Sci. Rep. Kanazawa Univ.
Vol. 41, No. 2, pp. 69-75
December 1996

An application of transformation of drift to weak convergence of quasimartingale probability measures

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(Recieved December 10, 1996)

Abstract: Central limit theorems are very important in probability theory and statistics. We show that we can obtain by the transformation of drift the movement of the mean vector in a central limit theorem for a sequence of continuous quasimartingale probability measures and apply this result to homogenizations of diffusions.

1. Introduction

The transformation of drift (Girsanov formula) is a Radon-Nikodym derivative theory in Ito stochastic calculus for quasimartingales and is used in many fields in stochastic analysis. The most typical example is to be used in a construction method of solutions of stochastic differential equations (cf. [2]). The aim of this paper is to apply the transformation of drift to a central limit theorem for a sequence of continuous quasimartingale probability measures (for example, a homogenization of diffusion processes) (cf. [4], [6]) and to catch the movement of the mean vector of the limiting Wiener process by using the transformation of drift.

We shall state the main theorem and introduce some notations. Let T be a positive number and set $I = [0, T]$. For $n \in \mathbf{N}$, we denote by $C_0(I, \mathbf{R}^n)$ the space of all continuous functions w

$$w = (w_1, w_2, \cdots, w_n) : I \ni t \mapsto w(t) \in \mathbf{R}^n$$

with $w(0) = 0$ and define the usual metric on $C_0(I, \mathbf{R}^n)$: that is, $\{w_n\}$ converges to w if and only if $\{w_n\}$ converges to w uniformly on I. The topological σ-algebra on $C_0(I,\mathbf{R}^n)$ is denoted by $\mathcal{B}(C_0(I,\mathbf{R}^n))$. Let d be a natural number and fix d in this paper. For $w=(w_1,w_2,\cdots,w_{2d+1})\in C_0(I,\mathbf{R}^{2d+1})$ and $t\in I$, set

$$\begin{aligned} X(t) &= (w_1(t), w_2(t), \cdots, w_d(t)),\\ Y(t) &= (w_{d+1}(t), w_{d+2}(t), \cdots, w_{2d}(t)),\\ Z(t) &= w_{2d+1}(t). \end{aligned}$$

We define the increasing family $\{\mathcal{F}_t\}$ of sub-σ-algebras of $\mathcal{F}=\mathcal{B}(C_0(I,\mathbf{R}^{2d+1}))$ by $\mathcal{F}_t=\sigma(w(s); s\le t)$ $(t\in I)$: that is, $\mathcal{F}_t$ is the smallest σ-algebra such that all $w(s)$ $(s\le t)$ are measurable.

Let $\mathcal{P}$ be the space of all probability measures P on $(C_0(I,\mathbf{R}^{2d+1}),\mathcal{F})$ such that $(X(t),P)$ and $(Z(t),P)$ are square integrable martingales. For $P\in\mathcal{P}$, we can consider the continuous process Γ_t defined by

$$\Gamma_t=\exp\{Z(t)-\frac{1}{2}\langle Z\rangle_P(t)\} \qquad (0\le t\le T), \tag{1.1}$$

where $\langle Z\rangle_P$ is the quadratic variational process of $(Z(t),P)$. We define the subspace of $\mathcal{P}$ by

$$\tilde{\mathcal{P}}=\{P\in\mathcal{P}\ ;\ \Gamma_t \text{ is a martingale under } P\}.$$

For $P\in\tilde{\mathcal{P}}$, we can define the probability measure $\hat{P}$ on $(C_0(I,\mathbf{R}^{2d+1}),\mathcal{F})$ by

$$\hat{P}(F)=E^P[\Gamma_T;F] \qquad \text{for} \quad F\in\mathcal{F},$$

where $E^P[\ \cdot\ ;A]$ is the expectation on A with respect to the probability measure P and Γ_t is the martingale given by (1.1).

