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On the Generalized Divisor Problem in Arithmetic Progressions

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Abstract. In this paper we investigate the asymptotic behavier of the summatory function $D_z(x,q,l)$ and $\pi_k(x,q,l)$, and its relation to the Riemann hypothesis for the Dirichlet L-function.

Introduction

One of the classical problems in analytic number theory which is now called "the Dirichlet divisor problem" is concerned with investigating the asymptotic behavier of $D_k(x) \equiv \sum_{n\le x} d_k(n)$ where $d_k(n)$ means the number of ways of expressing n as a product of k natural numbers. Namely, $d_k(n)$ is a multiplicative arithmetical function such that

$$d_k(p^m) = \binom{k+m-1}{m} \equiv \frac{k(k+1)\cdots(k+m-1)}{m!}.$$

It is well-known that $D_k(x)$ has an expression in the form

$$D_k(x) = xP_{k-1}(\log x) + \Delta_k(x)$$

where $P_k(x)$ is some polynomial of degree k, and $\Delta_k(x)$ is the error term. It seems that the essence of this problem is to establish some relationship between the order of $\Delta_k(x)$ and the property of $\zeta(s)$ since

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} \qquad (\sigma > 1),$$

where $s = \sigma + it$ and $\zeta(s)$ is the Riemann zeta function.

It is known that

$$\Delta_k(x) \ll x^{(k-1)/(k+1)+\varepsilon} \tag{0}$$

for every positive ε, and that the statement

$$\Delta_k(x) \ll x^{1/2+\varepsilon}$$

for every k is equivalent to the Lindelöf hypothesis. Finally it is conjectuered that

$$\Delta_k(x) \ll x^{(k-1)/2k+\varepsilon},$$

but any corresponding propertis of $\zeta(s)$ is not revealed yet.

On the other hand some mathematician tried to generalize the divisor problem. Let $d_z(n)$ be a multiplicative function defined by

$$d_z(p^m) \equiv \binom{z+m-1}{m}$$

where z is a complex number. Then we have

$$\zeta^z(s) = \prod_p (1 - \frac{1}{p^s})^{-z} = \sum_{n=1}^{\infty} \frac{d_z(n)}{n^s} \qquad (\sigma > 1)$$

where $\zeta^z(s) \equiv \exp(z \log \zeta(s))$ and let $\log \zeta(s)$ take real values for real $s > 1$.

The generalized divisor problem is to find an asymptotic formula for $\sum_{n \leq x} d_z(n)$, which was observed for real $z > 0$ by A. Kienast [7] and K. Iseki [5] independently. A. Selberg [15] considered it for all complex z, his result being

$$D_z(x) \equiv \sum_{n \leq x} d_z(n) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O(x(\log x)^{\Re z - 2}) \qquad (1)$$

uniformly for $|z| \leq A$, where A is any fixed positive number. This was extended by Rieger [14] to arithmetic progressions such that

$$D_z(x, q, l) \equiv \sum_{\substack{n \leq x \\ n \equiv l \ (mod\ q)}} d_z(n) = (\frac{\varphi(q)}{q})^z \frac{x}{\Gamma(z)\varphi(q)} (\log x)^{z-1}$$
$$+ O((\frac{\varphi(q)}{q})^z \frac{x}{\varphi(q)} (\log x)^{\Re z - 2} \log \log 4q)$$

uniformly for $|z| \leq A$, $q \leq (\log x)^{\tau}, (q, l) = 1$, where A and τ are any fixed positive numbers. We note that when z is a natural number, $d_z(n)$ coincides with the divisor function in the Dirichlet divisor problem, and $d_{-1}(n)$ with the Möbius function.

Next, let $\pi_k(x)$ be the number of integers $\leq x$ which are product of k distinct primes. For $k = 1$, $\pi_k(x)$ reduces to $\pi(x)$, the number of primes not exceeding x.

C.F. Gauss stated empirically that $\pi_2(x) \sim x(\log\log x)/\log x$, and E. Landau proved that $\pi_k(x) \sim x(\log\log x)^{k-1}/(k-1)!\log x$ by using the prime number theorem. Selberg considered $D_z(x)$ not only for its own sake but also with an intension to derive

$$\pi_k(x) = \frac{xQ(\log\log x)}{\log x} + O(\frac{x(\log\log x)^k}{k!(\log x)^2}) \tag{2}$$

uniformly for $1 \leq k \leq A\log\log x$, where $Q(x)$ is polynomial of degree $k-1$. Now we define $\pi_k(x,q,l)$ as a generalization of $\pi_k(x)$ by

$$\pi_k(x,q,l) \equiv \sum_{\substack{n\leq x \\ n\equiv l\,(mod\,q) \\ n=p_1\cdots p_k\,(p_i\neq p_j)}} 1.$$

In this paper we shall consider the connections between the asymptotic formulas of $D_z(x,q,l)$, $\pi_k(x,q,l)$ and the location of zeros of the Dirichlet L-function. In particular we shall establish some necessary and sufficient conditions for the truth of the Riemann hypothesis, so that this paper gives a generalization of [1] to arithmetic progressions.

The main term of (1) and (2) is, however, inconvenient for our aim so that we introduce the following integrals as the main terms of $D_z(x,q,l)$ and $\pi_k(x,q,l)$ respectively :

$$\Phi_z(x,q) = \frac{1}{2\pi i}\int_L (L(s,\chi_0))^z \frac{x^s}{s}\,ds,$$

$$F_{k,\delta}(x,q) = \frac{1}{(2\pi i)^2}\int_{|z|=1}\int_{L_\delta} (L(s,\chi_0))^z \times \{\prod_p (1+\frac{z\chi_0(p)}{p^s})(1-\frac{\chi_0(p)}{p^s})^z\}\frac{1}{z^{k+1}}\frac{x^s}{s}\,ds\,dz$$

where L is, for any r $(0<r<1/2)$, the path which begins at $1/2$, moves to $1-r$ along the real axis, encircle the point 1 with radius r in the counterclockwise direction, and returns to $1/2$ along the real axis, and L_δ is, for every δ and any r $(\delta>0,\ r>0,\ \delta+r<1/2)$, the path which begins at $1/2+\delta$, moves to $1-r$ along the real axis, encircle the point 1 with radius r in the counterclockwise direction, and returns to $1/2+\delta$ along the real axis. Here we denote by χ_0 the principal character *mod* q.

The error terms are defined by

$$\Delta_z(x,q,l) = D_z(x) - \frac{1}{\varphi(q)}\Phi_z(x,q),$$

$$R_{k,\delta}(x,q,l) = \pi_k(x,q,l) - \frac{1}{\varphi(q)}F_{k,\delta}(x,q).$$

Let

$$\Theta(\chi) = \sup\{\sigma : L(\sigma + it, \chi) = 0\}, \quad \Theta_q = \max_{\chi(mod q)} \Theta(\chi).$$

Then the following Theorem 1 and 2 follow from more general results proved in sections 1 and 2 below.

THEOREM 1. *There exists some constant c such that*

$$\Delta_z(x, q, l) \ll xe^{-c\sqrt{\log x}}$$

uniformly for $|z| \leq A$, $q \leq (\log x)^\tau$, $(q, l) = 1$ *where* A *and* τ *are any fixed positive numbers.*

Further we have

$$\Delta_z(x, q, l) \ll x^{\Theta_q + \varepsilon}$$

uniformly for $|z| \leq A$, $q \leq x$, $(q, l) = 1$.

