Girsanov formula in Dirichlet space theory

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Abstract: A Girsanov formula (Girsanov transformation) is a method to construct a new Markov process by changing the probability measure of the given Markov process. Using Fukushima's decomposition formula of additive functionals, we shall give a Girsanov formula which is applied to a Markov process associated with a Dirichlet space.

1. Introduction

Let $B(t)$ be a d-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ with a reference family $(\mathcal{F}_t)$. We consider a d-dimensional stochastic differential equation

$$dX(t) = \alpha(X(t))dB(t) + \beta(X(t))dt \qquad (1.1)$$

where $\alpha(x) = (\alpha_{ij}(x))_{i,j=1,2,\dots,d} : \mathbf{R}^d \to \mathbf{R}^d \otimes \mathbf{R}^d$ and $\beta(x) = (\beta^i(x))_{i=1,2,\dots,d} : \mathbf{R}^d \to \mathbf{R}^d$ are Borel measurable functions. If the uniquness in law of solutions for the equation (1.1) holds, the solution $X(t)$ on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ of the equation (1.1) is a diffusion process on $\mathbf{R}^d$ whose generator A is given by

$$A = \frac{1}{2}\sum_{i,j=1}^{d} (\alpha(x)^t\alpha(x))_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \beta^i(x)\frac{\partial}{\partial x_i}$$

where ${}^t\alpha(x)$ is the transposed matrix of $\alpha(x)$ and $(\alpha(x){}^t\alpha(x))_{i,j}$ $(1 \le i, j \le d)$ is the (i, j)-component of the matrix $\alpha(x){}^t\alpha(x)$.

Give a bounded Borel measurable function $\gamma(x) = (\gamma^i(x))_{i=1,2,\ldots,d}$ and set

$$\left.\begin{array}{l} M(t) = \exp\Big\{\int_0^t \gamma(X(s))dB(s) - \dfrac{1}{2}\int_0^t \parallel \gamma(X(s)) \parallel^2 ds\Big\} \\ d\overline{P} = M \cdot dP \end{array}\right\} \tag{1.2}$$

then we have by the Girsanov formula (the transformation of drift) (cf. R. H. Cameron and W. T. Martin [2], G. Maruyama [7], M. Motto [9] and I. V. Girsanov [5])

$$\overline{B}(t) = B(t) - \int_0^t \gamma(X(s))ds$$

is a d-dimensional Brownian motion on $(\Omega, \mathcal{F}, \overline{P}, (\mathcal{F}_t))$ and the solution $X(t)$ of the equation (1.1) on $(\Omega, \mathcal{F}, \overline{P})$ is a solution of the following stochastic differential equation

$$dX(t) = \alpha(X(t))d\overline{B}(t) + \Big[\beta\left(X(t)\right) + \alpha(X(t))\gamma(X(t))\Big]dt.$$

Therefore, if the uniqueness in law of solutions for the equation (1.1) holds, $X(t)$ on $(\Omega, \mathcal{F}, \overline{P}, (\mathcal{F}_t))$ is a diffusion process on $\mathbf{R}^d$ whose generator $\overline{A}$ is given by

$$\overline{A} = A + \sum_{i=1}^d (\alpha(x)\gamma(x))^i \frac{\partial}{\partial x^i}$$

where $(\alpha(x)\gamma(x))^i$ $(1 \le i \le d)$ is the i-th component of $\alpha(x)\gamma(x)$.

In this paper, we shall discuss a Girsanov formula for the diffusion process on $\mathbf{R}^d$ whose generator L is a second order differential operator of divergence form. Let L be

$$L = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big) + \sum_{i=1}^d b^i(x)\frac{\partial}{\partial x^i} \tag{1.3}$$

where $a(x) = (a_{ij}(x))_{i,j=1,2,\ldots,d} : \mathbf{R}^d \to \mathbf{R}^d \otimes \mathbf{R}^d$ is bounded Borel measurable and uniformly elliptic and $b(x) = (b^i(x))_{i=1,2,\ldots,d} : \mathbf{R}^d \to \mathbf{R}^d$ is bounded Borel measurable. The Markov transition function corresponding to the generator L exists uniquely (D. G. Aronson [1]). In this case, the drift coefficient of L is written formally

$$\sum_{i=1}^{d} \frac{\partial}{\partial x_i} a_{ij}(x) + b^j(x) \quad (1 \leq j \leq d)$$

but they are not functions in general. Therefore we can not apply the above Girsanov formula to the diffusion process with generator L. But we shall show a new Girsanov formula which can be applied to the last diffusion process (section 3).

