C_1}+\frac{(n+2:m)_2}{C_2}+\frac{(n+2:m)_3}{C_3}.$$

(3) Suppose $m=1$. In Figure 4-5, $C_{(n+2:1)}$ inscribes $C_{(n+1:1)}$ and circumscribes C_3, where $C_{(n+1:1)}$ is the 1^{st} tangent circle from a_1 with rank $n+1$ in C_1. Then the inverse image of $C_{(n+1:1)}$, $C_{(n+2:1)}$ under F_1 are $K_{(n:2^{n-1})}$, $K_{(n+1:2^n)}$ respectively, where $K_{(n:2^{n-1})}$ is the $(2^{n-1})^{th}$ tangent circle from a_3 with rank n in C_3, and $K_{(n+1:2^n)}$ is the $(2^n)^{th}$ tangent circle from a_3 with rank $n+1$ in C_3. Note that the inverse image of C_3 under F_1 is C_{31}. By (4:7) and the assumption of induction $(4:8),(4:9),(4:10),(4:11),(4:12),(4:13)$,

$$\frac{1}{C_{(n+1:1)}}=\frac{(n+1:1)_1}{C_1}+\frac{(n+1:1)_2}{C_2}+\frac{(n+1:1)_3}{C_3}=\frac{n+1}{C_1}+\frac{0}{C_2}+\frac{n}{C_3},$$

$$\frac{1}{C_3}=\frac{0}{C_1}+\frac{0}{C_2}+\frac{1}{C_3},$$

$$\frac{1}{K_{(n:2^{n-1})}}=\frac{(n:2^{n-1})_1}{C_3}+\frac{(n:2^{n-1})_2}{C_1}+\frac{(n:2^{n-1})_3}{C_2}=\frac{n}{C_3}+\frac{n-1}{C_1}+\frac{0}{C_2},$$

$$\frac{1}{C_{31}}=\frac{2}{C_1}+\frac{0}{C_2}+\frac{1}{C_3},$$

$$\frac{1}{K_{(n+1:2^n)}}=\frac{(n+1:2^n)_1}{C_3}+\frac{(n+1:2^n)_2}{C_1}+\frac{(n+1:2^n)_3}{C_2}=\frac{n+1}{C_3}+\frac{n}{C_1}+\frac{0}{C_2}.$$

Hence, also in this case, we can calculate $C_{(n+2:1)}$ by Corollary 3.3 as follows:

$$\frac{1}{C_{(n+2:1)}}=\frac{\frac{1}{C_{(n+1:1)}}\Big(\frac{1}{K_{(n+1:2^n)}}+\frac{1}{C_{31}}\Big)+\frac{1}{C_3}\Big(\frac{1}{K_{(n+1:2^n)}}-\frac{1}{K_{(n:2^{n-1})}}\Big)}{\Big(\frac{1}{K_{(n:2^{n-1})}}+\frac{1}{C_{31}}\Big)}. \tag{4:36}$$

In this identity,

$$\frac{1}{K_{(n:2^{n-1})}}+\frac{1}{C_{31}}=\frac{n+1}{C_1}+\frac{n+1}{C_3}=(n+1)\Big(\frac{1}{C_1}+\frac{1}{C_3}\Big),$$

$$\frac{1}{K_{(n+1:2^n)}}+\frac{1}{C_{31}}=\frac{n+2}{C_1}+\frac{n+2}{C_3}=(n+2)\Big(\frac{1}{C_1}+\frac{1}{C_3}\Big),$$

$$\frac{1}{K_{(n+1:2^n)}}-\frac{1}{K_{(n:2^{n-1})}}=\frac{1}{C_1}+\frac{1}{C_3}.$$