Conversely if $\Delta_z(x, q, l) \ll x^{\Xi + \varepsilon}$ *for any* l $((q, l) = 1)$ *and for some* $z \in C - Q^+$, *where* Q^+ *denotes the set of all non negative rational numbers, then any* $L(s, \chi)$ $(mod\, q)$ *has no zeros for* $\sigma > \Xi$.

The main term $\Phi_z(x, q)$ *has an asymptotic expansion*

$$\Phi_z(x, q) = x(\log x)^{z-1} \sum_{m=0}^{N-1} \frac{B_m(z, q)}{(\log x)^m \Gamma(z - m)} + O(x(\log x)^{\Re z - N - 1})$$

uniformly for $|z| \leq A$. *Here* N *is any fixed positive integer and* $B_m(z, q)$ $(0 \leq m \leq N - 1)$ *are regular functions of* z, *especially* $B_0(z, q) = (\varphi(q)/q)^z$.

THEOREM 2. *There is some constant c such that*

$$R_{k,\delta}(x, q, l) \ll xe^{-c\sqrt{\log x}}$$

uniformly for $k \geq 1$, $q \leq (\log x)^\tau$, $(q, l) = 1$.

Further we have

$$R_{k,\delta}(x, q, l) \ll x^{\Theta_q + \varepsilon}$$

uniformly for $k \geq 1$, $q \leq x$, $(q, l) = 1$.

Conversely if $R_{k,\delta}(x, q, l) \ll x^{\Xi + \varepsilon}$ *for any* l $((q, l) = 1)$ *and for some* $k \geq 1$, *then any* $L(s, \chi)$ $(mod\, q)$ *has no zeros for* $\sigma > \Xi$.

The main term $F_{k,\delta}(x,q)$ has an asymptotic expansion

$$F_{k,\delta}(x,q) = \frac{x}{\log x}\sum_{m=0}^{N-1}\frac{Q_{m,q}(\log\log x)}{(\log x)^m} + O(\frac{x(\log\log x)^{k-1}}{(\log x)^{N+1}})$$

for every k and q. Here N is any fixed positive integer and $Q_{m,q}(x)$ are polynomials of degree not exceeding $k-1$, especially the coefficient of x^{k-1} of $Q_{0,q}(x)$ is 1.

Remark.

1. If we define $r_{k,q,l}$ by

$$r_{k,q,l} = \inf_{\delta}\ \inf\{r : R_{k,\delta}(x,q,l) \ll x^r\}$$

Theorem 2 shows that $\max_l r_{k,q,l} = \Theta_q$. The statement $\Theta_q = 1/2$ for every q is equivalent to the truth of the Riemann hypothesis for Dirichlet L-function.

2. For $k=1$, we can express the main term in terms of the logalithmic integral. Namely,

$$F_{1,\delta}(x,q) = \int_2^x \frac{du}{\log u} + O(x^{1/2+\delta}),$$

so that

$$\pi_1(x,q,l) = \frac{1}{\varphi(q)}\int_2^x \frac{du}{\log u} + O(xe^{-c\sqrt{\log x}}).$$

3. Similar results hold for $\omega_k(x,q,l)$ and $\Omega_k(x,q,l)$. Here

$$\omega_k(x,q,l) \equiv \sum_{\substack{n\le x\\ n\equiv l\,(\mathrm{mod}\,q)\\ \omega(n)=k}} 1, \quad \Omega_k(x,q,l) \equiv \sum_{\substack{n\le x\\ n\equiv l\,(\mathrm{mod}\,q)\\ \Omega(n)=k}} 1 .$$

where $\omega(n)$ means the number of distinct prime factors of n, and $\Omega(n)$ means the number of total prime factors allowing multiplicity.

§1. The Generalized Divisor Problem

Actually we prove a more general statement than Theorem 1 and Theorem 2.

Suppose $b_z(n)$ is an arithmetical function with a complex number z and let

$$f(s,z)=\sum_{n=1}^{\infty}\frac{b_z(n)}{n^s}\qquad(\sigma>1/2)$$

be absolutely convergent, and that $f(s,0)=1$.

We define the multiplicative function $a_z(n)$ by

$$\zeta^z(s)f(s,z)=\sum_{n=1}^{\infty}\frac{a_z(n)}{n^s}\qquad(\sigma>1).$$

If we put

$$f(s,z,\chi)=\sum_{n=1}^{\infty}\frac{b_z(n)\chi(n)}{n^s}\qquad(\sigma>1/2)$$

where χ is a Dirichlet character $mod\, q$, then

$$(L(s,\chi))^z f(s,z,\chi)=\sum_{n=1}^{\infty}\frac{a_z(n)\chi(n)}{n^s}\qquad(\sigma>1).$$

Non negative number δ represent 0 or arbitrary small positive number according as

$$\lim_{s\to 1/2} f(s,z)<\infty$$

or not.

LEMMA 1.1. *We have*

$$A_z(x,q,l)\equiv\sum_{\substack{n\le x\\ n\equiv l\ (mod\, q)}}a_z(n)=\frac{1}{\varphi(q)}\frac{1}{2\pi i}\int_{L_\delta}(L(s,\chi_0))^z f(s,z,\chi_0)\frac{x^s}{s}ds$$
$$+O(xe^{-c\sqrt{\log x}})$$

uniformly for $|z|\le A$, $q\le(\log x)^\tau$, $(q,l)=1$, *where* χ_0 *is the principal character* $mod\, q$.

Further, if we put

$$\Phi_{z,\delta}(x,q)\equiv\frac{1}{2\pi i}\int_{L_\delta}(L(s,\chi_0))^z f(s,z,\chi_0)\frac{x^s}{s}\,ds,$$

then $\Phi_{z,\delta}(x,q)$ *has the following asymptotic expansion*

$$\Phi_{z,\delta}(x,q) = x(\log x)^{z-1}\sum_{m=0}^{N-1}\frac{B_m(z,q)}{(\log x)^m\Gamma(z-m)} + O(x(\log x)^{\Re z-N-1})$$

uniformly for $|z| \leq A$. *Here* N *is any fixed positive integer,* $B_m(z,q)$ *are regular functions of* z *to be defined in the proof, especially* $B_0(z,q) = (\varphi(q)/q)^z f(1,z,\chi_0)$.

Proof. We put $A_z(x,\chi) \equiv \sum_{n\leq x} a_z(n)\chi(n)$ and restrict that $q \leq \exp\{\sqrt{\log x}\}$. Absolute constants C, c, and so on, are not necessary the same at each occurrence.

First, it should be shown that

$$A_z(x,\chi) = \xi^{-1}\int_x^{x+\xi} A_z(u,\chi)du + O(\xi\log^A x) + O(x^{(A+2)/(A+1)}) \tag{3}$$

where $\xi = \xi(x)$ satisfies that $1 < \xi < x/2$.