In section 2, we shall state a Girsanov formula for a general Markov process associated with a (non-symmetric) Dirichlet space.

2. Girsanov formula in Dirichlet space theory

Let X be a locally compact Hausdorff space with countable base and m a non-negative Radon measure on X such that $\mathrm{supp}[m] = X$. $L^2(X, m)$ denotes the space of all square m-integrable real-valued functions with inner product

$$(u, v)_{L^2} = \int_X u(x)v(x)dm(x) \quad \text{for} \quad u, v \in L^2(X, m)$$

and the norm on $L^2(X, m)$ is introduced by $\|u\|_{L^2} = \sqrt{(u, v)_{L^2}}$ for $u \in L^2(X, m)$. Let H be a dense linear subspace of $L^2(X, m)$ which forms a Hilbert space with norm $\|\cdot\|_H$ such that for some positive constant K, $\|u\|_{L^2} \leq K\|u\|_H$ $(u \in H)$. We assume that the Hilbert space H satisfies $|u| \in H, u \wedge 1 \in H$ for any $u \in H$. Let us consider a bilinear form $\epsilon : H \times H \to \mathbf{R}$ which satisfies the following three conditions;

(ϵ.1) ϵ_α is coercive for any $\alpha > 0$, where $\epsilon_\alpha(u, v) = \epsilon(u, v) + \alpha(u, v)_{L^2}$ $(u, v \in H)$. That is, there exists a positive constant $K_1 = K_1(\alpha)$ such that $\epsilon_\alpha(u, u) \geq K_1\|u\|_H^2$ for any $u \in H$.

(ϵ.2) ϵ is continuous. That is, there exists a constant $K_2 > 0$ such that $|\epsilon(u, v)| \leq K_2\|u\|_H\|v\|_H$ for any $u, v \in H$.

(ϵ.3) $\quad \epsilon(T_0^1 u, u - T_0^1 u) \geq 0,\ \epsilon(u - T_0^1 u, T_0^1 u) \geq 0$ for $u \in H,$ where $T_0^1 u = 0 \vee u \wedge 1.$

The pair (ϵ, H) which satisfies the above – mentioned conditions is called a Dirichlet space on $L^2(X, m)$ in this paper. Moreover we assume that a Dirichlet space (ϵ, H) on $L^2(X, m)$ is C_0-regular, that is, $C_0(X) \cap H$ is dense in H with $\|\cdot\|_H$ and $C_0(X) \cap H$ is dense in $C_0(X)$ with $\|\cdot\|_\infty$. Here $C_0(X)$ is the space of all continuous real-valued functions on X with compact support and is equipped with the supremum norm.

From now on (ϵ, H) is a C_0-regular Dirichlet space on $L^2(X, m)$. S. Carrillo Menendez [8] showed that there exists a Hunt process $\boldsymbol{M} = (P_x, X_t)$ associated with (ϵ, H). M. Fukushima [4] developed the theory of stochastic calculus for additive functionals of a Hunt process associated with a C_0-regular symmetric Dirichlet space and J. H. Kim [6] also showed the same stochastic calculus of Fukushima for additive functionals of $\boldsymbol{M} = (P_x, X_t)$ associated with the (non-symmetric) Dirichlet space (ϵ, H).

Let M_t be a square integrable martingale additive functional of the Hunt process $\boldsymbol{M} = (P_x, X_t)$. There exists a unique positive continuous additive functional $\langle M \rangle_t$ of $\boldsymbol{M} = (P_x, X_t)$ such that for any $t > 0$

$$E^{P_x}[\langle M \rangle_t] = E^{P_x}[M_t^2] \qquad q.e.\ x \in X,$$

where E^{P_x} denotes the expectation with respect to the probability measure P_x and q.e. is a abbreviation of quasi-everywhere. The energy measure of M_t (the Revuz measure of $\langle M \rangle_t$) is denoted by $\mu_{\langle M \rangle}$. For $u \in H$, the additive functional $A_t^{[u]} = \tilde{u}(X_t) - \tilde{u}(X_0)$ ($\tilde{u}$ is a quasi-continuous version of u) is decomposed uniquely

$$A_t^{[u]} = M_t^{[u]} + N_t^{[u]} \tag{2.1}$$

where $M_t^{[u]}$ is a square integrable martingale additive functional of finite energy and $N_t^{[u]}$ is a continuous additive functional of zero energy (cf. M. Fukushima [4] and J. H. Kim [6]).

