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On Cellular Decompositions of the Boundaries of Handlebodies and Wall's Invariants

Dedicated to Professor Akio Hattori on his sixtieth birthday

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Abstract Cellular decompositions of the boundaries of handlebodies are accomplished in terms of Wall's invariants which determine handlebodies up to diffeomorphism ([12]). The results correspond to a globalization of those of James-Whitehead [10, I].

Introduction

In [6, I, II], the author generalized the James-Whitehead theorems [10, I, II] for sphere bundles over spheres to the theorems for connected sums of sphere bundles over spheres. Those results seem to be globalized furthermore to those for manifolds which are homotopy equivalent to the boundaries of handlebodies since these contain such connected sums naturally as special cases. In fact, the author announced some results in this thought in [7], and gave applications in [8] and [9]. To solve such problems, it is important to know the cellular decompositions of the boundaries of handlebodies. In this paper, we discuss those in the metastable range. We can describe such cellular decompositions in terms of Wall's invariant systems for handlebodies [12], and the results will be used in the subsequent papers. The proofs of Lemma 4, Lemma 5 of [7] are given in this paper. We obtain those from Theorem 4.3 and Proposition 5.4 at once.

Let $W=D^{p+q+1}\cup_{\{f_i\}}(\cup_{i=1}^{r}D_i^q\times D_i^{p+1})$ $(2p>q>1)$ be a handlebody obtained by gluing q-handles smoothly, r in number, to a $(p+q+1)$-disk, and let $\mathscr{H}(p+q+1, r, q)$ be the set of such handlebodies. There exists an invariant system $(H\ ;\ \phi,\ \alpha)$ which determines W up to diffeomorphism (cf. [12]). Here, $H=H_q(W)$, $\phi : H\times H\longrightarrow\pi_q(S^{p+1})$ is a certain bilinear form symmetric up to sign. For the attaching maps f_i, $i=1, 2, \cdots, r$, of handles, ϕ determines $f_i \mid \partial D_i^q\times o$, $i=1, 2, \cdots, r$, that is, the link of attaching spheres up to isotopy. The map $\alpha : H\longrightarrow\pi_{q-1}(SO_{p+1})$ is a quadratic form which assigns, to each $x\epsilon H\cong\pi_q(W)$, the characteristic element of the normal bundle of the imbedded q-sphere representing x.

Let $e_1, e_2, \cdots, e_r$ be the basis of H corresponding to the handles. Then, in Lemma 1.1, we show that ∂W has a cellular decomposition

$$\partial W \simeq (\vee_{i=1}^{r} S_i^p) \cup_{\{\varphi_j\}} (\cup_{j=1}^{r} D_j^q) \cup_{\omega} D^m \quad (m=p+q)$$

and $\varphi_j \in \pi_{q-1}(\vee_{i=1}^{r} S_i^p) = \Sigma_{i=1}^{r} \pi_{q-1}(S_i^p)$ is given by $\varphi_j = \Sigma_{i=1}^{r} \varphi_{ij}$, $\varphi_{ij} = E^{-1}\phi(e_i, e_j)$, $i, j=1, 2, \cdots, r$, where $E : \pi_{q-1}(S^p) \to \pi_q(S^{p+1})$ is the suspension isomorphism $(2p>q>1)$. Under the assumption that $\phi(x, x)=0$ for any $x \in H$, in Theorem 4.3, it is shown that the attaching map ω is determined as follows: Let $\beta_i \in \pi_{q-1}(SO_p)$, $i=1, 2, \cdots, r$, be the elements such that $S\beta_i = \alpha(e_i)$, $i=1, 2, \cdots, r$, where $S : \pi_{q-1}(SO_p) \to \pi_{q-1}(SO_{p+1})$ is the homomorphism induced from the inclusion $SO_p \subset SO_{p+1}$. Such β_i exists since $E\pi_* \alpha(e_i) = \phi(e_i, e_i) = 0$ by Lemma 3 of [12] for $\pi_* : \pi_{q-1}(SO_{p+1}) \longrightarrow \pi_{q-1}(S^p)$ induced from the projection. Then,

$$\omega = \mu + i_*(\iota_p^1 \circ J\beta_1 + \iota_p^2 \circ J\beta_2 + \cdots + \iota_p^r \circ J\beta_r),$$

where i_* is induced from the inclusion $i : \vee_{i=1}^{r} S_i^p \longrightarrow K_\varphi = (\vee_{i=1}^{r} S_i^p) \cup_{\{\varphi_j\}} (\cup_{j=1}^{r} D_j^q)$, $\iota_p^i : S^p \longrightarrow S_i^p \subset \vee_{i=1}^{r} S_i^p$, $i=1, 2, \cdots, r$, are canonical maps, and $J : \pi_{q-1}(SO_p) \longrightarrow \pi_{p+q-1}(S^p)$ is the J-homomorphism. μ is an element which is independent of $\alpha(e_i)$ and the choice of β_i, $i=1, 2, \cdots, r$, and is unique up to isomorphisms of H which preserve ϕ and keep the basis corresponding to the handles (Lemma 5.1). μ is of infinite order if $q \neq p+1$ and is called the *fundamental homotopy class*.

We remember that for an arbitrarily given basis of H, W can be represented to a handlebody again so that the given basis corresponds to the new handles (cf. Theorem 7.6 of [11]). So, in the above cellular decompositions, we have freedom of choice of the basis $e_1, \cdots, e_r$ of H and the elements $\beta_1, \cdots, \beta_r$ such that $S\beta_i = \alpha(e_i)$, $i=1, 2, \cdots, r$.

Let $f : \partial W \longrightarrow \partial W'$ be an orientation preserving homotopy equivalence for W, $W' \in \mathscr{H}(p+q+1, r, q)$ $(2p>q>1)$. To compare the attaching maps of the top dimensional cells of those cellular decompositions as above, in Proposition 5.4, we show that after a suitable choice of representation of W' to a handlebody, $f_*\mu - \mu'$ can be expressed using Whitehead products in $\vee_{i=1}^{r} S_i'^p$, where μ' is the fundamental homotopy class for the cellular decomposition of $\partial W'$.

The assumption that $\phi(x, x)=0$ for any $x \in H$ means that the normal p-sphere bundle of the imbedded q-sphere representing x admits a cross-section for any $x \in H$. So, the above results correspond to a globalization of the cellular decompositions of sphere bundles over spheres which admit cross-sections and those connected sums given by [10, I] and [6, I]. But, differently from such cases, the assumption does not always mean that ∂W is homotopy equivalent to a cellular complex consisting of one point union of spheres and an m-cell $(m=p+q)$. Here is the difficulty to determine the attaching map of the top dimensional cell. For example, if $(p, q)=(n-1, n+1)$ $(n \geq 4)$ or $(n-2, n+1)$ $(n \geq 6)$ and rank $\phi = r$, then the assumption yields a basis of H symplectic with respect to ϕ (cf. [4], [5], [7]). So, by Lemma 1.1, ∂W has a cellular decomposition $\partial W \simeq K \cup D^m$ as the simplest one, where

$$K = \vee_{i=1}^{c} \{(S_{2i-1}^p \cup_\theta D_{2i}^q) \vee (S_{2i}^p \cup_\theta D_{2i-1}^q)\} \quad (r=2c)$$

and θ is the non-trivial element of $\pi_n(S^{n-1})\cong Z_2$ if $(p, q)=(n-1, n+1)$ $(n\geq 4)$ or of $\pi_n(S^{n-2})\cong Z_2$ if $(p, q)=(n-2, n+1)$ $(n\geq 6)$.

§1. Cellular Decompositions

Let $W=D^{m+1}\cup_{\{f_i\}}(\cup_{i=1}^r D_i^q\times D_i^{p+1})$ $(m=p+q)$ be a handlebody of $\mathscr{H}(m+1, r, q)$ $(2p>q>1)$ with the invariant system $(H; \phi, \alpha)$. In this section, we study cellular decompositions of the boundary ∂W and determine the attaching maps of q-cells.

We denote $f_i(S_i^{q-1}\times o)$, $f_i(S_i^{q-1}\times D_i^{p+1})$ respectively by T_i^{q-1}, N_i^m, $i=1, 2, \cdots, r$. Let $X=S^m-\cup_{i=1}^r T_i^{q-1}$ be the complement of the link $T_1^{q-1}\cup T_2^{q-1}\cup\cdots\cup T_r^{q-1}$ in $S^m=\partial D^{m+1}$, and let $Y=S^m-\cup_{i=1}^r \mathrm{Int}\, N_i$. Then, Y is a deformation retract of X, and $\partial W=Y\cup_{\{\bar f_i\}}\{\cup_{i=1}^r D_i^q\times S_i^p\}$, where $\bar f_i=f_i\mid S_i^{q-1}\times S_i^p$, $i=1, 2, \cdots, r$. We can construct the boundary connected sum $N_1\natural N_2\natural\cdots\natural N_r$ in S^m by connecting N_i, N_{i+1} with a thin band b_i for $i=1, 2, \cdots, r-1$. The spheres T_i^{q-1}, $i=1, 2, \cdots, r$, are tied with the paths w_i, $i=1, 2, \cdots, r-1$, suitably chosen so that each w_i goes through the axis of b_i. Then, $N_1\natural\cdots\natural N_r$ is a neighborhood of the connected link $L=T_1^{q-1}\cup w_1\cup T_2^{q-1}\cup w_2\cup\cdots\cup T_{r-1}^{q-1}\cup w_{r-1}\cup T_r^{q-1}$.