If we denote $D_z(x,\chi) \equiv \sum_{n\leq x} d_z(n)\chi(n)$,

$$\begin{aligned}
&\sum_{x<n\leq x+\rho} a_z(n)\chi(n)\\
&= \sum_{n\leq x+\rho} b_z(n)\chi(n)D_z((x+\rho)/n,\chi) - \sum_{n\leq x} b_z(n)\chi(n)D_z(x/n,\chi)\\
&=(\sum_{n\leq x+\rho} b_z(n)\chi(n)D_z((x+\rho)/n,\chi) - \sum_{n\leq x+\rho} b_z(n)\chi(n)D_z(x/n,\chi))\\
&\quad+(\sum_{n\leq x+\rho} b_z(n)\chi(n)D_z(x/n,\chi) - \sum_{n\leq x} b_z(n)\chi(n)D_z(x/n,\chi))\\
&= \sum_{n\leq x+\rho} b_z(n)\chi(n) \sum_{x/n<m\leq(x+\rho)/n} d_z(m)\chi(m)\\
&\qquad\qquad + \sum_{x<n\leq x+\rho} b_z(n)\chi(n)D_z(x/n,\chi).
\end{aligned}$$

for $1 < \rho \leq \xi$. Here the second term is 0 whereas the first term is

$$\ll \rho\log^A x + x^{(A+1)/(A+2)}.$$

Because that $|d_z(n)| \leq d_k(n)$ where $k = [|z|] + 1$, and the well known result (0) make $\sum_{x<n\leq x+\rho} d_z(n) \ll \rho\log^{k-1} x + x^{k/(k+1)}$. Hence we have

$$\sum_{x<n\le x+\rho} a_z(n)\chi(n) \ll \rho \log^A x + x^{(A+1)/(A+2)}.$$

On the other hand

$$\xi^{-1}\int_x^{x+\xi} A_z(u,\chi)du = A_z(x,\chi) + O(\xi^{-1}\int_x^{x+\xi}\sum_{x<n\le u} a_z(n)\chi(n)du)$$
$$= A_z(x,\chi) + O(\sup_{0<\rho\le\xi} |\sum_{x<n\le x+\rho} a_z(n)\chi(n)|)$$

Hence we obtain (3).

We start from the expression

$$\int_0^x A_z(u,\chi)du = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{2-iT}^{2+iT}(L(s,\chi))^z f(s,z,\chi)\frac{x^{s+1}}{s(s+1)}ds.$$

By the Cauchy integral theorem, the path of integration can be deformed within the domain where the integrand is single-valued.

For $\chi = \chi_0$, we replace the path of integration to $\sum_{i=1}^7 L_i$ which is defined as follows :

L_1 is the segment $[2 - iT,\ 1 - \eta(-T,q) - iT]$,
L_2 is the curve $s = 1 - \eta(t,q) + it \quad (-T \le t \le -t_0)$
and two segments
$[\eta - it_0,\ \eta - i(1-\eta)\tan\theta] + [\eta - i(1-\eta)\tan\theta,\ 1/2+\delta - i(1/2-\delta)\tan\theta]$,
L_3 is the segment $[1/2+\delta - i(1/2-\delta)\tan\theta,\ 1 + re^{-i(\pi-\theta)}]$,
L_4 is the arc $s = 1 + re^{i\varphi}\ (-(\pi-\theta) \le \varphi \le \pi - \theta)$,
L_5 is the segment $[1 + re^{i(\pi-\theta)},\ 1/2+\delta + i(1/2-\delta)\tan\theta]$,
L_6 is two segments
$[1/2+\delta+i(1/2-\delta)\tan\theta,\ \eta+i(1-\eta)\tan\theta] + [\eta+i(1-\eta)\tan\theta,\ \eta+it_0]$
and the curve $s = 1 - \eta(t,q) + it \quad (t_0 \le t \le T)$,
L_7 is the segment $[1 - \eta(T,q) + iT,\ 2 + iT]$.

Here $\eta(t,q) = c/\log q(|t|+2)$, $\eta = 1 - \eta(t_0,q)$ and t_0 is suffciently large number to make $1/2 < \eta < 1$. Non negative number δ and any positive numbers r and θ are satisfying $1/2+\delta \le \eta < 1 - r$, $0 < (1-\eta)\tan\theta < t_0$.

The contributions from the integral along $L_1 + L_2 + L_6 + L_7$ are seen to give the error term, while the integral along $L_3 + L_4 + L_5$ gives the principal term since that path becomes L_δ allowing $\theta \downarrow 0$.

We can see

$$\log L(s,\chi_0) \ll \log\log q(|t|+3)$$

for $s \in L_1 + L_2 + L_6 + L_7$ by Hilfssatz 4 and 7 of Rieger [13], so that

$$(L(s,\chi_0))^z f(s,z,\chi_0) \ll (\log q(|t|+2))^A.$$

Then,

$$\int_{L_1} + \int_{L_7} \ll \int_{1-\eta(T,q)}^{2} (\log qT)^A \frac{x^3}{T(T+1)} d\sigma$$

which tend to 0 by $T \to \infty$, and

$$\begin{aligned}\int_{L_2} + \int_{L_6} &\ll \int_0^T (\log q(|t|+2))^A \frac{x^{2-\eta(t,q)}}{t^2} dt \\ &\quad + \int_0^{t_0} x^{1+\eta} dt + \int_{1/2+\delta}^{\eta} x^{\eta} d\sigma \\ &\ll x^2 e^{-c\sqrt{\log x}}\end{aligned}$$

uniformly for $T \geq 1$, $q \leq \exp\{\sqrt{\log x}\}$.

Hence we have

$$\begin{aligned}&\int_0^x A_z(u,\chi_0) du \\ &= \frac{1}{2\pi i} \int_{L_\delta} (L(s,\chi_0))^z f(s,z,\chi_0) \frac{x^{s+1}}{s(s+1)} ds + O(x^2 e^{-c\sqrt{\log x}}) \\ &= \int_0^x \Phi_{z,\delta}(u,q) du + O(x^2 e^{-c\sqrt{\log x}}).\end{aligned}$$

By the way, since

$$\log\{(s-1)L(s,\chi_0)\} \ll \sqrt{\log 2q}$$

for $s \in L_\delta$ because of $L(s,\chi_0) = \zeta(s) \prod_{p|q}(1-p^{-s})$, we have

$$\begin{aligned}\Phi'_{z,\delta}(x,q) &= \frac{d}{dx}\Phi_{z,\delta}(x,q) \\ &= \frac{1}{2\pi i} \int_{L_\delta} \{(s-1)L(s,\chi_0)\}^z f(s,z,\chi_0)(s-1)^{-z} x^{s-1} ds \\ &\ll e^{CA\sqrt{\log 2q}} \Big(\int_{1/2+\delta}^{1-r} + \int_{|s-1|=r} \Big)(s-1)^{-z} x^{s-1} ds \\ &\ll e^{CA\sqrt{\log 2q}} \Big\{ (\log x)^{A-1} \int_{r\log x}^{(\log x)/2} u^{-A} e^{-u} du + r^{1-A} x^{r} \Big\}\end{aligned}$$

where we put $(1-s)\log x = u$. Now we choose $r = 1/\log x$, this is

$$\ll e^{CA\sqrt{\log 2q}}(\log x)^{A-1} \ll \exp\{C'A(\log x)^{1/4}\}.$$

Hence

$$\begin{aligned}
&\int_x^{x+\xi} A_z(u,\chi)du \\
&= \int_0^{x+\xi} \Phi_{z,\delta}(u,q)du - \int_0^{x} \Phi_{z,\delta}(u,q)du + O(x^2 e^{-c\sqrt{\log x}}) \\
&= \xi\Phi_{z,\delta}(x,q) + \xi^2\Phi'_{z,\delta}(x+\xi\theta,\, q) + O(x^2 e^{-c\sqrt{\log x}}) \qquad (0<\theta<1) \\
&= \xi\Phi_{z,\delta}(x,q) + O(\xi^2 \exp\{C'A(\log x)^{1/4}\}) + O(x^2 e^{-c\sqrt{\log x}}).
\end{aligned}$$