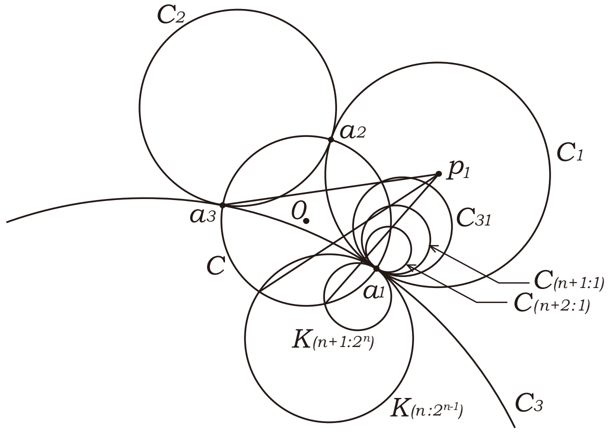

Fig 4-5

We substitute these identities into (4:36). Then we have

$$\frac{1}{C_{(n+2:1)}}=\frac{\frac{1}{C_{(n+1:1)}}(n+2)\Big(\frac{1}{C_1}+\frac{1}{C_3}\Big)+\frac{1}{C_3}\Big(\frac{1}{C_1}+\frac{1}{C_3}\Big)}{(n+1)\Big(\frac{1}{C_1}+\frac{1}{C_3}\Big)}=\frac{n+2}{n+1}\frac{1}{C_{(n+1:1)}}+\frac{1}{n+1}\frac{1}{C_3}$$

$$=\frac{n+2}{n+1}\Big(\frac{n+1}{C_1}+\frac{n}{C_3}\Big)+\frac{1}{n+1}\frac{1}{C_3}=\frac{n+2}{C_1}+\frac{n+1}{C_3}$$

$$=\frac{(n+2:1)_1}{C_1}+\frac{(n+2:1)_2}{C_2}+\frac{(n+2:1)_3}{C_3}.$$

(4) Suppose $m=2^n$. In Figure 4-6, $C_{(n+2:2^n)}$ inscribes C_{31} in C_1 and circumscribes C_{21} in C_1. Then, the inverse image of C_{31} under F_1 is C_3, the inverse image of C_{21} under F_1 is C_2, and the inverse image of $C_{(n+2:2^n)}$ under F_1 is $K_{(n+1:1)}$, which is the 1^{st} tangent circle from a_3 with rank $n+1$ in C_3. By (4:7) and the assumption of induction,

$$\frac{1}{C_{31}}=\frac{2}{C_1}+\frac{0}{C_2}+\frac{1}{C_3},\qquad \frac{1}{C_3}=\frac{0}{C_1}+\frac{0}{C_2}+\frac{1}{C_3},$$

$$\frac{1}{C_{21}}=\frac{2}{C_1}+\frac{1}{C_2}+\frac{0}{C_3},\qquad \frac{1}{C_2}=\frac{0}{C_1}+\frac{1}{C_2}+\frac{0}{C_3},$$

$$\frac{1}{K_{(n+1:1)}}=\frac{(n+1:1)_1}{C_3}+\frac{(n+1:1)_2}{C_1}+\frac{(n+1:1)_3}{C_2}=\frac{n+1}{C_3}+\frac{n}{C_2}.$$

Hence, we can calculate $C_{(n+2:2^n)}$ by Corollary 3.3 as follows:

$$\frac{1}{C_{(n+2:2^n)}}=\frac{\frac{1}{C_{31}}\Big(\frac{1}{K_{(n+1:1)}}+\frac{1}{C_2}\Big)+\frac{1}{C_{21}}\Big(\frac{1}{K_{(n+1:1)}}-\frac{1}{C_3}\Big)}{\Big(\frac{1}{C_3}+\frac{1}{C_2}\Big)}. \tag{4:37}$$

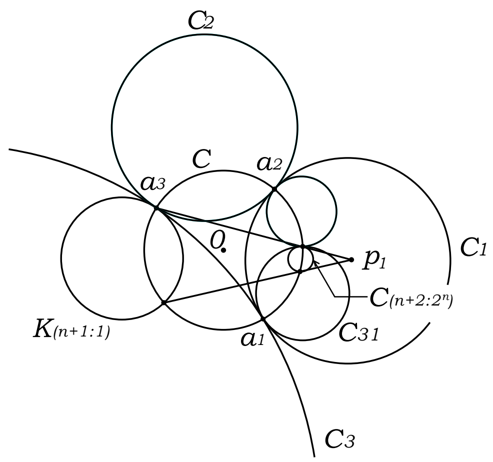

Fig 4-6

We substitute the following identities into (4:37).

$$\frac{1}{K_{(n+1:1)}}+\frac{1}{C_2}=\Big(\frac{n}{C_2}+\frac{n+1}{C_3}\Big)+\frac{1}{C_2}=(n+1)\Big(\frac{1}{C_2}+\frac{1}{C_3}\Big),$$
$$\frac{1}{K_{(n+1:1)}}-\frac{1}{C_3}=\Big(\frac{n}{C_2}+\frac{n+1}{C_3}\Big)-\frac{1}{C_3}=n\Big(\frac{1}{C_2}+\frac{1}{C_3}\Big).$$