For a positive-definite symmetric $d\times d$ matrix C and a d-dimensional vector $\mathbf{b}\in\mathbf{R}^d$, we denote by $P^W(C,\mathbf{b})$ the Wiener measure on $(C_0(I,\mathbf{R}^d),\mathcal{B}(C_0(I,\mathbf{R}^d)))$ whose covariance matrix is C and mean vector is $\mathbf{b}$. For a d-dimensional vector $\mathbf{b}\in\mathbf{R}^d$, the probability measure P on $(C_0(I,\mathbf{R}^d),\mathcal{B}(C_0(I,\mathbf{R}^d)))$ with $P(\{w\in C_0(I,\mathbf{R}^d); w(t)=\mathbf{b}t \quad (0\le t\le T)\})=1$ is denoted by $P^D(\mathbf{b})$. Then we can obtain the folowing theorem.

Theorem. *Let $\{P_n\}_{n=1,2\cdots}$ be a sequence in $\tilde{\mathcal{P}}$ such that $\{(\Gamma_T,P_n)\}$ is uniformly integrable. We assume that $\{(X(t),P_n)\}$ converges weakly to $P^W(C,\mathbf{0})$ and $\{(Y(t),P_n)\}$ converges weakly to $P^D(\mathbf{0})$. Further we assume that the sequence $\{(\langle X,Z\rangle_{P_n}(t),P_n)\}$ of the quadratic variational processes of X and Z converges weakly to $P^D(\mathbf{b})$. Then $\{(X(t)+Y(t),\hat{P}_n)\}$ converges weakly to $P^W(C,\mathbf{b})$.*

We shall prove the above theorem in section 2. In section 3 we shall discuss an application of the above theorem to a homogenization problem for diffusion processes which are expressed as solutions of the stochastic differential equations with periodic coefficients.

2. Proof of Theorem

We shall prepare a lemma to prove Theorem. Let E be a metric space with a metric d and $(\Omega, \mathcal{G})$ be a measurable space. We consider a sequence $\{Q_n\}$ of probability measures on $(\Omega, \mathcal{G})$ and a sequence $\{g_n\}$ of non-negative real valued $\mathcal{G}$-measurable functions defined on Ω with $E^{Q_n}[g_n] = 1$ for $n \in \mathbf{N}$. Define the probability measure $\hat{Q}_n$ $(n \geq 1)$ on $(\Omega, \mathcal{G})$ by

$$\hat{Q}_n(G) = E^{Q_n}[g_n; G] \qquad \text{for} \quad G \in \mathcal{G}.$$

Then we can get the following lemma.

Lemma. *Let $e_0 \in E$ and f be a E-valued $\mathcal{G}$-measurable function of Ω. Assume that $\{(g_n, Q_n)\}$ is uniformly integrable. If $\{(f, Q_n)\}$ converges weakly to the Dirac probability measure δ_{e_0} at e_0, then $\{(f, \hat{Q}_n)\}$ also converges weakly to δ_{e_0}.*

Proof. Let $B_r(e_0)$ be the open ball with the center e_0 and the radius r. Since the sequence $\{(f, Q_n)\}$ converges weakly to δ_{e_0}, we have for $r > 0$ (cf. [1])

$$\varliminf_{n \to \infty} Q_n(f \in B_r(e_0)) \geq \delta_{e_0}(B_r(e_0)) = 1.$$

Consequently there exists $N(r) \in \mathbf{N}$ such that

$$Q_n(f \in B_r(e_0)^c) \leq r \qquad \text{for} \quad n \geq N(r).$$

Let h be a bounded continuous function on E and $\varepsilon > 0$. Since $\{(g_n, Q_n)\}$ is uniformly integrable, there exists $r_0 > 0$ such that

$$E^{Q_n}[g_n; \Lambda] \leq \varepsilon \qquad \text{for} \quad n \geq 1 \text{ and } Q_n(\Lambda) \leq r_0 \tag{2.1}$$

and

$$|h(e) - h(e_0)| \leq \varepsilon \qquad \text{for} \quad d(e, e_0) \leq r_0. \tag{2.2}$$