Let $\tilde X=S^m-L$ and let $\tilde Y=S^m-\mathrm{Int}(N_1\natural\cdots\natural N_r)$. Then, $\tilde Y$ is a deformation retract of $\tilde X$, and $\tilde X$ is simply connected since $p>1$. By the Alexander duality theorem, we have

$$\tilde H_j(\tilde X)\cong\tilde H^{m-j-1}(L)\cong\tilde H^{m-j-1}(\vee_{i=1}^r S_i^{q-1}),$$

and therefore,

$$H_j(\tilde X)\cong\begin{cases} Z & (j=0)\\ \overbrace{Z+\cdots+Z}^{r} & (j=p)\\ 0, & \text{otherwise.}\end{cases}$$

Thus, $\tilde X$ and $\tilde Y$ have the homotopy type of $\vee_{i=1}^r S_i^p$. In fact, we can define a map e: $\vee_{i=1}^r S_i^p\longrightarrow\tilde Y\subset\tilde X$ so that each $e(S_i^p)$ rounds $f_i(x_i\times S_i^p)$, where $x_i\in S_i^{q-1}$ is a fixed point. Then, $e_*: H_j(\vee_{i=1}^r S_i^p)\longrightarrow H_j(\tilde Y)$ is isomorphic for all j, and therefore, e is a homotopy equivalence since $\tilde Y$ is simply connected.

Identifying S_i^p with $D_i^p/\partial D_i^p$, each handle $D_i^q\times S_i^p$ of $\partial W=Y\cup_{\{\bar f_i\}}(\cup_{i=1}^r D_i^q\times S_i^p)$ is considered as $D_i^q\times y_i\cup D_i^q\times D_i^p$ with identification $(x, y)=(x, y_i)$ for $(x, y)\in D_i^q\times\partial D_i^p$, where $y_i\in S_i^p$ is a fixed point. We denote $D_i^q\times y_i$, $D_i^q\times D_i^p$ simply by D_i^q, D_i^m respectively. Then, we have

$$\partial W=Y\cup(\cup_{i=1}^r D_i^q)\cup(\cup_{i=1}^r D_i^m). \tag{1}$$

On the other hand, $\bar f_i(S_i^{q-1}\times S_i^p)=\partial N_i$, $\bar f_{i+1}(S_{i+1}^{q-1}\times S_{i+1}^p)=\partial N_{i+1}$ are connected by the band b_i for $i=1, 2, \cdots, r-1$. We may assume that those are connected near the points $\bar f_i(x_i, y_i)$, $\bar f_{i+1}(x_{i+1}, y_{i+1})$ but not at those points. Hence, $D_i^m=D_i^q\times D_i^p$, $D_{i+1}^m=D_{i+1}^q\times D_{i+1}^p$ are

connected by b_i in ∂W for $i=1, 2, \cdots, r-1$, that is, we have the boundary connected sum $D^m = D_1^m \natural D_2^m \natural \cdots \natural D_r^m$ in ∂W. Thus, each b_i is excluded from Y and included into D^m (See Fig. 1). Hence, we have

$$\partial W = \tilde{Y} \cup (\cup_{i=1}^r D_i^q) \cup D^m, \tag{2}$$

where each D_i^q is attached to $\tilde{Y}$ by $f_i \mid S_i^{q-1} \times y_i$ $(y_i \in S_i^p)$, $i=1, 2, \cdots, r$. Since $\tilde{Y}$ has the homotopy type of $\vee_{i=1}^r S_i^p$, we have the homotopy equivalence

$$\partial W \simeq (\vee_{i=1}^r S_i^p) \cup (\cup_{j=1}^r D_j^q) \cup D^m, \tag{3}$$

where each D_j^q is attached to $\vee_{i=1}^r S_i^p$ by $d \circ (f_j \mid S_j^{q-1} \times y_j)$ and d is the homotopy inverse of e. We note that if we take $e : \vee_{i=1}^r S_i^p \longrightarrow \tilde{Y}$ as an embedding, then $\tilde{Y}$ is deformed and retracted to $e(\vee_{i=1}^r S_i^p)$ by d.

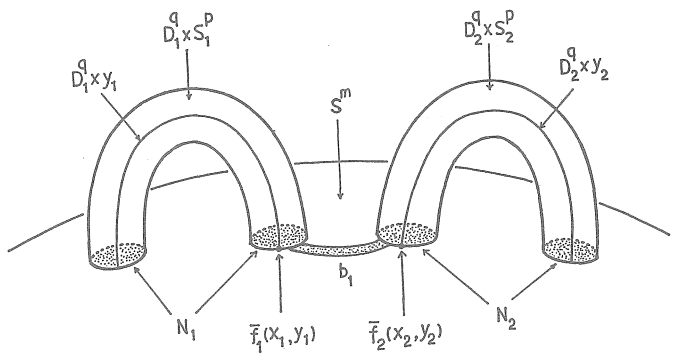

Fig.1 (r=2)

Let's remember the definition of linking elements for a differentiable link defined by Haefliger [2]. Let $T'^{q-1}_j = f_j(S_j^{q-1} \times y_j)$ $(y_j \in S_j^p)$, $j=1, 2, \cdots, r$. For each j, we have a link $T_1^{q-1} \cup \cdots \cup T_j^{q-1} \cup T'^{q-1}_j \cup T_{j+1}^{q-1} \cup \cdots \cup T_r^{q-1}$ of $(q-1)$-spheres, $r+1$ in number, in S^m. Since the maps

$$\vee_{i=1}^r S_i^p \xrightarrow{e} \tilde{Y} \hookrightarrow \tilde{X} \hookrightarrow X$$

induce the isomorphisms of homotopy groups up to dimension $m-3$, each T'^{q-1}_j determines an element $\varphi_j \in \pi_{q-1}(\vee_{i=1}^r S_i^p)$ which is the homotopy class of $d \circ (f_j \mid S_j^{q-1} \times y_j)$. Since $\pi_{q-1}(\vee_{i=1}^r S_i^p) = \sum_{i=1}^r \pi_{q-1}(S_i^p)$ $(2p > q)$ by [3], we have $\varphi_j = \sum_{i=1}^r \varphi_{ij}$ for cetain $\varphi_{ij} \in \pi_{q-1}(S_i^p)$, $i=1, 2, \cdots, r$, where each $\pi_{q-1}(S_i^p)$ is imbedded by the injection induced from the inclusion $S_i^p \subset \vee_{i=1}^r S_i^p$. φ_{ij} is just the linking element of T'^{q-1}_j to T_i^{q-1} of the link $T_i^{q-1} \cup T'^{q-1}_j \subset S^m$. Since T'^{q-1}_j is homotopic to T_j^{q-1} in $S^m - T_i^{q-1}$ if $i \neq j$, φ_{ij} $(i \neq j)$ is equal to the linking element of T_j^{q-1} to T_i^{q-1} of the link $T_i^{q-1} \cup T_j^{q-1} \subset S^m (i \neq j)$. φ_{ii} is the linking element of T'^{q-1}_i to T_i^{q-1} of the link $T_i^{q-1} \cup T'^{q-1}_i \subset S^m$.

Let $e_1, e_2, \cdots, e_r$ be the basis of $H=H_q(W)$ corresponding to $[D_i^q \times o]$, $i=1, 2, \cdots, r$, by the isomorphism $H_q(W) \cong H_q(W, D^{m+1})$. Each e_i is represented by a q-sphere S_i^q with $D_i^q \times o$ as one hemisphere and a disk E_i^q in D^m as the other. $\alpha(e_i)$ is the characteristic element of the normal bundle of S_i^q. Then, since we may assume that T'^{q-1}_i is very close to T_i^{q-1} up to homotopy in $S^m - T_i^{q-1}$, it is easily seen that $\varphi_{ii} = \pi_* \alpha(e_i)$, where $\pi_* : \pi_{q-1}(SO_{p+1}) \longrightarrow \pi_{q-1}(S^p)$ is induced from the projection $\pi : SO_{p+1} \longrightarrow S^p$. On the otherhand, by Lemma 3 and Lemma 7 of [12], we have $\phi(e_i, e_i) = E\pi_* \alpha(e_i)$ and $\phi(e_i, e_j) = E\varphi_{ij}$ $(i \neq j)$, where E is the suspension homomorphism. Hence, we know that $\phi(e_i, e_j) = E\varphi_{ij}$ for $i, j = 1, 2, \cdots, r$. Since $E : \pi_{q-1}(S^p) \longrightarrow \pi_q(S^{p+1})$ is isomorphic $(2p > q)$, we have $\varphi_{ij} = E^{-1}\phi(e_i, e_j)$, $i, j = 1, 2, \cdots, r$, that is, the homotopy classes of the attaching maps of D_j^q, $j=1, 2, \cdots, r$, in (3) are determined by ϕ. Thus, we have

LEMMA 1.1. *Let $W = D^{m+1} \cup_{\{f_i\}} (\cup_{i=1}^r D_i^q \times D_i^{p+1})$ $(m=p+q,\ 2p>q>1)$ be a handlebody with the invariant system $(H\ ;\ \phi,\ \alpha)$, and let $e_1, e_2, \cdots, e_r$ be the basis of H corresponding to the q-handles. Then, the boundary of W has a cellular decomposition*

$$\partial W \simeq (\vee_{i=1}^r S_i^p) \cup (\cup_{j=1}^r D_j^q) \cup D^m$$

up to homotopy equivalence. Each D_j^q is attached by the map $\varphi_j = \sum_{i=1}^r \varphi_{ij} \in \pi_{q-1}(\vee_{i=1}^r S_i^p) = \sum_{i=1}^r \pi_{q-1}(S_i^p)$ which is determined by $\varphi_{ij} = E^{-1}\phi(e_i, e_j)$, $i, j = 1, 2, \cdots, r$.