Hence using (3) and choosing $\xi = x\exp\{-c\sqrt{\log x}/2\}/2$ make

$$\begin{aligned}
A_z(x,\chi_0) =&\Phi_{z,\delta}(x,q) + O(\xi\exp\{C'A(\log x)^{1/4}\}) + O(\xi^{-1}x^2 e^{-c\sqrt{\log x}}) \\
&+ O(\xi\log^A x) + O(x^{(A+1)/(A+2)}) \\
=&\Phi_{z,\delta}(x,q) + O(xe^{-c\sqrt{\log x}/4}).
\end{aligned}$$

Next we consider for $\chi = \chi_1$ which is the exceptional character with respect to the zero on the real axis.

If Siegel zero β_1 of $L(s,\chi_1)$ exists in the range $\beta_1 > 1-c/(2\log 2q)$, we replace the path of integration to $\sum_{i=1}^{7} L_i$ which is defined as follows.:

L_1 is the segment $[\,2-iT,\ 1-\eta(-T,q)-iT\,]$,
L_2 is the curve $s = 1-\eta(t,q)+it \quad (-T \le t \le -t_0)$
and the segment $[\,\eta - it_0, \eta - i(\beta_1-\eta)\tan\theta\,]$,
L_3 is the segment $[\,\eta - i(\beta_1-\eta)\tan\theta,\ \beta_1 + r_1e^{-i(\pi-\theta)}\,]$,
L_4 is the arc $s = \beta_1 + r_1 e^{i\varphi}\,(-(\pi-\theta) \le \varphi \le \pi-\theta\,)$,
L_5 is the segment $[\,\beta_1 + r_1e^{i(\pi-\theta)},\ \eta + i(\beta_1-\eta)\tan\theta\,]$,
L_6 is the segment $[\,\eta + i(\beta_1-\eta)\tan\theta,\ \eta + it_0\,]$
and the curve $s = 1-\eta(t,q)+it \quad (\,t_0 \le t \le T\,)$,
L_7 is the segment $[\,1-\eta(T,q)+iT,\ 2+iT\,]$.

Here $\eta(t,q) = c/\log q(|t|+2)$, $\eta = 1-\eta(t_0,q)$ and t_0 is suffciently large number to make $1-c/(\log 2q) < \eta < 1-c/(2\log 2q)$. Any positive numbers r_1 and θ are satisfying $\eta < \beta_1 - r_1$, $0 < (\beta_1-\eta)\tan\theta < t_0$. Let $\lim_{\theta\to 0}(L_3+L_4+L_5) = \ell$.

As the same as the case for $\chi = \chi_0$, we have

$$\int_{L_1+L_2+L_6+L_7} (L(s,\chi_1))^z f(s,z,\chi_1)\frac{x^{s+1}}{s(s+1)}ds \ll x^2 e^{-c\sqrt{\log x}}.$$

Hence

$$\int_0^x A_z(u,\chi_1)du$$
$$= \frac{1}{2\pi i}\int_l (L(s,\chi_1))^z f(s,z,\chi_1)\frac{x^{s+1}}{s(s+1)}ds + O(x^2 e^{-c\sqrt{\log x}})$$
$$= \int_0^x \Phi_z(u,\chi_1)du + O(x^2 e^{-c\sqrt{\log x}}), \quad say.$$

We can see

$$\log\frac{L(s,\chi_1)}{s-\beta_1} \ll \log\log 4q$$

for $s \in l$ by Hilfssats 7 of Rieger [13], so that

$$(\frac{L(s,\chi_1)}{s-\beta_1})^z f(s,z,\chi_1) \ll (\log 4q)^{CA}.$$

Hence

$$\Phi_z(x,\chi_1) = \frac{1}{2\pi i}\int_l (L(s,\chi_1))^z f(s,z,\chi_1)\frac{x^s}{s}ds$$
$$= x^{\beta_1}\frac{1}{2\pi i}\int_l (\frac{L(s,\chi_1)}{s-\beta_1})^z f(s,z,\chi_1)(s-\beta_1)^z x^{s-\beta_1}\frac{ds}{s}$$
$$\ll x^{\beta_1}(\log 4q)^{CA}(\int_\eta^{\beta_1-r_1} + \int_{|s-\beta_1|=r_1})(s-\beta_1)^z x^{s-\beta_1}ds$$
$$\ll x^{\beta_1}(\log 4q)^{CA}\{(\log x)^{A-1}\int_{r_1\log x}^{(\log x)/2} u^{-A}e^{-u}du + r_1^{1-A}x_1^{r}\}$$

where we put $(\beta_1 - s)\log x = u$. Now we choose $r_1 = 1/\log x$, this is

$$\ll x^{\beta_1}(\log 4q)^{CA}(\log x)^{A-1} \ll x^{\beta_1}(\log x)^{C'A}.$$

Here we have

$$\int_x^{x+\xi} A_z(u,\chi_1)du$$
$$= \int_0^{x+\xi}\Phi_z(u,\chi_1)du - \int_0^x \Phi_z(u,\chi_1)du + O(x^2 e^{-c\sqrt{\log x}})$$
$$= \xi\Phi_z(x+\xi\theta,\, \chi_1) + O(x^2 e^{-c\sqrt{\log x}}) \qquad (0<\theta<1)$$
$$\ll \xi x^{\beta_1}(\log x)^{C'A} + x^2 e^{-c\sqrt{\log x}}.$$

Hence using (3) and choosing $\xi = x\exp\{-c\sqrt{\log x}/2\}/2$ make

$$\begin{aligned} A_z(x,\chi_1) \ll & x^{\beta_1}(\log x)^{C'A} + \xi^{-1}x^2e^{-c\sqrt{\log x}} \\ & + \xi\log^A x + x^{(A+1)/(A+2)} \\ \ll & x^{\beta_1}(\log x)^{C'A} + xe^{-c\sqrt{\log x}/4} \end{aligned}$$

On the other hand if $\beta_1 \le 1 - c/(2\log 2q)$, we replace the path of integration to $\sum_{i=1,2,6,7} L_i$ which is defined as follows.:

L_1 is the segment $[2 - iT,\ 1 - \eta(-T,q) - iT]$,
L_2 is the curve $s = 1 - \eta(t,q) + it \quad (-T \le t \le 0)$
L_6 is the curve $s = 1 - \eta(t,q) + it \quad (0 \le t \le T)$,
L_7 is the segment $[1 - \eta(T,q) + iT,\ 2 + iT]$.
Here $\eta(t,q) = c/(4\log q(|t|+2))$.