Then we have for $n \geq N(r_0)$ by the inequalities (2.1) and (2.2)

$$\begin{aligned}
&|E^{\hat{Q}_n}[h \circ f] - E^{\delta_{e_0}}[h]| \\
&= |E^{Q_n}[(h \circ f - h(e_0))g_n]| \\
&\leq E^{Q_n}[|h \circ f - h(e_0)|g_n; f \in B_{r_0}(e_0)] + E^{Q_n}[|h \circ f - h(e_0)|g_n; f \in B_{r_0}(e_0)^c] \\
&\leq \varepsilon + 2\|h\|_\infty \varepsilon,
\end{aligned}$$

where $\|h\|_\infty = \max_{e \in E} |h(e)|$. Thus we get the weak convergence of $\{(f, \hat{Q}_n)\}$ to δ_{e_0}. □

Proof of Theorem. Since $P_n \in \tilde{\mathcal{P}}$ $(n = 1, 2, \cdots)$, we can define the quadratic variational process $\langle X, Z\rangle_{P_n}$ of (X, Z) and have

$$X(t) = (X(t) - \langle X, Z\rangle_{P_n}(t)) + \langle X, Z\rangle_{P_n}(t)$$
$$= \hat{X}_n(t) + \langle X, Z\rangle_{P_n}(t).$$

Then $\hat{X}_n(t)$ is a continuous square integrable martingale under the probability measure $\hat{P}_n$ satisfying that (cf. [2])

$$\langle \hat{X}_n\rangle_{\hat{P}_n} = \langle \hat{X}_n\rangle_{P_n} = \langle X\rangle_{P_n},$$

because P_n and $\hat{P}_n$ are mutually absolutely continuous. We denote by R the probability measure on $C_0(I, \mathbf{R}^{d^2})$ such that

$$R(\{w = (w_{ij})_{i,j=1,2,\cdots,d}\ ;\ w_{ij}(t) = c_{ij}t \qquad (t \in I, i, j = 1, 2, \cdots, d)\}) = 1,$$

where $C = (c_{ij})$.

Since $\{(X(t), P_n)\}$ converges weakly to $P^W(C, 0)$, we get (cf. [3], [5]) that $\{(\langle X\rangle_{P_n}(t), P_n)\}$ converges weakly to R. The above lemma implies $\{(\langle \hat{X}_n\rangle_{\hat{P}_n}(t), \hat{P}_n)\}$ also converges weakly to R. Consequently we have that $\{(\hat{X}_n, \hat{P}_n)\}$ converges weakly to $P^W(C, 0)$. Using the assumption of Theorem and the above lemma, $\{(\langle X, Z\rangle_{P_n}(t) + Y(t), \hat{P}_n)\}$ converges weakly to $P^D(\mathbf{b})$. Noting that

$$(X(t) + Y(t), \hat{P}_n) = (\hat{X}_n(t) + \langle X, Z\rangle_{P_n}(t) + Y(t), \hat{P}_n),$$

we can obtain that $\{(X(t) + Y(t), \hat{P}_n)\}$ converges weakly to $P^W(C, \mathbf{b})$. □

Finally we note a converse assertion of Theorem.

Remark. Let $\{P_n\}_{n=1,2,\cdots}$ be a sequence in $\tilde{\mathcal{P}}$ such that $\{(\Gamma_T^{-1}, \hat{P}_n)\}$ is uniformly integrable. We assume that $\{(X(t) + Y(t), \hat{P}_n)\}$ and $\{(Y(t), \hat{P}_n)\}$ converge weakly to $P^W(C, \mathbf{b})$ and $P^D(0)$ respectively. Then we have that $\{(\langle X, Z\rangle_{P_n}(t), P_n)\}$ converges weakly to $P^D(\mathbf{b})$.

3. An application of Theorem to homogenization of diffusions

We consider a d-dimensional stochastic differential equation with periodic coefficients:

$$dx_t = \alpha(x_t)dB_t + \beta(x_t)dt, \tag{3.1}$$

where B_t is a d-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{B}, P)$ starting at 0 and the coefficients $\alpha(x)$ and $\beta(x)$ satisfy the following conditions:

(i) $\alpha(x) = (\alpha_{ij}(x))_{i,j=1,2,\cdots,d}$ ($x \in \mathbf{R}^d$) is uniformly elliptic and its components $\alpha_{ij}(x)$ ($i,j = 1,2,\cdots,d$) are smooth and periodic (period $= 1$).

