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An extension of Stieltjes-Young integrals

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Abstract: Let $\phi(t)$ and $f(t)$ be real-valued functions defined on a closed interval $[a,b]$. The Riemann-Stieltjes integral of f with respect to ϕ is usually denoted by $\int_a^b f(t)d\phi(t)$. When $\phi(x)$ is of bounded variation on the interval $[a,b]$, we can treat this integral in the framework of measure theory. Let p and q are positive numbers such that $1/p+1/q>1$. L. C. Young showed that the integral $\int_a^b f(t)d\phi(t)$ in the case that $f(t)$ and $\phi(t)$ have finite mean variation of order p and q, respectively. In this paper we shall try to extend the Stieltjes-Young integration theory when $f(t)$ and $\phi(t)$ are stochastic processes.

1. Introduction

Let $\phi(t)$ and $f(t)$ be real-valued functions defined on a closed interval $[a,b]$. The Riemann-Stieltjes integral of f with respect to ϕ (which is usually denoted by $\int_a^b f(t)d\phi(t)$) is defined by the limit of the Riemann-Stieltjes sums of f and ϕ whenever the limit exists. When $\phi(t)$ is of bounded variation on the interval $[a,b]$, there exists a unique measure on $([a,b],\mathcal{B}([a,b]))$ corresponding to ϕ. Therefore the Riemann-Stieltjes integral with respect to a function ϕ of bounded variation can be treated in the integration theory based on the measures. But the measure theory is powerless to discuss the existence of the Riemann-Stieltjes integral with respect to ϕ which is not of bounded variation. Let p and q be positive numbers. L. C. Young [2] showed that if $\phi(t)$ has the finite mean variation of order p (cf. [1]), $f(t)$ has finite mean variation of order q and $1/p+1/q>1$, the Riemann-Stieltjes integral of f with respect to ϕ exists. We shall try an extension of the results about the Riemann-Stieltjes integral shown by L. C. Young [2] in this paper.

2. Result

Let $(\Omega, \mathcal{F}, P)$ be a measure space and $X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n$ be $\mathcal{F}$-measurable real-valued functions defined on the measure space $(\Omega, \mathcal{F}, P)$. Put $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_n)$. The operation of replacing by " + " certain of the " , " separating consecutive terms of a finite sequence $a = (a_1, a_2, \cdots, a_n)$ may be termed a partition Q. Let p and q be positive numbers such that $1/p + 1/q = 1$ and $\alpha > 0$. Define

$$S_{p,q,\alpha}(\mathbf{X}, \mathbf{Y}) = \max_{Q} \left(\sum_{k=1}^{m} E[|X'_k|^p] \right)^{1/p} \left(\sum_{k=1}^{m} E[|Y'_k|^q]^{\frac{1}{(1+\alpha)}} \right)^{(1+\alpha)/q},$$

where $Q\mathbf{X} = (X'_1, X'_2, \cdots, X'_m)$ and $Q\mathbf{Y} = (Y'_1, Y'_2, \cdots, Y'_m)$. Then we get the following lemma.

Lemma.

$$E[| \sum_{1 \le k \le l \le n} X_k X_l |] \le \left\{ 1 + \zeta \left(\frac{1}{p} + \frac{1+\alpha}{q} \right) \right\} S_{p,q,\alpha}(\mathbf{X}, \mathbf{Y}),$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

Let T be a positive number. For a $\mathcal{B}([0,T]) \times \mathcal{F}$-measurable real-valued function $X(t,\omega)$ ($= X(t)$ in abbreviation) defined on the product space $[0,T] \times \Omega$ and positive numbers p and α, we introduce the following two variations of $X(t)$:

$$V_p(X) = \sup_{\Delta} \left(\sum_{i=1}^{n} E[|X_{t_i} - X_{t_{i-1}}|^p] \right)^{1/p}$$

$$V_{p,\alpha}(X) = \sup_{\Delta} \left(\sum_{i=1}^{n} E[|X_{t_i} - X_{t_{i-1}}|^p]^{1+\alpha} \right)^{(1+\alpha)/p},$$

where $\Delta = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ is a partition of the interval $[0,T]$. We call $V_p(X)$ and $V_{p,\alpha}(X)$ the integrated mean variation on $[0,T]$ of $X(t)$ of order p and the integrated mean variation on $[0,T]$ of $X(t)$ of order (p,α), respectively.

Let $X(t)$ and $Y(t)$ be $\mathcal{B}([0,T]) \times \mathcal{F}$ - measurable real-valued functions defined on the product space $[0,T] \times \Omega$. We give the Riemann-Stieltjes sum of (X,Y) over the partition $\Delta = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ and $u = (u_1, u_2, \cdots, u_n)$ $(t_{i-1} \le u_i \le t_i, 1 \le i \le n)$ by

$$S(X,Y;\Delta,u) = \sum_{i=1}^{n} Y_{u_i}(X_{t_i} - X_{t_{i-1}}).$$

We give the condition (A) for $X(t)$ and $Y(t)$.

$$(A)\begin{cases} V_p(X) < \infty \quad \text{and} \quad V_{q,\alpha}(Y) < \infty. \\ E[|Y_t - Y_s|^q] \quad \text{is continuous in } (s,t). \end{cases}$$

For a partition $\Delta = \{0 = t_0 < t_1 < \cdots < t_n = T\}$, put $|\Delta| = \max |t_i - t_{i-1}|$. Then we get the following theorem.

Theorem. Let p, q and α be positive numbers such that $1/p + 1/q = 1$. Suppose that $X(t)$ and $Y(t)$ satisfy the condition (A). Then there exists an integral $\int_0^T Y(t)dX(t)$ $(= I)$ such that $E[|I|] < \infty$. That is, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$E[|I - S(X,Y;\Delta,u)|] \leq \varepsilon \qquad \text{for} \quad \forall\Delta : |\Delta| \leq \delta,\ \forall u.$$

References

[1] M. Breneau, Variation Totale d'une Function, Springer-Verlag, (Lecture Notes in Mathematics, 413), 1974.

[2] L. C. Young, *An inequality of Hölder type, connected with Stieltjes integration,* Acta. Math., **67** (1936), 251–282.