The above gives a simpler proof of Lemma 1.1 of [6, II] in complete form. Using such cellular decompositions, we can show the following, which means that ϕ is a homotopy invariant of ∂W (cf. §2 of [6, II]).

THEOREM 1.2. *Let W, W' be handlebodies of $\mathscr{H}(p+q+1, r, q)$ $(2p>q>1)$ with the invariant systems $(H\ ;\ \phi,\ \alpha)$, $(H'\ ;\ \phi',\ \alpha')$ respectively and assume that $p \neq q-1$. If $f : \partial W \longrightarrow \partial W'$ is a homotopy equivalence, then $\phi = \phi' \circ (h \times h)$ for the isomorphism $h = k'_* \circ f_* \circ k_*^{-1} : H \longrightarrow H'$, where $k : \partial W \longrightarrow W$, $k' : \partial W' \longrightarrow W'$ are inclusions.*

§2. The attaching map of the top dimensional cell

We continue to discuss cellular decompositions of the boundary of a handlebody $W = D^{m+1} \cup_{\{f_i\}} (\cup_{i=1}^r D_i^q \times D_i^{p+1})$ $(m=p+q,\ 2p>q>1)$ with the invariant system $(H\ ;\ \phi,\ \alpha)$. In this section, we investigate the attaching map of D^m in Lemma 1.1 under the assumption that $\phi(x, x) = 0$ for any $x \in H$. Since $\phi(x, x) = E\pi_* \alpha(x)$ by Lemma 3 of [12], the assumption implies that the normal p-sphere bundle of the imbedded q-sphere representing x admits a cross-section.

We may assume, up to diffeomorphism, that each q-handle $D_i^q \times D_i^{p+1}$ of W is thin enough so that $D_i^q \times D_i^{p+1}$ is just one half of the tubular neighbourhood of $S_i^q = D_i^q \times o \cup E_i^q$

($E_i^q \subset D^m$) which represents $e_i \in H$, where $e_1, \cdots, e_r$ is the basis of H corresponding to the q-handles. Since the trivialization of the tubular neighbourhood of E_i^q is unique up to orientation, we have standard product structures $E_i^q \times E_i^{p+1}$ in D^m and $T_i^{q-1} \times E_i^{p+1}$ in S^m by restricting it to $T_i^{q-1} = \partial E_i^q$, where E_i^{p+1} is a copy of the standard $(p+1)$-disk. Then, $f_i : \partial D_i^q \times D_i^{p+1} \longrightarrow S^m$ is given by $f_i(x, y) = (x, \alpha_i(x)y)$, where $\alpha_i = \alpha(e_i) \in \pi_{q-1}(SO_{p+1})$, $i=1, 2, \cdots, r$, and T_i^{q-1} is identified with $S_i^{q-1} \times o$ by f_i. Since $\pi_* \alpha(e_i) = 0$ by assumption, there exists $\beta_i \in \pi_{q-1}(SO_p)$ such that $S\beta_i = \alpha_i$, $i=1, 2, \cdots, r$, where $S : \pi_{q-1}(SO_p) \longrightarrow \pi_{q-1}(SO_{p+1})$ is induced from the inclusion $SO_p \subset SO_{p+1}$. We take the north pole $(0, \cdots, 0, 1)$ of the p-sphere as y_i. Then y_i is fixed under the operation of $S\beta_i$, and we have a cross-section $\tilde{S}_i^q = D_i^q \times y_i \cup E_i^q \times y_i$ of the normal p-sphere bundle of S_i^q. Hence, we may assume that each $D_i^q = D_i^q \times y_i$ of (1) in §1 is just one half of the cross-section.

Let $S_i^p = D_{+i}^p \cup D_{-i}^p$ be the union of upper and lower hemispheres. Let $T_i^p = \partial E_i^{p+1}$ and let $T_i^p = E_{+i}^p \cup E_{-i}^p$ similarly. Then, the normal p-sphere bundle of S_i^q is divided into two p-disk bundles $\mathscr{B}_i^+, \mathscr{B}_i^-$ which are equivalent by the reflection, where we note that $T_i^{q-1} \times T_i^p = \bar{f}_i(S_i^{q-1} \times S_i^p)$, $\bar{f}_i = f_i \mid S_i^{q-1} \times S_i^p$, apart from the product structures. $\mathscr{B}_i^+$ has $\tilde{S}_i^q$ as the zero cross-section and is given by $D_i^q \times D_{+i}^p \cup E_i^q \times E_{+i}^p$ with the relation $(x, y) = (x, \beta_i(x)y)$ for $x \in \partial D_i^q$, $y \in D_{+i}^p$, where the hemispheres are considered as p-disks. In §1, we have seen that by identifying S_i^p with $D_i^p / \partial D_i^p$ ($y_i = [\partial D_i^p]$), each handle $D_i^q \times S_i^p$ of $\partial W = Y \cup_{\{\bar{f}_i\}} (\cup_{i=1}^r D_i^q \times S_i^p)$ is given by $D_i^q \times y_i \cup D_i^q \times D_i^p$ with identification $(x, y) = (x, y_i)$ for each $(x, y) \in D_i^q \times \partial D_i^p$. The upper hemisphere D_{+i}^p of S_i^p is identified with a collar neighbourhood of ∂D_i^p in D_i^p under the identification $S_i^p = D_i^p / \partial D_i^p$. Let $A_i^p = S_i^{p-1} \times [0, 1]$ be such a collar neighbourhood of $S_i^{p-1} = \partial D_i^p$ suitably chosen in D_i^p. Then, $tv \in D_{+i}^p$ ($v \in S_i^{p-1}$, $0 \leq t \leq 1$) is identified with $[(v, t)] \in A_i^p / S_i^{p-1} \times 0 \subset D_i^p / \partial D_i^p$. Let $B_i^p =$ the closure of $D_i^p - A_i^p$. Then, $D_i^m = D_i^q \times D_i^p = D_i^q \times A_i^p \cup D_i^p \times B_i^p$ and $\partial D_i^m = \partial(D_i^q \times D_i^p) = S_i^{q-1} \times A_i^p \cup S_i^{q-1} \times B_i^p \cup D_i^q \times S_i^{p-1}$. Let $\psi_i : \partial D_i^m \longrightarrow Y \cup (\cup_{i=1}^r D_i^q)$ be the attaching map of D_i^m in (1) of §1. ψ_i maps ∂D_i^m to $T_i^{q-1} \times T_i^p \cup D_i^q \times y_i \subset Y \cup (\cup_{i=1}^r D_i^q)$, where $D_i^q = D_i^q \times y_i$, $i=1, 2, \cdots, r$. On $S_i^{q-1} \times A_i^p \subset \partial D_i^m$, ψ_i corresponds to the pasting map of $\mathscr{B}_i^+ = D_i^q \times D_{+i}^p \cup E_i^q \times E_{+i}^p$, and on $D_i^q \times S_i^{p-1} \subset \partial D_i^m$, corresponds to the identification map of $D_i^q \times y_i \cup D_i^q \times D_i^p$ for the handle $D_i^q \times S_i^p$ (See Fig. 1). Thus, we have

$$\psi_i(x, y) = (x, t(\beta_i(x)v)) \in T_i^{q-1} \times E_{+i}^p \subset T_i^{q-1} \times T_i^p \tag{4}$$

for $(x, y) \in S_i^{q-1} \times A_i^p \subset \partial D_i^m$, $y = (v, t) \in A_i^p$, and

$$\psi_i(x, y) = (x, y_i) \in D_i^q \times y_i \text{ for } (x, y) \in D_i^q \times S_i^{p-1} \subset \partial D_i^m. \tag{5}$$

Let $U_i \supset E_{-i}^p$ be an open neighbourhood of E_{-i}^p in T_i^p and let $u_t^{(i)} : T_i^p \longrightarrow T_i^p$ be a homotopy of the identity map such that $u_1^{(i)}(E_{-i}^p) = z_i = (0, \cdots, 0, -1)$ and $u_t^{(i)}$ fixes $T_i^p - U_i$. Then, $1 \times u_t^{(i)}$ is a homotopy of the identity map of $T_i^{q-1} \times T_i^p = \bar{f}_i(S_i^{q-1} \times S_i^p)$ such that $1 \times u_1^{(i)}(T_i^{q-1} \times E_{-i}^p) = T_i^{q-1} \times z_i$ and $1 \times u_t^{(i)}$ fixes $T_i^{q-1} \times (T_i^p - U_i)$. Let $g_t : Y \longrightarrow Y$ be the homotopy extension of $1 \times u_t^{(1)} \cup 1 \times u_t^{(2)} \cup \cdots \cup 1 \times u_t^{(r)}$ in Y. Then, g_1 is homotopic to 1_Y, the identity map of Y, and $g_1(T_i^{q-1} \times E_{-i}^p) = T_i^{q-1} \times z_i$ and $g_1 = 1_Y$ on $T_i^{q-1} \times (T_i^p - U_i)$ for $i=1$,

2, $\cdots$, r. Since $g_1 : Y \longrightarrow Y$ fixes $T_i^{q-1}\times y_i$, $i=1, 2, \cdots, r$, it is extended to a map $\bar{g}_1 : Y \cup (\cup_{i=1}^r D_i^q) \longrightarrow Y \cup (\cup_{i=1}^r D_i^q)$ by defining $\bar{g}_1 = g_1$ on Y and $\bar{g}_1 =$ identity on D_i^q, $i=1, 2, \cdots, r$. Since B_i^p is considered as the lower hemisphere D_{-i}^p of S_i^p under the identification $S_i^p = D_i^p / \partial D_i^p$, the attaching map ψ_i corresponds to the pasting map of $\mathscr{B}_i^- = D_i^q \times D_{-i}^p \cup E_i^q \times E_{-i}^p$ on $S_i^{q-1}\times B_i^p$. Since $\bar{g}_1(T_i^{q-1}\times E_{-i}^p) = T_i^{q-1}\times z_i$ by construction, we have

$$\bar{g}_1 \circ \psi_i(x, y) = (x, z_i) \in T_i^{q-1}\times T_i^p \quad \text{for } (x, y) \in S_i^{q-1}\times B_i^p \subset \partial D_i^m. \tag{6}$$

Let $p : T_i^{q-1}\times T_i^p \cup D_i^q \times y_i \longrightarrow T_i^p$ be the projection map which is well defined. Then, by (4), (5), (6), and (2. 1) of [10, I], it is easily seen that $p\circ(\bar{g}_1\circ\psi_i)$ represents $\tilde{\iota}_p^i \circ J\beta_i \in \pi_{p+q-1}(T_i^p)$, *where* $\tilde{\iota}_p^i \in \pi_p(T_i^p)$ is the orientation generator.