Then we have

$$\frac{1}{C_{(n+2:2^n)}}=\frac{\frac{1}{C_{31}}(n+1)\Big(\frac{1}{C_2}+\frac{1}{C_3}\Big)+\frac{1}{C_{21}}n\Big(\frac{1}{C_2}+\frac{1}{C_3}\Big)}{\Big(\frac{1}{C_2}+\frac{1}{C_3}\Big)}=(n+1)\frac{1}{C_{31}}+n\frac{1}{C_{21}}$$

$$=(n+1)\Big(\frac{2}{C_1}+\frac{1}{C_3}\Big)+n\Big(\frac{2}{C_1}+\frac{1}{C_2}\Big)=\frac{2(2n+1)}{C_1}+\frac{n}{C_2}+\frac{n+1}{C_3}.$$

Meanwhile,

$$\begin{aligned}(n+2:2^n)_1 &= [2^{n+2}:2^{n+1}+2^n-1][2^{n+2}:2^{n+1}+2^n]\\ &= [2^{n+2}:2^{n+1}+2^n-1][4:3]=2[2^{n+2}:3\cdot 2^n-1].\end{aligned}$$

Here we have

$$\begin{aligned}[2^{n+2}:3\cdot 2^n-1] &= [2^{n+1}:3\cdot 2^{n-1}-1]+[2^{n+1}:3\cdot 2^{n-1}]\\ &= [2^{n+1}:3\cdot 2^{n-1}-1]+[4:3]=[2^{n+1}:3\cdot 2^{n-1}-1]+2,\end{aligned}$$

which implies that $\Big\{[2^{n+2}:3\cdot 2^n-1]\Big\}_{n=1}^{\infty}$ is an arithmetic sequence with the initial term $[2^3:3\cdot 2-1]=[8:5]=3$ and common difference 2. Hence we have

$$[2^{n+2}:3\cdot 2^n-1]=3+(n-1)\cdot 2=2n+1.$$

Therefore we have

$(n+2:2^n)_1 = 2(2n+1)$,

$(n+2:2^n)_2 = [2^{n+2}:2^n-1][2^{n+2}:2^n] = [2^n:2^n-1] = n$,

$(n+2:2^n)_3 = [2^{n+2}:2^{n+1}-(2^n-1)][2^{n+2}:2^{n+1}-2^n]$

$\quad = [2^{n+2}:2^n+1][2^{n+2}:2^n] = [2^{n+1}:2^n+1] = n+1$.

Hence, also in this case,

$$\frac{1}{C_{(n+2:2^n)}} = \frac{(n+2:2^n)_1}{C_1} + \frac{(n+2:2^n)_2}{C_2} + \frac{(n+2:2^n)_3}{C_3}.$$

From the above mentioned, all the radii of the m^{th} tangent circles ($1 \le m \le 2^n$) from a_1 with rank $n+2$ in C_1,

$$\frac{1}{C_{(n+2:m)}} = \frac{(n+2:m)_1}{C_1} + \frac{(n+2:m)_2}{C_2} + \frac{(n+2:m)_3}{C_3}. \tag{4:38}$$

To complete the proof of Theorem 4.4, we must prove that all the radii of the m^{th} tangent circles ($2^n+1 \le m \le 2^{n+1}$) from a_1 with rank $n+2$ in C_1 are also expressed by (4:38). Although it can be proved in the same way as in the case $1 \le m \le 2^n$, we will prove it by using the result in the case $1 \le m \le 2^n$.