As the same as the last case we have

$$\int_{L_1+L_2+L_6+L_7} (L(s,\chi_1))^z f(s,z,\chi_1)\frac{x^{s+1}}{s(s+1)}ds \ll x^2e^{-c\sqrt{\log x}}.$$

Hence

$$\int_0^x A_z(u,\chi_1)du \ll x^2e^{-c\sqrt{\log x}}.$$

By using (3) , we have

$$\begin{aligned} A_z(x,\chi_1) \ll & \xi^{-1}x^2e^{-c\sqrt{\log x}} + \xi\log^A x + x^{(A+1)/(A+2)} \\ \ll & xe^{-c\sqrt{\log x}/4}. \end{aligned}$$

Hence in either case we have

$$A_z(x,\chi_1) \ll x^{\beta_1}(\log x)^{C'A} + xe^{-c\sqrt{\log x}/4},$$

where we note that

$$x^{\beta_1}(\log x)^{C'A} \ll xe^{-c\sqrt{\log x}/4}$$

for the case $\beta_1 \le 1 - c/(2\log 2q)$.

Finally we consider for the case $\chi \ne \chi_0, \chi_1$. The deformation of the path of integration is as the same as the case $\beta_1 \le 1 - c/(2\log 2q)$ for $\chi = \chi_1$, apart from $\eta(t,q) = c/\log q(|t|+2)$.

Similar observation leads

$$A_z(x,\chi) \ll xe^{-c\sqrt{\log x}/4}.$$

As a consequence of these, we have, by replacing C' to C and $c/4$ to c, we have

$$A_z(x,\chi) = E_0\Phi_{z,\delta}(x,q) + E_1 O(x^{\beta_1}(\log x)^{CA}) + O(xe^{-c\sqrt{\log x}}),$$

where E_0 takes 1 or 0 according as $\chi = \chi_0$ or not, and E_1 does 1 or 0 according as $\chi = \chi_1$ or not.

Hence we have

$$\begin{aligned} A_z(x,q,l) &= \frac{1}{\varphi(q)} \sum_{\chi (mod q)} \overline{\chi}(l) A_z(x,\chi) \\ &= \frac{1}{\varphi(q)}\Phi_{z,\delta}(x,q) + O(\frac{x^{\beta_1}}{\varphi(q)}(\log x)^{CA}) + O(xe^{-c\sqrt{\log x}}). \end{aligned}$$

Siegel's theorem makes

$$\frac{x^{\beta_1}}{\varphi(q)}(\log x)^{CA} \ll xe^{-c\sqrt{\log x}}$$

uniformly for $q \le (\log x)^{\tau}$.

Now it remains the asymptotic expansion of $\Phi_{z,\delta}(x,q)$. We define regular functions $B_m(z,q)$ as Taylor coefficients

$$\{(s-1)L(s,\chi_0)\}^z f(s,z,\chi_0)\, s^{-1} = \sum_{m=0}^{N-1} B_m(z,q)(s-1)^m + R_N(s,z,q)$$

for $s \in L_\delta$, $|z| \le A$, where N is any fixed positive integer, especially

$$\begin{aligned} B_0(z,q) &= (\frac{\varphi(q)}{q})^z f(1,z,\chi_0) \\ B_1(z,q) &= (\frac{\varphi(q)}{q})^z \{(z\gamma + z\sum_{p|q}\frac{\log p}{p-1} - 1) f(1,z,\chi_0) + f_s(1,z,\chi_0)\}, \end{aligned}$$

where γ is Euler constant.

Since we can see $R_N(s,z,q) \ll (s-1)^N$,

$$\Phi_{z,\delta}(x,q) = \frac{1}{2\pi i}\int_{L_\delta} \{(s-1)L(s,\chi_0)\}^z f(s,z,\chi_0)\, s^{-1}(s-1)^{-z}x^s ds$$
$$= \frac{x}{2\pi i}\Big(\int_\Gamma - \int_{\Gamma - L_\delta}\Big)\sum_{m=0}^{N-1} B_m(z,q)(s-1)^{m-z}x^{s-1}ds$$
$$+ \frac{x}{2\pi i}\int_{L_\delta} R_N(s,z,q)(s-1)^{-z}x^{s-1}ds$$

where Γ is the path which consist of the segment $(-\infty, i-r]$, the arc $s = i + re^{i\varphi}$ $(-\pi \le \varphi \le \pi)$ and the segment $[1-r, -\infty)$. By substituting $(s-1)\log x = \omega$, we find

$$\frac{x}{2\pi i}\int_\Gamma (s-1)^{m-z}x^{s-1}ds = (\log x)^{z-m-1}\frac{1}{2\pi i}\int_{\Gamma'} \omega^{m-z}e^{\omega}d\omega$$
$$= \frac{(\log x)^{z-m-1}}{\Gamma(z-m)}$$

The remaining integrals are to be the error term which is proved in Ivić[6].

LEMMA 1.2. *We have*

$$\log L(s,\chi) \ll (\log q(|t|+2))^{1+2\Theta(\chi)-2\sigma+\varepsilon}$$

uniformly for $\Theta(\chi) < \sigma_0 \le \sigma \le 1$, $|t| \ge E_0$, $q \ge 1$.

This is proved in the same way as Theorem 14.2 in Titchmarsh [16], which is the case of $q = 1$ and $\sigma = 1/2$.

The Lemma is also a generalization of Lemma 1.2 in [11] which is the case of $q = 1$.

We define the error terms

$$\Delta_{z,\delta}(x,\chi) = A_z(x,\chi) - E_0\Phi_{z,\delta}(x,q),$$
$$\Delta_{z,\delta}(x,q,l) = A_z(x,q,l) - \frac{1}{\varphi(q)}\Phi_{z,\delta}(x,q),$$

and let

$$\alpha_z(\chi) = \inf_\delta \inf\{\alpha : \Delta_{z,\delta}(x,\chi) \ll x^\alpha\},$$
$$\alpha_{z,q,l} = \inf_\delta \inf\{\alpha : \Delta_{z,\delta}(x,q,l) \ll x^\alpha\}.$$

THEOREM 1.3. *We have*

$$\alpha_z(\chi) \le \Theta(\chi)$$

for any $z \in C$ under the assumption that $a_z(n) \ll n^{1/2+\varepsilon}$.

Remark. Theorem 1.3 leads us easily to that

$$\alpha_{z,q,l} \le \Theta_q$$

because of the relation

$$\Delta_{z,\delta}(x,q,l) = \frac{1}{\varphi(q)} \sum_{\chi(mod q)} \overline{\chi}(l)\, \Delta_{z,\delta}(x,\chi).$$

Proof. By Lemma 3.12 in Titchmarsh [16], $A_z(x,\chi)$ has the expression

$$A_z(x,\chi) = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} (L(s,\chi))^z f(s,z,\chi) \frac{x^s}{s} ds + O(\frac{x^2}{T}) + O(x^{1/2+\varepsilon})$$

uniformly for $T \ge 1$.

For $\chi = \chi_0$, the path of integration can be replaced by $\sum_{i=1}^{7} L_i$ which is defined as follows :

L_1 is the segment $[\,2 - iT,\ \eta - iT\,]$,

L_2 is two segments $[\,\eta - iT,\ \eta - i(1-\eta)\tan\theta\,] + [\,\eta - i(1-\eta)\tan\theta,\ 1/2 + \delta - i(1/2-\delta)\tan\theta\,]$,

L_3, L_4 and L_5 are the same as in Lemma1.1,

L_6 is two segments $[\,1/2 + \delta + i(1/2-\delta)\tan\theta,\ \eta + i(1-\eta)\tan\theta\,] + [\,\eta + i(1-\eta)\tan\theta,\ \eta + iT\,]$,

L_7 is the segment $[\,\eta + iT,\ 2 + iT\,]$.