(ii) $\beta(x) = (\beta_i(x))_{i=1,2,\cdots,d}$ ($x \in \mathbf{R}^d$) has smooth and periodic (period $= 1$) components.

Let denote by $\mathcal{L}$ the generator of the diffusion process associated with the stochastic differential equation (3.1):

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{d}\beta_i(x)\partial_i,$$

where $a(x) = (a_{ij}(x))_{i,j=1,2,\cdots,d} = \alpha(x)^t\alpha(x)$. Let m be the invariant probability measure of $\mathcal{L}$-diffusion process on the torus T^d. In this section we assume that the centering condition:

$$\int_{T^d}\beta(x)dm(x) = 0.$$

Under these conditions we discuss two homogenization problems. The first is a homogenization of $\mathcal{L}^{(\varepsilon)}$-diffusion processes. For $\varepsilon > 0$ let consider the following stochastic differential equation:

$$dx_t^\varepsilon = \alpha\left(\frac{x_t^\varepsilon}{\varepsilon}\right)dB_t + \frac{1}{\varepsilon}\beta\left(\frac{x_t^\varepsilon}{\varepsilon}\right)dt$$
$$x_0^\varepsilon = 0. \tag{3.2}$$

The solution x_t^ε of the stochastic differential equation (3.2) is a diffusion process on $\mathbf{R}^d$ starting at 0 whose generator is:

$$\mathcal{L}^{(\varepsilon)} = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\left(\frac{x}{\varepsilon}\right)\partial_i\partial_j + \frac{1}{\varepsilon}\sum_{i=1}^{d}\beta_i\left(\frac{x}{\varepsilon}\right)\partial_i.$$

Let $\chi = (\chi_1, \chi_2, \cdots, \chi_d)$ be the periodic solution of the differential equation $\mathcal{L}\chi_i = -\beta_i$ ($1 \le i \le d$) and define the positive-definite symmetric $d \times d$ matrix C by

$$C = \int_{T^d}(I + \nabla\chi)a^t(I + \nabla\chi)dm,$$

where

$$\nabla\chi = \begin{pmatrix} \partial_1\chi_1 & \cdots & \partial_d\chi_1 \\ \partial_1\chi_2 & \cdots & \partial_d\chi_2 \\ \vdots & \ddots & \vdots \\ \partial_1\chi_d & \cdots & \partial_d\chi_d \end{pmatrix}$$

Then the sequence of the distributions on $C_0(I, \mathbf{R}^d)$ of (x_t^ε, P) converges weakly to $P^W(C, \mathbf{0})$ as $\varepsilon \to 0$ (cf. [4] [6]).

The second is a homogenization of $\hat{\mathcal{L}}^{(\varepsilon)}$-diffusion processes, where $\hat{\mathcal{L}}^{(\varepsilon)}$ is given by

$$\hat{\mathcal{L}}^{(\varepsilon)} = \mathcal{L}^{(\varepsilon)} + \sum_{i=1}^{d} b_i\left(\frac{x}{\varepsilon}\right)\partial_i$$

and $b(x) = (b_i(x))_{i=1,2,\cdots,d}$ $(x \in \mathbf{R}^d)$ has smooth and periodic (period $= 1$) components. The corresponding stochastic differential equations to $\hat{\mathcal{L}}^{(\varepsilon)}$ is:

$$\begin{aligned} dy_t^\varepsilon &= \alpha\left(\frac{y_t^\varepsilon}{\varepsilon}\right)dB_t + \frac{1}{\varepsilon}\beta\left(\frac{y_t^\varepsilon}{\varepsilon}\right)dt + b\left(\frac{y_t^\varepsilon}{\varepsilon}\right)dt, \\ y_0^\varepsilon &= 0. \end{aligned} \tag{3.3}$$

Let define $\mathbf{b} \in \mathbf{R}^d$ by

$$\mathbf{b} = \int_{T^d} (I + \nabla\chi(x))^t b(x) dm(x).$$

Then we have that the sequence of the distributions on $C_0(I, \mathbf{R}^d)$ of the solutions (y_t^ε, P) of (3.3) converges weakly to $P^W(C, \mathbf{b})$ as $\varepsilon \to 0$ (cf. [4], [6]).