Let $\chi_i = \bar{g}_1 \circ \psi_i$, $i=1, 2, \cdots, r$. Then, the decomposition (1) in §1 may be replaced by

$$\partial W \simeq Y \cup (\cup_{i=1}^r D_i^q) \cup_{\{\chi_i\}} (\cup_{i=1}^r D_i^m)$$

up to homotopy equivalence. To obtain the decomposition (2), we choose the bands b_i, $i=1, 2, \cdots, r-1$, so that each b_i is enough thin and connects $\partial N_i = T_i^{q-1}\times T_i^p$, $\partial N_{i+1} = T_{i+1}^{q-1}\times T_{i+1}^p$ near the points $f_i(x_i, y_i)$, $f_{i+1}(x_{i+1}, y_{i+1})$ but not at those points, where $x_i \in S_i^{q-1}$, $x_{i+1} \in S_{i+1}^{q-1}$ are base points. Then, $\bar{g}_1$ does not affect the connecting parts and so we have a similar decomposition

$$\partial W \simeq \tilde{Y} \cup (\cup_{i=1}^r D_i^q) \cup D^m,$$

where $D^m = D_1^m \natural D_2^m \natural \cdots \natural D_r^m$ and the attaching map of D^m is given as follows. Let $\tilde{Q}_i = T_i^{q-1}\times T_i^p \cup D_i^q \times y_i$ which is included in $Y \cup (\cup_{i=1}^r D_i^q)$, $i=1, 2, \cdots, r$. Then, each χ_i maps ∂D_i^m into $\tilde{Q}_i$. $\tilde{Q} = \tilde{Q}_1 \# \tilde{Q}_2 \# \cdots \# \tilde{Q}_r$ connected at $T_i^{q-1}\times T_i^p$, $i=1, 2, \cdots, r$, is included in $\tilde{Y} \cup (\cup_{i=1}^r D_i^q)$ and ∂D^m is mapped by $\chi = \chi_1 \# \chi_2 \# \cdots \# \chi_r$ into $\tilde{Q}$. Since each χ_i maps ∂D_i^m into $\tilde{Q}_i$, which is a subspace of $D_i^q \times T_i^p$, we can put χ_i as $\chi_i = (\chi_i^{(1)}, \chi_i^{(2)})$, where $\chi_i^{(1)} : \partial D_i^m \longrightarrow D_i^q$, $\chi_i^{(2)} : \partial D_i^m \longrightarrow T_i^p$, and $\chi_i^{(2)} = p\circ\bar{g}_1\circ\psi_i$. Then, by (4), (5), and (6) and from the above argument, we know

(7) $\chi_i^{(1)}(x, y) = x$ for any $(x, y) \in \partial D_i^m = S_i^{q-1}\times D_i^p \cup D_i^q \times S_i^{p-1}$, and $\chi_i^{(2)}$ represents $\tilde{\iota}_p^i \circ J\beta_i$.

Let $\bar{d} : \tilde{Y} \cup (\cup_{i=1}^r D_i^q) \longrightarrow (\vee_{j=1}^r S_j^p) \cup_{\{\varphi_i\}} (\cup_{i=1}^r D_i^q)$ be the canonical extension of $d : \tilde{Y} \longrightarrow \vee_{j=1}^r S_j^p$, the homotopy inverse of e. Then, $\bar{d}$ is a homotopy equivalence. Let $\tilde{\iota} : \tilde{Q} \longrightarrow \tilde{Y} \cup (\cup_{i=1}^r D_i^q)$ be the inclusion map. From the above, we have

LEMMA 2.1. Let $W = D^{m+1} \cup_{\{f_i\}} (\cup_{i=1}^r D_i^q \times D_i^{p+1})$ $(m = p+q,\ 2p > q > 1)$ *be a handlebody with the invariant system* $(H\ ;\ \phi,\ \alpha)$ *which has the property* $\phi(x, x) = 0$ *for any* $x \in H$. Let $e_1, e_2, \cdots, e_r$ *be the basis of* H *corresponding to the* q*-handles, and choose* $\beta_i \in \pi_{q-1}(SO_p)$ *such that* $S\beta_i = \alpha(e_i)$, $i=1, 2, \cdots, r$. *Then, in the cellular decomposition of Lemma* 1.1, *the attaching map* ω *of* D^m *is given by* $\omega = \bar{d}\circ\tilde{\iota}\circ\chi$, *where* χ *is represented as* $\chi = \chi_1 \# \chi_2 \# \cdots \# \chi_r$ *and each* χ_i *satisfies* (7).

§3. Determination of the map χ_i

Let D^q, S^p be the standard q-disk and the standard p-sphere respectively, and let y_0 be the north pole of S^p. Let $Q_0=S^{q-1}\times S^p\cup D^q\times y_0$ which is a subspace of $D^q\times S^p$, and let $Q_i=S_i^{q-1}\times S_i^p\cup D_i^q\times y_i$, $i=1, 2, \cdots, r$, be the copies of Q_0. Then, as we have seen in §2, each Q_i is embedded in $Y\cup(\cup_{i=1}^r D_i^q)$ as $\tilde{Q}_i=T_i^{q-1}\times T_i^p\cup D_i^q\times y_i$ for $i=1, 2, \cdots, r$. The sum $Q=Q_1\#\cdots\#Q_r$ connected at $S_i^{q-1}\times S_i^p$, $i=1, 2, \cdots, r$, is also embedded in $\tilde{Y}\cup(\cup_{i=1}^r D_i^q)$ as $\tilde{Q}=\tilde{Q}_1\#\cdots\#\tilde{Q}_r$.

Q_0 has the homotopy type of $S^{m-1}\vee S^p$ $(m=p+q)$. Because, the top dimensional cell of $S^{q-1}\times S^p$ is attached to $S^{q-1}\vee S^p=S^{q-1}\times y_0\cup x_0\times S^p(x_0\in S^{q-1})$ by the Whitehead product $[\iota_{q-1}, \iota_p]$, which is trivial here since $S^{q-1}\times y_0$ bounds $D^q\times y_0$. More precisely, we can look it as follows, where we note that Q_0 is homotopy equivalent to $Q_0/D^q\times y_0=S^{q-1}\times S^p/S^{q-1}\times y_0$. Let $\pi: S^{q-1}\times S^p/S^{q-1}\times y_0\longrightarrow S^{q-1}\times S^p/S^{q-1}\vee S^p=S^{m-1}$ be the projection and let $\rho: S^{q-1}\times S^p/S^{q-1}\times y_0\longrightarrow S^p$ be the projection to the second component of the product, which is well defined. Then, by reason of dimensions, we have the following maps and the diagram is commutative up to homotopy:

$$\begin{tikzcd}
 & S^{m-1}\times S^p \\
S^{q-1}\times S^p/S^{q-1}\times y_0 \arrow[ur, "\pi\times\rho"] \arrow[r, "(\pi\times\rho)'"] \arrow[dr, "g"'] & S^{m-1}\times S^p-\{*\} \arrow[u, hook] \\
 & S^{m-1}\vee S^p \arrow[u, hook]
\end{tikzcd}$$

We know that g is a homotopy equivalece $(p, q>1)$. In fact, g induces isomorphisms between the homology groups: Let $i: S^{m-1}\vee S^p\longrightarrow S^{m-1}\times S^p$ be the inclusion map and let $p_1: S^{m-1}\times S^p\longrightarrow S^{m-1}$, $p_2: S^{m-1}\times S^p\longrightarrow S^p$ be projections. Then, $i_*: H_j(S^{m-1}\vee S^p)\longrightarrow H_j(S^{m-1}\times S^p)$ is an isomorphism for $j<2p+q-1$ and the inverse is given by $(p_1)_*\oplus(p_2)_*: H_j(S^{m-1}\times S^p)\longrightarrow H_j(S^{m-1})\oplus H_j(S^p)=H_j(S^{m-1}\vee S^p)$ for $0<j<2p+q-1$. Since $g_*=i_*^{-1}\circ(\pi\times\rho)_*=((p_1)_*\oplus(p_2)_*)\circ(\pi\times\rho)_*=\pi_*\oplus\rho_*$, it is easily seen that $g_*([x_0\times S^p])=[S^p]$ and $g_*([S^{q-1}\times S^p])=[S^{m-1}]$.