Here η is a constant such that $\Theta(\chi_0) < \eta < 1$, and non negative number δ and any positive numbers r and θ are such that $1/2 + \delta \le \eta < 1 - r,\ 0 < (1-\eta)\tan\theta < 1$.

As in Lemma 1.1, $L_3 + L_4 + L_5$ becomes L_δ by allowing $\theta \downarrow 0$.

From Lemma 1.2, we have

$$(L(s,\chi_0))^z f(s,z,\chi_0) \ll (q(|t|+2))^\varepsilon$$

for $s \in L_1, L_2, L_6, L_7$. Therefore,

$$\int_{L_1} + \int_{L_7} \ll \int_\eta^2 (qT)^\varepsilon \frac{x^2}{T} d\sigma \ll q^\varepsilon T^{\varepsilon-1} x^2,$$

$$\int_{L_2} + \int_{L_6} \ll \int_0^T (qt)^\varepsilon \frac{x^\eta}{t+1} dt + \int_{1/2+\delta}^{\eta} q^\varepsilon x^\eta d\sigma \ll q^\varepsilon T^\varepsilon x^\eta.$$

Hence

$$A_z(x,\chi_0) = \Phi_{z,\delta}(x,q) + O(q^\varepsilon T^{\varepsilon-1} x^2) + O((qT)^\varepsilon x^\eta) + O(\frac{x^2}{T}) + O(x^{1/2+\varepsilon}).$$

By taking $T = x^2$, $\eta = \Theta(\chi_0) + \varepsilon$ we have

$$A_z(x,\chi_0) = \Phi_{z,\delta}(x,q) + O(x^{\Theta(\chi_0)+4\varepsilon})$$

uniformly for $q \le x$.

For $\chi \ne \chi_0$, the path of integration is replaced by $\sum_{i=1,2,6,7} L_i$.

L_1 is the segment $[2 - iT,\ \eta - iT]$, L_2 is the segment $[\eta - iT,\ \eta]$, L_6 is the segment $[\eta,\ \eta + iT]$, L_7 is the segment $[\eta + iT,\ 2 + iT]$, where η is a constant such that $\Theta(\chi) < \eta < 1$.

By similar way, we find

$$A_z(x,\chi) \ll x^{\Theta(\chi)+4\varepsilon}.$$

Hence

$$A_z(x,\chi) = E_0 \Phi_{z,\delta}(x,q) + O(x^{\Theta(\chi)+4\varepsilon}),$$

this proves the theorem.

THEOREM 1.4. *We have*

$$\Theta(\chi) \le \alpha_z(\chi)$$

for any $z \in C - Q^+$.

Remark. Theorem 1.4 leads us easily to that

$$\Theta(\chi) \le \max_l \ \alpha_{z,q,l}$$

because of the relation

$$\Delta_{z,\delta}(x,\chi) = \sum_l \chi(l)\, \Delta_{z,\delta}(x,q,l).$$

Proof. First, we suppose that $\sigma > 2$. Then,

$$s\int_1^\infty \frac{\Phi_{z,\delta}(x,q)}{x^{s+1}}dx$$
$$= s\int_1^\infty \Big(\frac{1}{2\pi i}\int_{L_\delta}(L(\omega,\chi_0))^z f(\omega,z,\chi_0)\frac{x^\omega}{\omega}d\omega\Big)\frac{1}{x^{s+1}}\,dx$$
$$= \frac{s}{2\pi i}\int_{L_\delta}(L(\omega,\chi_0))^z f(\omega,z,\chi_0)\Big(\int_1^\infty x^{\omega-s-1}dx\Big)\frac{d\omega}{\omega}$$
$$= \frac{s}{2\pi i}\int_{L_\delta}(L(\omega,\chi_0))^z f(\omega,z,\chi_0)\frac{d\omega}{\omega(s-\omega)}$$
$$= \frac{1}{2\pi i}\int_{L_\delta}(L(\omega,\chi_0))^z f(\omega,z,\chi_0)\frac{d\omega}{\omega}$$
$$+\frac{1}{2\pi i}\int_{L_\delta}(L(\omega,\chi_0))^z f(\omega,z,\chi_0)\frac{d\omega}{s-\omega}.$$

The interchange of the order of integration is justified because of the absolute convergence. Hence

$$(L(s,\chi))^z f(s,z,\chi) = s\int_1^\infty \frac{A_z(x,\chi)}{x^{s+1}}\,dx$$
$$= E_0\; s\int_1^\infty \frac{\Phi_{z,\delta}(x,q)}{x^{s+1}}\,dx + s\int_1^\infty \frac{\Delta_{z,\delta}(x,\chi)}{x^{s+1}}\,dx$$
$$=\frac{E_0}{2\pi i}\int_{L_\delta}(L(\omega,\chi_0))^z f(\omega,z,\chi_0)\frac{d\omega}{\omega}$$
$$+\frac{E_0}{2\pi i}\int_{L_\delta}(L(\omega,\chi_0))^z f(\omega,z,\chi_0)\frac{d\omega}{s-\omega} + s\int_1^\infty \frac{\Delta_{z,\delta}(x,\chi)}{x^{s+1}}dx. \quad (4)$$

We put

$$L_{\sigma_0} = \{s:\sigma_0<\sigma\} - \{s:|s-1|\le r\} - \{s:t=0,\sigma<1\}$$

for $1/2 \le \sigma_0 \le 2$. Then on the right hand side of (4) , the first and the second term can be continued analytically for $s \in L_{1/2}$ as a function of s, while the third term can be continued for $\sigma > \alpha_z(\chi)$ since the involved integral converge uniformly for $\sigma > \alpha_z(\chi)$. Hence the right hand side of (4) is regular for $s \in L_{\alpha_z(\chi_0)}$ in the case for $\chi = \chi_0$, and for $\sigma > \alpha_z(\chi)$ in the case for $\chi \neq \chi_0$.

On the other hand the left hand side of (4) has logarithmic singularities at the zeros of $L(s,\chi)$ when $s \in C - Q^+$.

We therefore conclude that $\Theta(\chi) \leq \alpha_z(\chi)$ for any $z \in C - Q^+$, since $L(s, \chi_0)$ vanishes neither on the real axis ($\sigma \geq 1/2$) nor near the point $s = 1$.

Remark.

If we suppose that all the zeros of $L(s, \chi)$ are simple, Theorem 1.4 holds for all $z \in C - N$.

Now Theorem 1 follows by taking $f(s, z) \equiv 1$.

§2. The Asymptotic Formula for $\pi_k(x,q,l)$

Throughout this section, we suppose that $a_z(n)$ is regular for $|z| \le A$, and has Taylor expansion at $z=0$ such that $a_z(n) = \sum_{k=0}^{\infty} c_k(n) z^k$ for $|z| \le A$ with $A > 1$.