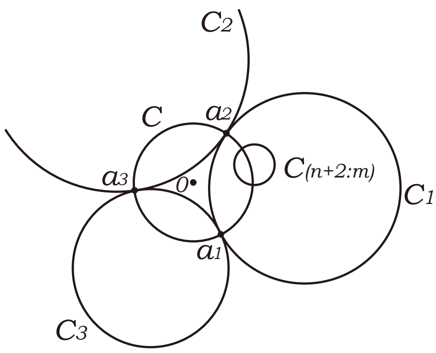

Fig 4-7

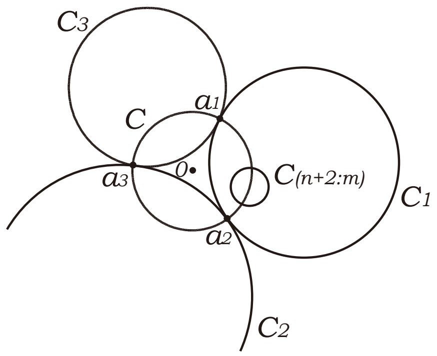

Fig 4-8

In Figure 4-7, let $C_{(n+2:m)}$ be the radius of the m^{th} tangent circle ($2^n+1 \le m \le 2^{n+1}$) from a_1 with rank $n+2$ in C_1. We symmetrically move Figure 4-7 with respect to the real axis to obtain Figure 4-8. Then, $C_{(n+2:m)}$ is the $(2^{n+1}-m+1)^{th}$ tangent circle ($1 \le 2^{n+1}-m+1 \le 2^n$) from a_2 with rank $n+2$ in C_1. Hence, by replacing C_3 with C_2, and C_2 with C_3 in (4:38), we have

$$\frac{1}{C_{(n+2:m)}} = \frac{(n+2:2^{n+1}-m+1)_1}{C_1} + \frac{(n+2:2^{n+1}-m+1)_2}{C_3} + \frac{(n+2:2^{n+1}-m+1)_3}{C_2}.$$

In this identity,

$$\begin{aligned}(n+2:2^{n+1}-m+1)_1 &= [2^{n+2}:2^{n+1}+2^{n+1}-m][2^{n+2}:2^{n+1}+2^{n+1}-m+1]\\ &= [2^{n+2}:2^{n+2}-m][2^{n+2}:2^{n+2}-(m-1)]\\ &= [2^{n+2}:2^{n+1}+m][2^{n+2}:2^{n+1}+(m-1)] = (n+2:m)_1\,,\\ (n+2:2^{n+1}-m+1)_2 &= [2^{n+2}:2^{n+1}-m][2^{n+2}:2^{n+1}-(m-1)] = (n+2:m)_3\,,\\ (n+2:2^{n+1}-m+1)_3 &= [2^{n+2}:2^{n+1}-(2^{n+1}-m)][2^{n+2}:2^{n+1}-(2^{n+1}-m+1)]\\ &= [2^{n+2}:m][2^{n+2}:m-1] = (n+2:m)_2\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{C_{(n+2:m)}} &= \frac{(n+2:2^{n+1}-m+1)_1}{C_1}+\frac{(n+2:2^{n+1}-m+1)_2}{C_3}+\frac{(n+2:2^{n+1}-m+1)_3}{C_2}\\ &= \frac{(n+2:m)_1}{C_1}+\frac{(n+2:m)_2}{C_2}+\frac{(n+2:m)_3}{C_3}.\end{aligned}$$

Now, we have completed the proof of Theorem 4.4. □

Corollary 4.5. *For* $(n:m)_1,(n:m)_2$ *and* $(n:m)_3\,(m=1,2,\cdots,2^{n-1})$ *in Definition* 4.3,

$$\frac{\pi}{3\sqrt{3}} = \lim_{n\to\infty}\sum_{m=1}^{2^{n-1}}\frac{1}{(n:m)_1+(n:m)_2+(n:m)_3}. \qquad (4:39)$$

Proof. Under the setting in Theorem 4.4, let $\triangle a_1a_2a_3$ be a regular triangle. Then since $C_1 = C_2 = C_3 = \sqrt{3}$, we have

$$\frac{2\pi}{3} = \lim_{n\to\infty}\sum_{m=1}^{2^{n-1}}2C_{(n:m)} = \lim_{n\to\infty}\sum_{m=1}^{2^{n-1}}\frac{2\sqrt{3}}{(n:m)_1+(n:m)_2+(n:m)_3}.$$

Hence we have the result. □

One of the referees pointed out that Corollary 4.5 may have relationship with the results in [1, 3]. We would like to study this problem for future research.

Acknowledgments. I have studied SDIs and Markov Conjecture for almost 30 years. Under the guidance of Prof. Akito Nomura, I have tried to complete my papers since 6 years ago. However, regrettably, he passed away in June 2016. Afterwards, under the guidance of Prof. Teruhisa Kadokami and Prof. Hidemitsu Wadade, I have completed the present paper. For their useful advice, I am sincerely grateful to them, and wish to acknowledge some very helpful comments and suggestions from the referees.

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