Let $h: Q_i\longrightarrow Q_i/D_i^q\times y_i=S_i^{q-1}\times S_i^p/S_i^{q-1}\times y_i$ be the canonical homotopy equivalence. Then, $g\circ h: Q_i\longrightarrow S^{m-1}\vee S^p$ is a homotopy equivalence and hence induce an isomorphism $(g\circ h)_*: \pi_{m-1}(Q_i)\longrightarrow\pi_{m-1}(S^{m-1}\vee S^p)=\pi_{m-1}(S^{m-1})\oplus\pi_{m-1}(S^p)$, where $m=p+q$ and $p>1$ (cf. [3]). Let $\tilde{\iota}_p^i: S^p\longrightarrow x_i\times T_i^p\subset\tilde{Q}_i$, $i=1, 2, \cdots, r$, be canonical homeomorphisms. Here, we denote each fixed point $f_i(x_i, o)$ of $T_i^{q-1}=f_i(S_i^{q-1}\times o)$ by x_i again if there arises no confusion. We may identify each $\tilde{Q}_i=T_i^{q-1}\times T_i^p\cup D_i^q\times y_i$ with Q_i. Then, we have

LEMMA 3.1. *The map* $\chi_i : \partial D_i^m \longrightarrow \tilde{Q}_i$ *represents the element* $\varkappa_i + \tilde{\iota}_p^i \circ J\beta_i \in \pi_{m-1}(\tilde{Q}_i)$ *which corresponds to* $\iota_{m-1} \oplus J\beta_i$ *under the isomorphism* $(g \circ h)_*$, $i=1, 2, \cdots, r$.

Proof. $i_* : \pi_{m-1}(S^{m-1} \vee S^p) \longrightarrow \pi_{m-1}(S^{m-1} \times S^p)$ is an isomorphism with the inverse $(p_1)_* \oplus (p_2)_*$ $(p>1)$, and so $g_* : \pi_{m-1}(S^{q-1} \times S^p / S^{q-1} \times y_0) \longrightarrow \pi_{m-1}(S^{m-1} \vee S^p)$ is given also by $g_* = i_*^{-1} \circ (\pi \times \rho)_* = \pi_* \oplus \rho_*$. Hence, for the composition

$$g \circ h : \tilde{Q}_i \longrightarrow T_i^{q-1} \times T_i^p / T_i^{q-1} \times y_i \longrightarrow S^{m-1} \vee S^p$$

and the map $\chi_i : \partial D_i^m \longrightarrow \tilde{Q}_i$, we have $(g \circ h)_*(\chi_i) = g_*(h \circ \chi_i) = (\pi_* \oplus \rho_*)(h \circ \chi_i) = (\pi \circ h \circ \chi_i) \oplus (\rho \circ h \circ \chi_i)$. Here, $\rho \circ h \circ \chi_i = (\tilde{\iota}_p^i)^{-1} \circ p \circ \chi_i = (\tilde{\iota}_p^i)^{-1} \circ \chi_i^{(2)}$ and represents $J\beta_i$ by (7) of §2.

On the other hand, $\pi \circ h \circ \chi_i : \partial D_i^m \longrightarrow S^{m-1}$ $(m=p+q)$ is a map of degree 1. In fact, $\pi \circ h \circ \chi_i$ maps $D_i^q \times S_i^{p-1}$ of $\partial D_i^m = S_i^{q-1} \times D_i^p \cup D_i^q \times S_i^{p-1}$ to the base point of $S^{m-1}$by (5), where we note that $\bar{g}_1 =$ identity on $D_i^q \times y_i$. On $S_i^{q-1} \times D_i^p$, $\pi \circ h \circ \chi_i$ is given by the composition of the following maps (cf. §2) :

$$S_i^{q-1} \times D_i^p \xrightarrow{\psi_i} T_i^{q-1} \times T_i^p \xrightarrow{1 \times u_1^{(i)}} T_i^{q-1} \times T_i^p$$

$$\xrightarrow{\pi \circ h} T_i^{q-1} \times T_i^p / T_i^{q-1} \vee T_i^p = S^{m-1}$$

Let F be an arc in T_i^p combining y_i and $z_i = -y_i$ along a meridian. Then, $C^q = [T_i^{q-1} \times F]$ in $S^{m-1} = T_i^{q-1} \times T_i^p / T_i^{q-1} \vee T_i^p$ is a q-ball with the boundary $T_i^{q-1} \times z_i$. Let $\tilde{\pi} : S^{m-1} \longrightarrow S^{m-1}/C^q = S^{m-1}$ be the canonical projection, which is a homotopy equivalence. Then, we know that $\tilde{\pi} \circ (\pi \circ h \circ \chi_i)$ maps $S_i^{q-1} \times B_i^p \cup D_i^q \times S_i^{p-1}$ of $\partial D_i^m = S_i^{q-1} \times A_i^p \cup S_i^{q-1} \times B_i^p \cup D_i^q \times S_i^{p-1}$ to the base point of S^{m-1} and maps $S_i^{q-1} \times A_i^p$ to S^{m-1} in degree one way by (4) in §2. Hence, $\tilde{\pi} \circ (\pi \circ h \circ \chi_i)$ and so $\pi \circ h \circ \chi_i$ are maps of degree 1. Thus, $\pi \circ h \circ \chi_i$ represents ι_{m-1}. We define $\varkappa_i$ by $(g \circ h)_*^{-1} \iota_{m-1}$. □

It is easily seen that $\pi_q(\tilde{Q}_i, T_i^{q-1} \times T_i^p) \cong H_q(\tilde{Q}_i, T_i^{q-1} \times T_i^p) \cong Z$ for $p, q > 1$ with the generator $\tilde{\sigma}_q^i$ represented by $D_i^q \times y_i$.

LEMMA 3.2. *Let* $j_* : \pi_{m-1}(\tilde{Q}_i) \longrightarrow \pi_{m-1}(\tilde{Q}_i, T_i^{q-1} \times T_i^p)$ *be the homomorphism induced from the inclusion map. Then, we have* $j_*(\varkappa_i) = [\tilde{\sigma}_q^i, \tilde{\iota}_p^i]$, $i=1, 2, \cdots, r$.

Proof. Let $p : T_i^{q-1} \times T_i^p \longrightarrow T_i^p = S^p$ be the projection. Then, p is extended to a map $\bar{p} : \tilde{Q}_i \longrightarrow S^q \vee S^p$ canonically since p maps $\partial(D_i^q \times y_i) = T^{q-1} \times y_i$ to y_0. Let $\bar{\bar{p}} : (\tilde{Q}_i, T_i^{q-1} \times T_i^p) \longrightarrow (S^q \vee S^p, S^p)$ be the relativization of $\bar{p}$. We have the following diagram :

$$\begin{array}{ccccccc}
\pi_{m-1}(T_i^{q-1}\times T_i^p) & \xrightarrow{i_*} & \pi_{m-1}(\tilde{Q}_i) & \xrightarrow{j_*} & \pi_{m-1}(\tilde{Q}_i\,,\ T_i^{q-1}\times T_i^p) & \xrightarrow{\partial} & \pi_{m-2}(T_i^{q-1}\times T_i^p) \\
\downarrow p_* & & \downarrow \bar{p}_* & & \downarrow \bar{\bar{p}}_* & & \\
0 \longrightarrow \pi_{m-1}(S^p) & \xrightarrow{i_*} & \pi_{m-1}(S^q\vee S^p) & \xrightarrow{j_*} & \pi_{m-1}(S^q\vee S^p,\ S^p) & \longrightarrow & 0
\end{array}$$

As we have seen, there is an isomorphism $(g\circ h)_* : \pi_{m-1}(\tilde{Q}_i)\cong\pi_{m-1}(S^{m-1}\vee S^p)$ and the latter homotopy group is isomorphic to $\pi_{m-1}(S^{m-1})\oplus\pi_{m-1}(S^p)$ $(p>1)$. The homomorphism $i_* : \pi_{m-1}(T_i^{q-1}\times T_i^p)\longrightarrow\pi_{m-1}(\tilde{Q}_i)$ annihilates the direct summand $\pi_{m-1}(T_i^{q-1})$ of $\pi_{m-1}(T_i^{q-1}\times T_i^p)\cong\pi_{m-1}(T_i^{q-1})\oplus\pi_{m-1}(T_i^p)$ and maps the summand $\pi_{m-1}(T_i^p)$ isomorphically to $(g\circ h)_*^{-1}(\pi_{m-1}(S^p))$. So, we have the diagram

$$\begin{array}{ccc}
Z\cong(g\circ h)_*^{-1}(\pi_{m-1}(S^{m-1})) & \xrightarrow[\cong]{j_*} & \mathrm{Im}\,j_* \\
\downarrow \bar{p}_* & & \downarrow \bar{\bar{p}}_* \\
\pi_{m-1}(S^q\vee S^p) & \xrightarrow{j_*} & \pi_{m-1}(S^q\vee S^p,\ S^p)
\end{array}$$