LEMMA 2.1. *We have*

$$C_k(x,q,l) \equiv \sum_{\substack{n \le x \\ n \equiv l \ (\mathrm{mod}\, q)}} c_k(n) = \frac{1}{\varphi(q)} \frac{1}{2\pi i} \int_{|z|=1} \frac{\Phi_{z,\delta}(x,q)}{z^{k+1}} dz + O(xe^{-c\sqrt{\log x}})$$

uniformly for $k \ge 1$, $q \le (\log x)^{\tau}$, $(q,l)=1$.

Further, if we put

$$F_{k,\delta}(x,q) \equiv \frac{1}{2\pi i} \int_{|z|=1} \frac{\Phi_{z,\delta}(x,q)}{z^{k+1}} dz,$$

then $F_{k,\delta}(x,q)$ *has the following asymptotic expansion*

$$F_{k,\delta}(x,q) = \frac{x}{\log x} \sum_{m=0}^{N-1} \frac{Q_{m,q}(\log\log x)}{(\log x)^m} + O\left(\frac{x(\log\log x)^{k-1}}{(\log x)^{N+1}}\right)$$

for every k*, where* $Q_{m,q}(x)$ *are polynomials of degree not exeeding* $k-1$*, especially the coefficient of* x^{k-1} *of* $Q_0(x)$ *is* 1.

Proof. Since $A_z(x,q,l)$ is regular for $|z| \le A$ as a function of z, and $C_k(x,q,l)$ is Taylor coefficient of z^k, it follows by using Lemma 1.1 that

$$\begin{aligned} C_k(x,q,l) &= \frac{1}{2\pi i} \int_{|z|=1} \frac{A_z(x,q,l)}{z^{k+1}} dz \\ &= \frac{1}{\varphi(q)} \frac{1}{2\pi i} \int_{|z|=1} \frac{\Phi_{z,\delta}(x,q)}{z^{k+1}} dz + \frac{1}{2\pi i} \int_{|z|=1} \frac{\Delta_{z,\delta}(x,q,l)}{z^{k+1}} dz, \end{aligned}$$

where

$$\begin{aligned} \frac{1}{2\pi i} \int_{|z|=1} \frac{\Delta_{z,\delta}(x,q,l)}{z^{k+1}} dz &\ll \max_{|z|=1} |\Delta_{z,\delta}(x,q,l)| \\ &\ll xe^{-c\sqrt{\log x}} \end{aligned}$$

which proves the first half.

Now, we expand the principal term asymptotically. By using the asymptotic expansion of $\Phi_{z,\delta}(x,q)$ proved in Lemma 1.1, $F_{k,\delta}(x,q)$ has the following expression

$$F_{k,\delta}(x,q) = \frac{x}{\log x}\frac{1}{2\pi i}\int_{|z|=1}\frac{(\log x)^z}{z^{k-1}}\sum_{m=0}^{N-1}\frac{B_m(z,q)}{(\log x)^m\Gamma(z-m)}dz + O(\frac{x}{(\log x)^{N+1}}\frac{1}{2\pi i}\int_{|z|=1}\frac{(\log x)^{\Re z}}{z^{k+1}}dz)$$

Then, if we denote

$$\frac{B_m(z,q)}{\Gamma(z-m)} = \sum_{i=1}^{\infty} e_{m,i,q}\, z^i, \qquad (\log x)^z = \sum_{l=0}^{\infty}\frac{(\log\log x)^l}{l!}z^l,$$

the leading term can be deformed

$$\begin{aligned}
&\frac{x}{\log x}\frac{1}{2\pi i}\int_{|z|=1}\frac{(\log x)^z}{z^{k+1}}\sum_{m=0}^{N-1}\frac{B_m(z,q)}{(\log x)^m\Gamma(z-m)}dz \\
&= \frac{x}{\log x}\sum_{m=0}^{N-1}\frac{1}{(\log x)^m} \\
&\qquad \times \frac{1}{2\pi i}\int_{|z|=1}\sum_{l=0}^{\infty}\frac{(\log\log x)^l}{l!}\sum_{i=1}^{\infty}e_{m,i,q}\, z^{l+i-k-1}dz \\
&= \frac{x}{\log x}\sum_{m=0}^{N-1}\frac{1}{(\log x)^m}\sum_{\substack{l+i=k\\ l\ge 0,\, i\ge 1}}\frac{e_{m,i,q}}{l!}(\log\log x)^l \\
&= \frac{x}{\log x}\sum_{m=0}^{N-1}\frac{1}{(\log x)^m}\sum_{l=0}^{k-1}\frac{e_{m,k-l,q}}{l!}(\log\log x)^l \\
&= \frac{x}{\log x}\sum_{m=0}^{N-2}\frac{Q_{m,q}(\log\log x)}{(\log x)^m} + O(\frac{x(\log\log x)^{k-1}}{(\log x)^{N+1}}),
\end{aligned}$$

where $Q_{m,q}(x) = \sum_{l=0}^{k-1} e_{m,k-l,q}(l!)^{-1}x^l$ are polynomials of degree not exceeding $k-1$.

On the other hand

$$\frac{x}{(\log x)^{N+1}}\frac{1}{2\pi i}\int_{|z|=1}\frac{(\log x)^{\Re z}}{z^{k+1}}dz \ll \frac{x}{(\log x)^{N+1}}\int_{|z|=1}\frac{\log x}{|z|^{k+1}}|dz| \ll \frac{x}{(\log x)^N}.$$

Hence

$$F_{k,\delta}(x,q) = \frac{x}{\log x}\sum_{m=0}^{N-2}\frac{Q_{m,q}(\log\log x)}{(\log x)^m} + O(\frac{x(\log\log x)^{k-1}}{(\log x)^N}) + O(\frac{x}{(\log x)^N}).$$

This proves the Lemma by replacing N to $N+1$.

Remark.

For $k=1$, we can express the main term in terms of the logalithmic integral. Namely, we start from the expression

$$\Phi_{z,\delta}(x,q) = \int_2^x (\frac{1}{2\pi i}\int_{L_\delta}(L(s,\chi_0))^z f(s,z,\chi_0)x^{s-1}\,ds)\,dx + O(1)$$

and define $\widetilde{B}_m(z,q)$ by Taylor coefficients of

$$\{(s-1)L(s,\chi_0)\}^z f(s,z,\chi_0) = (\frac{\varphi(q)}{q})^z f(1,z,\chi_0) + \sum_{m=0}^{N-1}\widetilde{B}_m(z,q)(s-1)^m + \widetilde{R}_N(s,z,q)$$

instead of $B_m(z,q)$. Then similar consideration to the asymptotic expansion in Lemma 1.1 and Lemma 1.3 make

$$F_{1,\delta}(x,q) = \int_2^x \frac{du}{\log u} + O(x^{1/2+\delta}),$$

so that

$$C_1(x,q,l) = \frac{1}{\varphi(q)}\int_2^x \frac{du}{\log u} + O(xe^{-c\sqrt{\log x}}).$$

This satisfies the assertion.

We define the error terms

$$R_{k,\delta}(x,\chi) = C_k(x,\chi) - E_0F_{k,\delta}(x,q)$$

$$R_{k,\delta}(x,q,l) = C_k(x,q,l) - \frac{1}{\varphi(q)}F_{k,\delta}(x,q)$$

and let

$$r_k(\chi) = \inf_\delta \inf\{r : R_{k,\delta}(x,\chi) \ll x^r\},$$

$$r_{k,q,l} = \inf_\delta \inf\{r : R_{k,\delta}(x,q,l) \ll x^r\}.$$

THEOREM 2.2. *We have*

$$r_k(\chi) = \Theta(\chi)$$

for any $k \geq 1$.