We explain that the homogenization of $\hat{\mathcal{L}}^{(\varepsilon)}$-diffusion process can be obtained from the homogenization of $\mathcal{L}^{(\varepsilon)}$-diffusion process by using Theorem. Let x_t be a solution of the stochastic differential equation (3.1) with $x_0 = 0$. Then we have for $\varepsilon > 0$

$$\begin{aligned} d\varepsilon x_{t/\varepsilon^2} &= \alpha\left(\frac{\varepsilon x_{t/\varepsilon^2}}{\varepsilon}\right) d\varepsilon B_{t/\varepsilon^2} + \frac{1}{\varepsilon}\beta\left(\frac{\varepsilon x_{t/\varepsilon^2}}{\varepsilon}\right)dt \\ \varepsilon x_{0/\varepsilon^2} &= 0 \end{aligned}$$

and $(\varepsilon B_{t/\varepsilon^2}, P)$ is a d-dimensional Brownian motion starting at 0. It implies that $(\varepsilon x_{t/\varepsilon^2}, P)$ has the same distribution on $C_0(I, \mathbf{R}^d)$ of (x_t^ε, P). Let define $\gamma(x) = (\gamma_i(x))_{i=1,2,\cdots,d}$ $(x \in \mathbf{R}^d)$ by ${}^t\gamma(x) = \alpha(x)^{-1}\,{}^tb(x)$ and define $\Gamma_t^{(\varepsilon)}$ by

$$\Gamma_t^{(\varepsilon)} = \exp\left\{\int_0^t \gamma\left(\frac{x_s^{(\varepsilon)}}{\varepsilon}\right) dB_s^{(\varepsilon)} - \frac{1}{2}\int_0^t \|\gamma(\frac{x_s^{(\varepsilon)}}{\varepsilon})\|^2 ds\right\},$$

where $x_t^{(\varepsilon)} = \varepsilon x_{t/\varepsilon^2}$ and $B_t^{(\varepsilon)} = \varepsilon B_{t/\varepsilon^2}$. Then $(x_t^{(\varepsilon)}, \hat{P^{(\varepsilon)}})$ satisfies the stochastic differential equation (3.3), where $\hat{P^{(\varepsilon)}}$ is the probability measure defined by $d\hat{P^{(\varepsilon)}} = \Gamma^{(\varepsilon)} dP$. By the Ito formula $M_t = x_t + \chi(x_t)$ is a continuous square-integrable martingale with the quadratic variational process $\langle M \rangle$:

$$\langle M \rangle(t) = \int_0^t (I + \nabla\chi(x_s))a(x_s)^t(I + \nabla\chi(x_s))ds$$

and

$$\langle M, \int_0^{\cdot} \gamma(x_s) dB_s \rangle(t) = \int_0^t (I + \nabla\chi(x_s))^t b(x_s) ds.$$

Then we have

$$x_t^{(\varepsilon)} = \varepsilon M_{t/\varepsilon^2} - \varepsilon\chi(x_{t/\varepsilon^2})$$

and the distribution of $(x_t^{(\varepsilon)}, P)$ and $(\varepsilon\chi(x_{t/\varepsilon^2}), P)$ converges weakly to $P^W(C, \mathbf{0})$ and $P^D(\mathbf{0})$. By the ergodic property of $\mathcal{L}$-diffusion on T^d, we get the sequence of

$$\langle \varepsilon M_{\cdot/\varepsilon^2}, \int_0^{\cdot} \gamma\left(\frac{x_s^{(\varepsilon)}}{\varepsilon}\right) dB_s^{(\varepsilon)} \rangle(t) = \varepsilon^2 \int_0^{t/\varepsilon^2} (I + \nabla\chi(x_s))^t b(x_s) ds$$

converges weakly to $P^D(\mathbf{b})$ as $\varepsilon \to 0$. Consequently Theorem implies that the distribution of $(x_t^{(\varepsilon)}, \hat{P^{(\varepsilon)}})$ converges weakly to $P^W(C, \mathbf{b})$ as $\varepsilon \to 0$.

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