On the other hand, by Lemma 3.1, a map $\chi_i^0 : \partial D^m=D^q\times S^{p-1}\cup S^{q-1}\times D^p\longrightarrow\tilde{Q}_i$ defined similarly to χ_i with trivial β_i represents $\varkappa_i$. That is, let $\chi_i^0(x, y)=(x, y_i)\in D_i^q\times y_i$ if $(x, y)\in D^q\times S^{p-1}$ and let $\chi_i^0(x, y)=(x, \tilde{p}(y))\in T_i^{q-1}\times T_i^p$ if $(x, y)\in S^{q-1}\times D^p$, where $\tilde{p} : D^p\longrightarrow D^p/\partial D^p=T_i^p$ is the canonical projection and we denote $f_i(x, o)$ by x again. Then, it is clear that $\bar{p}\circ\chi_i^0$ represents the Whitehead product $[\iota_q, \iota_p]\in\pi_{m-1}(S^q\vee S^p)$ and so $\bar{p}_*(\varkappa_i)=[\iota_q, \iota_p]$. Since we know that $\pi_{m-1}(S^q\vee S^p)\cong\pi_{m-1}(S^q)\oplus\pi_{m-1}(S^p)\oplus Z[\iota_q, \iota_p]$ $(p, q>1)$ by [3], this means that $j_*\circ\bar{p}_*$ is injective on the subgroup $(g\circ h)_*^{-1}(\pi_{m-1}(S^{m-1}))$. Thus, $\bar{\bar{p}}_*$ must be injective on the subgroup $\mathrm{Im}\,j_*$. Since $\partial[\tilde{\sigma}_q^i, \tilde{\iota}_p^i]=-[\tilde{\iota}_{q-1}^i, \tilde{\iota}_p^i]=0$ in $\pi_{m-2}(T_i^{q-1}\times T_i^p)$, $[\tilde{\sigma}_q^i, \tilde{\iota}_p^i]$ belongs to $\mathrm{Im}\,j_*$, and clearly $\bar{\bar{p}}_*[\tilde{\sigma}_q^i, \tilde{\iota}_p^i]=j_*[\iota_q, \iota_p]$. Thus, we have $j_*(\varkappa_i)=[\tilde{\sigma}_q^i, \tilde{\iota}_p^i]$ since $(\bar{\bar{p}}_*\circ j_*)(\varkappa_i)=(j_*\circ\bar{p}_*)(\varkappa_i)=j_*[\iota_q, \iota_p]$. □

Remark. If the handlebody W in question has only one handle, then $S^m-\mathrm{Int}\,f_1(S_1^{q-1}\times D_1^{p+1})=Y$ is considered as $D^q\times S^p$ up to diffeomorphism and so homotopy equivalent to S^p. Then, $T_1^{q-1}\times T_1^p$ in Y is mapped to S^p by the homotopy equivalence which corresponds to p on $T_1^{q-1}\times T_1^p$. Hence the decomposition $\partial W=Y\cup D_1^q\times y_1\cup_{\chi_1}D^m$ is homotopy equivalent to $(S^p\vee S^q)\cup_{\bar{p}\circ\chi_1}D^m$ and $\bar{p}\circ\chi_1=\bar{p}_*\chi_1=\bar{p}_*(\varkappa_1+\tilde{\iota}_p^1\circ J\beta_1)=\bar{p}_*\varkappa_1+\iota_p\circ J\beta_1$ by Lemma 3.1. Since $\bar{p}_*\varkappa_1=[\iota_q, \iota_p]$ from the above argument, we have $\bar{p}\circ\chi_1=[\iota_q, \iota_p]+\iota_p\circ J\beta_1$ and this coincides with (3.7) of [10, I].

§4. Determination of the attaching map ω

In this section, we determine the attaching maps χ and ω in Lemma 2.1. χ maps ∂D^m to the sum $\tilde{Q}=\tilde{Q}_1\#\cdots\#\tilde{Q}_r$ connected at $T_i^{q-1}\times T_i^p$, $i=1, 2, \cdots, r$, which is included in $\tilde{Y}\cup(\cup_{i=1}^r D_i^q)$. The attaching map ω of D^m in the cellular decomposition of ∂W is given by χ considering the deformation of $\tilde{Y}$ at the same time. In the following arguments, we may assume that $r=2$ without loss of generality.

$Q=Q_1\#Q_2$ connected near $(x_i, y_i)\in S_i^{q-1}\times S_i^p$, $i=1,2$, has the homotopy type of $S^{m-1}\vee(S_1^p\vee S_2^p)$ $(m=p+q)$. In fact, $S_1^{q-1}\times S_1^P \# S_2^{q-1}\times S_2^p$ has the cellular decomposition $(S_1^{q-1}\vee S_1^p)\vee(S_2^{q-1}\vee S_2^p)\cup e^{m-1}$, $m=p+q$, and e^{m-1} is attached by $[\iota_{q-1}^1, \iota_p^1]+[\iota_{q-1}^2, \iota_p^2]$ which is trivial in Q since $S_i^{q-1}\times y_i$ bounds $D_i^q\times y_i$, $i=1, 2$. (cf. [6, I]). More precisely, we can look it as follows. Let $\bar{w}$ be an arc of Q which ties the points (x_1, y_1), (x_2, y_2) passing through the cylinder connecting Q_1 and Q_2. Let $L=(S_1^{q-1}\vee S_1^p)\cup\bar{w}\cup(S_2^{q-1}\vee S_2^p)$ and let $\tilde{L}=(D_1^q\vee S_1^p)\cup\bar{w}\cup(D_2^q\vee S_2^p)$, Then, $Q/\tilde{L}=S_1^{q-1}\times S_1^p\#S_2^{q-1}\times S_2^p/L=S_1^{m-1}\# S_2^{m-1}/\bar{w}=S^{m-1}$ (cf. Fig. 2). Let $M=S_1^{q-1}\times y_1\cup\bar{w}\cup S_2^{q-1}\times y_2$ and let $\tilde{M}=D_1^q\times y_1\cup\bar{w}\cup D_2^q\times y_2$. We have the projection $\pi: Q/\tilde{M}=S_1^{q-1}\times S_1^p \# S_2^{q-1}\times S_2^p/M\longrightarrow S_1^{q-1}\times S_1^p \# S_2^{q-1}\times S_2^p/L=S^{m-1}$. Let $\rho_i: S_i^{q-1}\times S_i^p/S_i^{q-1}\times y_i\longrightarrow S_i^p$, $i=1, 2$, be canonical projections. We define a map $\rho: Q/\tilde{M}\longrightarrow S_1^p\vee S_2^p$ by composing the projection $\varepsilon: Q/\tilde{M}\longrightarrow Q/(\tilde{M}\cup\text{cylinder})$ with $\rho_1\vee\rho_2$, where $Q/(\tilde{M}\cup\text{cylinder})\simeq(S_1^{q-1}\times S_1^p/S_1^{q-1}\times y_1)\vee(S_2^{q-1}\times S_2^p/S_2^{q-1}\times y_2)$.

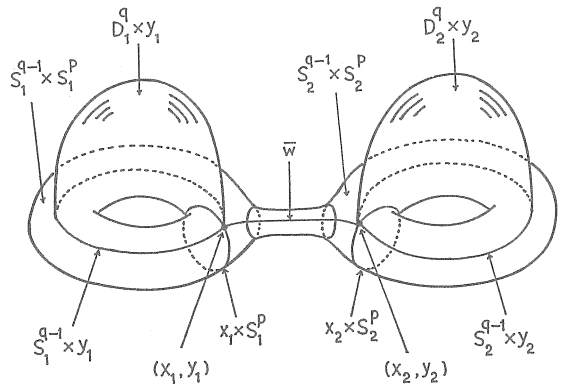

Fig 2 (r=2)

We have the following diagram which is commutative up to homotopy:

$$\begin{tikzcd}
 & S^{m-1}\times(S_1^p\vee S_2^p) \\
Q/\tilde{M} \arrow[ur, "\pi\times\rho"] \arrow[r, "(\pi\times\rho)'"] \arrow[dr, "g"'] & S^{m-1}\times(S_1^p\vee S_2^p)-\{*_1,\ *_2\} \arrow[u, hook] \\
 & S^{m-1}\vee(S_1^p\vee S_2^p) \arrow[u, hook]
\end{tikzcd}$$

Here, $*_i$ is a point suitably chosen from $S^{m-1}\times S_i^p$, $i=1, 2$. Then, by reason of dimensions, $\pi\times\rho$ is homotopic to a map $(\pi\times\rho)'$. $S^{m-1}\times(S_1^p\vee S_2^p)-\{*_1, *_2\}$ is a union of $S^{m-1}\times S_1^p-*_1$ and $S^{m-1}\times S_2^p-*_2$ with intersection $S^{m-1}\times y_0$ $(y_0=y_1=y_2)$. Since those spaces are deformed to $S^{m-1}\vee S_1^p$, $S^{m-1}\vee S_2^p$ respectively with the intersection fixed, we know that $S^{m-1}\vee(S_1^p\vee S_2^p)$ is a deformation retract of $S^{m-1}\times(S_1^p\vee S_2^p)-\{*_1, *_2\}$. So, we have a map $g: Q/\tilde{M}\longrightarrow S^{m-1}\vee(S_1^p\vee S_2^p)$.

We can see that g is a homotopy equivalence as follows: Let $i: S^{m-1}\vee(S_1^p\vee S_2^p)\longrightarrow S^{m-1}\times(S_1^p\vee S_2^p)$ be the inclusion map and let $p_1: S^{m-1}\times(S_1^p\vee S_2^p)\longrightarrow S^{m-1}$, $p_2: S^{m-1}\times(S_1^p\vee S_2^p)\longrightarrow S_1^p\vee S_2^p$ be projections. Then, $i_*: H_j(S^{m-1}\vee(S_1^p\vee S_2^p))\longrightarrow H_j(S^{m-1}\times(S_1^p\vee S_2^p))$ is isomorphic for $j<m+p-1$ and the inverse is given by $(p_1)_*\oplus(p_2)_*$ for $0<j<m+p-1$. Hence, we have $g_*=i_*^{-1}\circ(\pi\times\rho)_*=((p_1)_*\oplus(p_2)_*)\circ(\pi\times\rho)_*=\pi_*\oplus\rho_*$. So, we know that $g_*([x_i\times S_i^p])=[S_i^p]$, $i=1, 2$, and $g_*([S_1^{q-1}\times S_1^p\ \#\ S_2^{q-1}\times S_2^p])=\pi_*([S_1^{q-1}\times S_1^p\ \#\ S_2^{q-1}\times S_2^p])=[S^{m-1}]$. Thus, g is a homotopy equivalence. Let $h: Q\longrightarrow Q/\tilde{M}$ be the canonical homotopy equivalence. Then, $g\circ h: Q\longrightarrow S^{m-1}\vee(S_1^p\vee S_2^p)$ is a homotopy equivalence.