Remark. Theorem 2.2 leads us easily to that

$$\max_{l} r_{k,q,l} = \Theta_q$$

by the relations that

$$R_{k,\delta}(x,q,l) = \frac{1}{\varphi(q)} \sum_{\chi (mod q)} \overline{\chi}(l) R_{k,\delta}(x,\chi),$$
$$R_{k,\delta}(x,\chi) = \sum_{l} \chi(l)\, R_{k,\delta}(x,q,l).$$

Proof. From Theorem 1.3, we have

$$R_{k,\delta}(x,\chi) = \frac{1}{2\pi i}\int_{|z|=1} \frac{\Delta_{z,\delta}(x,\chi)}{z^{k+1}} dz \ll \max_{|z|=1} |\Delta_{z,\delta}(x,\chi)| \ll x^{\Theta(\chi)+4\varepsilon}.$$

Hence $r_k(\chi) \leq \Theta(\chi)$.

Conversely,

$$\begin{aligned} C_k(x,\chi) &= \frac{1}{2\pi i}\int_{|z|=1} \frac{A_z(x,\chi)}{z^{k+1}}\, dz \\ &= \frac{1}{2\pi i}\int_{|z|=1} \left(\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} (L(s,\chi))^z f(s,z,\chi)\frac{x^s}{s} ds\right)\frac{1}{z^{k+1}}\, dz \\ &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \left(\frac{1}{2\pi i}\int_{|z|=1} \frac{(L(s,\chi))^z f(s,z,\chi)}{z^{k+1}} dz\right)\frac{x^s}{s}\, ds \\ &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} G_k(s,\chi)\frac{x^s}{s}\, ds, \text{ say.} \end{aligned}$$

Here we have

$$G_k(s,\chi) = \sum_{\ell=0}^{k} \frac{1}{\ell!(k-\ell)!}(\log L(s,\chi))^{\ell} f^{(k-\ell)}(s,0,\chi)$$

where $f^{(n)}(s,z,\chi)$ means the n-th derivative of $f(s,z,\chi)$ with respect to z. It follows that $G_k(s,\chi)$ is regular for $s \in L_{\Theta(\chi)}$, and has the expression

$$G_k(s,\chi)=\sum_{n=1}^{\infty}\frac{c_k(n)\chi(n)}{n^s} \qquad (\sigma>1).$$

Thus

$$F_{k,\delta}(x,q)=\frac{1}{2\pi i}\int_{L_\delta} G_k(s,\chi)\frac{x^s}{s}\,ds.$$

If we suppose $\sigma > 2$, then

$$\begin{aligned} s\int_1^\infty \frac{F_{k,\delta}(x,q)}{x^{s+1}}\,dx &= s\int_1^\infty\Big(\frac{1}{2\pi i}\int_{L_\delta} G_k(\omega,\chi)\frac{x^\omega}{\omega}d\omega\Big)\frac{1}{x^{s+1}}\,dx \\ &= \frac{s}{2\pi i}\int_{L_\delta} G_k(\omega,\chi)\Big(\int_1^\infty x^{\omega-s-1}dx\Big)\frac{d\omega}{\omega} \\ &= \frac{1}{2\pi i}\int_{L_\delta} G_k(\omega,\chi)\frac{d\omega}{\omega}+\frac{1}{2\pi i}\int_{L_\delta} G_k(\omega,\chi)\frac{d\omega}{s-\omega}. \end{aligned}$$

Hence

$$\begin{aligned} G_k(s,\chi) &= E_0\, s\int_1^\infty \frac{C_k(x,\chi)}{x^{s+1}}\,dx \\ &= E_0\, s\int_1^\infty \frac{F_{k,\delta}(x,q)}{x^{s+1}}\,dx + s\int_1^\infty \frac{R_{k,\delta}(x,\chi)}{x^{s+1}}\,dx \\ &= \frac{E_0}{2\pi i}\int_{L_\delta} G_k(\omega,\chi)\frac{d\omega}{\omega} \\ &\quad + \frac{E_0}{2\pi i}\int_{L_\delta} G_k(\omega,\chi)\frac{d\omega}{s-\omega} + s\int_1^\infty \frac{R_{k,\delta}(x,\chi)}{x^{s+1}}\,dx. \end{aligned} \tag{5}$$

Now on the right hand side of (5), the first and the second term can be continued analytically for $s \in L_{1/2}$ as a function of s, while the third term can be continued for $\sigma > r_k(\chi)$, since the involved integral converges uniformly for $\sigma > r_k(\chi)$. Hence the right hand side of (5) is regular for $s \in L_{r_k(\chi_0)}$ in the case for $\chi=\chi_0$, and for $\sigma > r_k(\chi)$ in the case for $\chi \neq \chi_0$.

But $G_k(s,\chi)$ has singularities at zeros $\rho=\beta+i\gamma$, say, of $L(s,\chi)$. In fact, if we consider the limit $G_k(\sigma+i\gamma,\chi)$ as $\sigma \downarrow \beta$, under the assumption that ρ is a zero of order M,

$$
\begin{aligned}
&G_k(\sigma+i\gamma,\chi)\\
&\quad\sim\sum_{\ell=0}^{k}\frac{1}{\ell!(k-\ell)!}(\log L(\sigma+i\gamma,\chi))^{\ell} f^{(k-\ell)}(\sigma+i\gamma,0,\chi)\\
&\quad\sim\sum_{\ell=0}^{k}\frac{1}{\ell!(k-\ell)!}(M\log(\sigma-\beta))^{\ell} f^{(k-\ell)}(\rho,0,\chi)\\
&\quad\sim\sum_{\ell=0}^{k}\frac{1}{\ell!(k-\ell)!}M^{\ell}(-t)^{\ell} f^{(k-\ell)}(\rho,0,\chi) \qquad (\sigma-\beta=e^{-t})\\
&\quad\sim M^k t^k \qquad (t\to\infty),
\end{aligned}
$$

for $f^{(k-\ell)}(\rho,0,\chi)$ is bounded and $f(s,0,\chi)=1$. Hence we conclude that $\Theta(\chi)\le r_k(\chi)$ whether $\chi=\chi_0$, or not, for any $k\ge 1$ since $L(s,\chi_0)$ vanishes neither on the real axis $(\sigma\ge 1/2)$ nor near the point $s=1$.

This completes the theorem.

Now Theorem 2 follows by taking

$$
\begin{aligned}
f(s,z)&=\prod_p(1+\frac{z}{p^s})(1-\frac{1}{p^s})^z &&\textit{for}\quad \pi_k(x,q,l)\\
f(s,z)&=\prod_p(1+\frac{z}{p^s-1})(1-\frac{1}{p^s})^z &&\textit{for}\quad \omega_k(x,q,l)\\
f(s,z)&=\prod_p(1-\frac{z}{p^s})^{-1}(1-\frac{1}{p^s})^z &&\textit{for}\quad \Omega_k(x,q,l)
\end{aligned}
$$

which satisfy the assumptions on $f(s,z)$ at the top of sections 1 and 2.

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