From now on M_t is a continuous square integrable martingale additive functional such that $d\mu_{\langle M \rangle}(x) = \gamma(x) dm(x)$ and we assume that $\gamma(x)$ is bounded Borel measurable. Fix M_t and define a bilinear form $a(\cdot,\cdot)$ on $H \times H$ by

$$a(u,v) = \epsilon(u,v) + \int_X \tilde{v}(x) d\mu_{\langle M^{[u]}, M\rangle}(x) \qquad \text{for } u, v \in H \tag{2.2}$$

where

$$\mu_{\langle M^{[u]}, M\rangle} = \frac{1}{4}\Big\{\mu_{\langle M^{[u]}+M\rangle} - \mu_{\langle M^{[u]}-M\rangle}\Big\}.$$

Then we have the following theorem.

Theorem 1. *(a, H) is a C_0-regular Dirichlet space on $L^2(X, m)$.*

Put in the same way as (1.2)

$$\left.\begin{aligned} \Gamma_t &= \exp\{-M_t - \frac{1}{2}\langle M\rangle_t\} \\ d\overline{P}_x &= \Gamma \cdot dP_x. \end{aligned}\right\} \tag{2.3}$$

Then Γ_t is a multiplicative functional of the Hunt process $\boldsymbol{M} = (P_x, X_t)$ and $\overline{P}_x$ is the transformed probability measure by the multiplicative functional Γ_t (cf. E. B. Dynkin [3]) and we have the following theorem.

Theorem 2. *$(\overline{P}_x, X_t)$ is a Hunt process associated with the Dirichlet space (a, H).*

3. Girsanov formula for diffusion processes whose generator is of divergence form

Let us consider a second order differential operator L of divergence form given by (1.3) and assume that $a(x)$ is bounded Borel measurable and uniformly elliptic and $b(x)$ is bounded Borel measurable. We consider a diffusion process (P_x, X_t) on $\mathbf{R}^d$ corresponding to L. The Dirichlet space (ϵ, H) associated with (P_x, X_t) is given by

$$\left.\begin{aligned} H &= H^1(\mathbf{R}^d) \\ \epsilon(u,v) &= \sum_{i,j=1}^d \int_{\mathbf{R}^d} a_{ij}(x) \frac{\partial u}{\partial x_j}(x) \frac{\partial v}{\partial x_i}(x) dx \\ &\quad - \sum_{i=1}^d \int_{\mathbf{R}^d} b^i(x) v(x) \frac{\partial u}{\partial x^i}(x) dx \quad (u, v \in H^1(\mathbf{R}^d)) \end{aligned}\right\} \tag{3.1}$$

where $H^1(\mathbf{R}^d)$ is the Sobolev space of order 1. Put $\tilde{a}(x) = (a(x) + {}^t a(x))/2$ and $\alpha(x)$ is the symmetric square root of $2\tilde{a}(x)$. The inverse of $\alpha(x)$ is denoted by $\left(\alpha_{ij}^{-1}\right)_{i,j=1,2\dots,d}$. The decomposition of (2.1) can be extended locally and we get for q.e. $x = (x^1, x^2, \dots, x^d) \in \mathbf{R}^d$

$$X_t^i = x^i + M_t^{[x_i]} + N_t^{[x_i]} \qquad (1 \le i \le d),$$

where x_i is the i-th coordinate function of $\mathbf{R}^d$. Setting

$$B_t^i = \sum_{j=1}^d \int_0^t \alpha_{ij}^{-1}(X_s) dM_s^{[x_j]} \qquad (1 \le i \le d)$$

$(B_t = (B_t^1, B_t^2, \dots, B_t^d), P_x)$ is a d-dimensional Brownian motion. Let $\gamma(x)$ be a bounded Borel function on $\mathbf{R}^d$. Put

$$\left.\begin{aligned} \Gamma_t &= \exp\Big\{\int_0^t \gamma(X_s) dB_s - \frac{1}{2}\int_0^t \|\gamma(X_s)\|^2 ds\Big\} \\ d\overline{P}_x &= \Gamma \cdot dP_x, \end{aligned}\right\} \tag{3.2}$$

then we have the following theorem.

Theorem 3. *$(\overline{P}_x, X_t)$ is a diffusion process on $\mathbf{R}^d$ whose Dirichlet space (a, H) is given by*

$$H = H^1(\mathbf{R}^d)$$

$$a(u,v) = \epsilon(u,v) - \sum_{j=1}^d \int_{\mathbf{R}^d} c_j(x) \frac{\partial u}{\partial x_j}(x) v(x) dx \qquad (u, v \in H^1(\mathbf{R}^d)),$$

where $\epsilon(\cdot,\cdot)$ is a bilinear form of (3.1) *and $c(x) = \alpha(x)\gamma(x)$.*

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