Generally, for $Q=Q_1\#\cdots\#Q_r$ and $S^{m-1}\vee(\vee_{i=1}^r S_i^p)$, we have the homotopy equivalences h, g in a quite similar way. So, we have the homotopy equivalence $g\circ h: Q=Q_1\#\cdots\#Q_r\longrightarrow S^{m-1}\vee(\vee_{i=1}^r S_i^p)$ for any r. We may identify Q with $\tilde{Q}=\tilde{Q}_1\#\cdots\#\tilde{Q}_r$.

LEMMA 4.1. *The map* $\chi: \partial D^m\longrightarrow\tilde{Q}$ *represents an element*

$$\kappa+\tilde{\iota}_p^1\circ J\beta_1+\tilde{\iota}_p^2\circ J\beta_2+\cdots+\tilde{\iota}_p^r\circ J\beta_r$$

which corresponds to

$$\iota_{m-1}+\iota_p^1\circ J\beta_1+\iota_p^2\circ J\beta_2+\cdots+\iota_p^r\circ J\beta_r$$

under the isomorphism

$$(g\circ h)_*: \pi_{m-1}(\tilde{Q})\longrightarrow\pi_{m-1}(S^{m-1}\vee(\vee_{i=1}^r S_i^p))=\pi_{m-1}(S^{m-1})\oplus\pi_{m-1}(\vee_{i=1}^r S_i^p),$$

where $\sum_{i=1}^r\pi_{m-1}(S_i^p)\subset\pi_{m-1}(\vee_{i=1}^r S_i^p)$ *and* $m=p+q$, p, $q>1$.

Proof. We may assume that $r=2$ without loss of generality. Since $i_*: \pi_{m-1}(S^{m-1}\vee(S_1^p\vee S_2^p))\longrightarrow\pi_{m-1}(S^{m-1}\times(S_1^p\vee S_2^p))$ is an isomorphism with the inverse $(p_1)_*\oplus(p_2)_*$ $(p>1)$, $g_*: \pi_{m-1}(\tilde{Q}/\tilde{M})\longrightarrow\pi_{m-1}(S^{m-1}\vee(S_1^p\vee S_2^p))$ is given by $g_*=i_*^{-1}\circ(\pi\times\rho)_*=\pi_*\oplus\rho_*$ similarly to the homology case. So, we have $(g\circ h)_*=(\pi\circ h)_*\oplus(\rho\circ h)_*$. The attaching map $\chi: \partial D^m\longrightarrow\tilde{Q}$ is given by $\chi=\chi_1\#\chi_2$, $\chi_i: \partial D^m\longrightarrow\tilde{Q}_i$, $i=1, 2$, where $D^m=D_1^m\natural D_2^m$ and $\partial D^m=\partial D_1^m\#\partial D_2^m$. Then, we have

$$(\rho\circ h)_*(\chi)=\rho\circ h\circ\chi=(\rho_1\vee\rho_2)\circ\varepsilon\circ h\circ\chi$$
$$=(\rho_1\vee\rho_2)\circ(h_1\circ\chi_1+h_2\circ\chi_2)=\rho_1\circ h_1\circ\chi_1+\rho_2\circ h_2\circ\chi_2,$$

where $h_i:\tilde{Q}_i\longrightarrow\tilde{Q}_i/D_i^q\times y_i=T_i^{q-1}\times T_i^p/T_i^{q-1}\times y_i$, $i=1, 2$, are canonical homotopy equivalences and $\rho_i: T_i^{q-1}\times T_i^p/T_i^{q-1}\times y_i\longrightarrow T_i^p=S_i^p$, $i=1, 2$, are canonical projections. As we have seen in the proof of Lemma 3.1, $\rho_i\circ h_i\circ\chi_i=(\tilde{\iota}_p^i)^{-1}\circ\chi_i^{(2)}$, $i=1, 2$, which represent $J\beta_i$ by (7) of §2. Thus, we have $(\rho\circ h)_*(\chi)=\iota_p^1\circ J\beta_1\oplus\iota_p^2\circ J\beta_2$.

On the other hand, we have $(\pi\circ h)_*\chi=\pi\circ h\circ\chi=\pi\circ h\circ(\chi_1\#\chi_2)$, and $\pi\circ h\circ(\chi_1\#\chi_2)$ corresponds to the connected sum of $\pi_i\circ h_i\circ\chi_i:\partial D_i^m\longrightarrow S_i^{m-1}$, $i=1, 2$, which are of degree 1 as we have seen also in the proof of Lemma 3.1. Here, $\pi_i: T_i^{q-1}\times T_i^p/T_i^{q-1}\times y_i\longrightarrow T_i^{q-1}\times T_i^p/T_i^{q-1}\vee T_i^p=S_i^{m-1}$, $i=1, 2$, are canonical projections. Hence, we know that $(\pi\circ h)_*\chi$ is of degree 1 and so $(\pi\circ h)_*\chi=\iota_{m-1}$. Thus, we have $(g\circ h)_*\chi=\iota_{m-1}\oplus\iota_p^1\circ J\beta_1\oplus\iota_p^2\circ J\beta_2$. We define $\varkappa$ by $(g\circ h)_*^{-1}\iota_{m-1}$. □

It is easily seen that $\pi_q(\tilde{Q}, \#_{i=1}^r T_i^{q-1}\times T_i^p)\cong H_q(\tilde{Q}, \#_{i=1}^r T_i^{q-1}\times T_i^p)\cong Z+\cdots+Z$ $(p>1, q>2)$ with the generators $\tilde{\sigma}_q^i$, $i=1, 2, \cdots, r$, represented by $D_i^q\times y_i$, $i=1, 2, \cdots, r$.

LEMMA 4.2 *Let $j_*:\pi_{m-1}(\tilde{Q})\longrightarrow\pi_{m-1}(\tilde{Q}, \#_{i=1}^r T_i^{q-1}\times T_i^p)$ be the homomorphism induced from the inclusion map. Then, we have*

$$j_*(\varkappa)=[\tilde{\sigma}_q^1, \tilde{\iota}_p^1]+[\tilde{\sigma}_q^2, \tilde{\iota}_p^2]+\cdots+[\tilde{\sigma}_q^r, \tilde{\iota}_p^r],$$

which is of infinite order $(m=p+q, p>1, q>2)$.

Proof. We may assume again that $r=2$ without loss of generality. Let $c:\tilde{Q}\longrightarrow\tilde{Q}_1\vee\tilde{Q}_2$ be the map which crushes the cylinder connecting $\tilde{Q}_1$ and $\tilde{Q}_2$ *and the arc* $\bar{w}$ tying the base points to a point, and let $\bar{c}:(\tilde{Q}, T_1^{q-1}\times T_1^p\ \#\ T_2^{q-1}\times T_2^p)\longrightarrow(\tilde{Q}_1\vee\tilde{Q}_2, T_1^{q-1}\times T_1^p\vee T_2^{q-1}\times T_2^p)$ be the relativization. We have the following commutative diagram:

$$\begin{array}{ccccc}
\pi_{m-1}(T_1^{q-1}\times T_1^p\ \#\ T_2^{q-1}\times T_2^p) & \xrightarrow{i_*} & \pi_{m-1}(\tilde{Q}) & \xrightarrow{j_*} & \pi_{m-1}(\tilde{Q}, T_1^{q-1}\times T_1^p\ \#\ T_2^{q-1}\times T_2^p) \\
\downarrow c_* & & \downarrow c_* & & \downarrow \bar{c}_* \\
\pi_{m-1}(T_1^{q-1}\times T_1^p\vee T_2^{q-1}\times T_2^p) & \xrightarrow{i_*} & \pi_{m-1}(\tilde{Q}_1\vee\tilde{Q}_2) & \xrightarrow{j_*} & \pi_{m-1}(\tilde{Q}_1\vee\tilde{Q}_2, T_1^{q-1}\times T_1^p\vee T_2^{q-1}\times T_2^p) \\
\cup & & \cup & & \cup \\
\pi_{m-1}(T_1^{q-1}\times T_1^p) & & \pi_{m-1}(\tilde{Q}_1) & & \pi_{m-1}(\tilde{Q}_1, T_1^{q-1}\times T_1^p) \\
\oplus & \xrightarrow{(i_1)_*\oplus(i_2)_*} & \oplus & \xrightarrow{(j_1)_*\oplus(j_2)_*} & \oplus \\
\pi_{m-1}(T_2^{q-1}\times T_2^p) & & \pi_{m-1}(\tilde{Q}_2) & & \pi_{m-1}(\tilde{Q}_2, T_2^{q-1}\times T_2^p)
\end{array}$$

In the proof of Lemma 3.2, we have seen that a map $\chi_i^0:\partial D_i^m\longrightarrow\tilde{Q}_i$ with trivial β_i

represents $\varkappa_i$. So, the map $\chi^0=\chi_1^0\#\chi_2^0\colon \partial D^m=\partial(D_1^m\natural D_2^m)\longrightarrow\tilde{Q}_1\#\tilde{Q}_2$ represents $\varkappa$ by Lemma 4.1 since χ^0 has trivial β_1, β_2. So, it is clear that $c_*(\varkappa)=\varkappa_1\oplus\varkappa_2\in\pi_{m-1}(\tilde{Q}_1)\oplus\pi_{m-1}(\tilde{Q}_2)$, and we have $j_*c_*(\varkappa)=j_*(\varkappa_1\oplus\varkappa_2)=(j_1)_*(\varkappa_1)\oplus(j_2)_*(\varkappa_2)=[\tilde{\sigma}_q^1,\tilde{\iota}_p^1]\oplus[\tilde{\sigma}_q^2,\tilde{\iota}_p^2]$ by Lemma 3.2. On the other hand, we have $\partial_*([\tilde{\sigma}_q^1,\tilde{\iota}_p^1]+[\tilde{\sigma}_q^2,\tilde{\iota}_p^2])=-([\tilde{\iota}_{q-1}^1,\tilde{\iota}_p^1]+[\tilde{\iota}_{q-1}^2,\tilde{\iota}_p^2])=0$ for $\partial_*:\pi_{m-1}(\tilde{Q},T_1^{q-1}\times T_1^p\#T_2^{q-1}\times T_2^p)\longrightarrow\pi_{m-2}(T_1^{q-1}\times T_1^p\#T_2^{q-1}\times T_2^p)$ since the top cell of $T_1^{q-1}\times T_1^p\#T_2^{q-1}\times T_2^p$ is attached by $[\tilde{\iota}_{q-1}^1,\tilde{\iota}_p^1]+[\tilde{\iota}_{q-1}^2,\tilde{\iota}_p^2]$. Thus, $[\tilde{\sigma}_q^1,\tilde{\iota}_p^1]+[\tilde{\sigma}_q^2,\tilde{\iota}_p^2]$ belongs to $\mathrm{Ker}\,\partial_*=\mathrm{Im}\,j_*$ and is mapped to $[\tilde{\sigma}_q^1,\tilde{\iota}_p^1]\oplus[\tilde{\sigma}_q^2,\tilde{\iota}_p^2]$ by $\bar{c}_*$. We know that $\pi_{m-1}(\tilde{Q})=(g\circ h)_*^{-1}(\pi_{m-1}(S^{m-1}))\oplus(g\circ h)_*^{-1}(\pi_{m-1}(S_1^p\vee S_2^p))$ and that $\bar{c}_*\circ j_*=j_*\circ c_*$ is injective on the subgroup $(g\circ h)_*^{-1}(\pi_{m-1}(S^{m-1}))$ since $(j_1)_*$, $(j_2)_*$ are injective on the subgroups generated by $\varkappa_1$, $\varkappa_2$, respectively (cf. the proof of Lemma 3.2). So, j_* is injective on $(g\circ h)_*^{-1}(\pi_{m-1}(S^{m-1}))$ and $\bar{c}_*$ is injective on $j_*((g\circ h)_*^{-1}(\pi_{m-1}(S^{m-1})))$. Hence, if we show that $\mathrm{Im}\,j_*=j_*((g\circ h)_*^{-1}(\pi_{m-1}(S^{m-1})))$, that is, $\mathrm{Im}\,i_*=(g\circ h)_*^{-1}(\pi_{m-1}(S_1^p\vee S_2^p))$, then $j_*(\varkappa)=[\tilde{\sigma}_q^1,\tilde{\iota}_p^1]+[\tilde{\sigma}_q^2,\tilde{\iota}_p^2]$ and consequently it must be of infinite order. Because, $\bar{c}_*(j_*\varkappa)=\bar{c}_*([\sigma_q^1,\tilde{\iota}_p^1]+[\tilde{\sigma}_q^2,\tilde{\iota}_p^2])$ and $[\tilde{\sigma}_q^1,\tilde{\iota}_p^1]+[\tilde{\sigma}_q^2,\tilde{\iota}_p^2]$ belongs to $\mathrm{Im}\,j_*$.

We show that $\mathrm{Im}\,i_*=(g\circ h)_*^{-1}(\pi_{m-1}(S_1^p\vee S_2^p))$ for $i_*:\pi_{m-1}(T_1^{q-1}\times T_1^p\#T_2^{q-1}\times T_2^p)\longrightarrow\pi_{m-1}(\tilde{Q})$. In the homotopy exact sepuence of the pair $(T_1^{q-1}\times T_1^p\#T_2^{q-1}\times T_2^p,L)$

$$\pi_{m-1}(L)\xrightarrow{(i_0)_*}\pi_{m-1}(T_1^{q-1}\times T_1^p\#T_2^{q-1}\times T_2^p)$$
$$\xrightarrow{(j_0)_*}\pi_{m-1}(T_1^{q-1}\times T_1^p\#T_2^{q-1}\times T_2^p,L)\xrightarrow{(\partial_0)_*}\pi_{m-2}(L),$$

where $L=(T_1^{q-1}\vee T_1^p)\cup\bar{w}\cup(T_2^{q-1}\vee T_2^p)$, we know that $(\partial_0)_*$ is injective and therefore $(i_0)_*$ is surjective. Because, $\pi_{m-1}(T_1^{q-1}\times T_1^p\#T_2^{q-1}\times T_2^p,L)\cong H_{m-1}(T_1^{q-1}\times T_1^p\#T_2^{q-1}\times T_2^p,L)\cong Z$ $(q>2)$ and $(\partial_0)_*$ maps the positive generator to $[\tilde{\iota}_{q-1}^1,\tilde{\iota}_p^1]\oplus[\tilde{\iota}_{q-1}^2,\tilde{\iota}_p^2]$ which generates an infinite cyclic subgroup of $\pi_{m-2}(L)\cong\pi_{m-2}(T_1^{q-1}\vee T_1^p\vee T_2^{q-1}\vee T_2^p)$. (cf. [3]). we have the diagram

$$\begin{tikzcd}
\pi_{m-1}(T_1^{q-1}\times T_1^p\#T_2^{q-1}\times T_2^p) \arrow[r, "i_*"] & \pi_{m-1}(\tilde{Q}) \arrow[d, "(g\circ h)_*=(\pi\circ h)_*\oplus(\rho\circ h)_*"] \\
\pi_{m-1}(L) \arrow[u, "(i_0)_*"] \arrow[ur, "i'_*"] \arrow[r, "(g\circ h\circ i')_*"'] & \pi_{m-1}(S^{m-1}\vee(S_1^p\vee S_2^p)),
\end{tikzcd}$$

where i' is the composition $i\circ i_0$. From the above, $\mathrm{Im}\,i_*=\mathrm{Im}\,i'_*$, and $(\pi\circ h)_*(\mathrm{Im}\,i_*)=(\pi\circ h)_*(\mathrm{Im}\,i'_*)=0$ since $\pi\circ h$ collapses L to a point. Let $h'\colon L\longrightarrow S_1^p\vee S_2^p$ be the map which collapses $M=T_1^{q-1}\cup\bar{w}\cup T_2^{q-1}$ to the base point. Then, $(\rho\circ h)\circ i'=h'$ and so $(\rho\circ h)_*$

$(\mathrm{Im}\, i'_*)=\mathrm{Im}\, h'_*$. Since h'_* is surjective, we have $(\rho\circ h)_*(\mathrm{Im}\, i'_*)=\pi_{m-1}(S_1^p\vee S_2^p)$. Thus, $(g\circ h)_*(\mathrm{Im}\, i_*)=(g\circ h)_*(\mathrm{Im}\, i'_*)=((\pi\circ h)_*\oplus(\rho\circ h)_*)(\mathrm{Im}\, i'_*)=(\rho\circ h)_*(\mathrm{Im}\, i'_*)=\pi_{m-1}(S_1^p\vee S_2^p)$. □

In the cellular decomposition of Lemma 1.1, let $K_\varphi=(\vee_{i=1}^r S_i^p)\cup_{\{\varphi_j\}}(\cup_{j=1}^r D_j^q)$ for $\varphi=\{\varphi_j\}$. We have $\pi_q(K_\varphi, \vee_{i=1}^r S_i^p)\cong H_q(K_\varphi, \vee_{i=1}^r S_i^p)\cong Z+\cdots+Z$, and $\pi_q(K_\varphi, \vee_{i=1}^r S_i^p)$ has a basis $\{\sigma_q^j\,;\, j=1, 2, \cdots, r\}$ corresponding to $[D_j^q]\in H_q(K_\varphi, \vee_{i=1}^r S_i^p)$, $j=1, 2, \cdots, r$. Let $\iota_p^i: S^p\longrightarrow S_i^p\subset K_\varphi$, $i=1, 2, \cdots, r$, be standard maps. Now, we have the following theorem which is the conclusion of these sections.

THEOREM 4.3. *Let $W=D^{m+1}\cup_{\{f_i\}}(\cup_{i=1}^r D_i^q\times D_i^{p+1})$ $(m=p+q,\ 2p>q>1)$ be a handlebody with the invariant system $(H\,;\,\phi,\,\alpha)$ which has the property $\phi(x, x)=0$ for any $x\in H$.* Let $e_1, e_2, \cdots, e_r$ *be the basis of H corresponding to the q-handles*, and choose $\beta_i\in\pi_{q-1}(SO_p)$ *so that $S\beta_i=\alpha(e_i)$, $i=1, 2, \cdots, r$. Then, there exists a cellular decomposition of ∂W as in Lemma* 1.1 *with the attaching map ω of D^m given by*

$$\omega=\mu+i_*(\iota_p^1\circ J\beta_1+\cdots+\iota_p^r\circ J\beta_r),$$

where $i_*: \pi_{m-1}(\vee_{i=1}^r S_i^p)\longrightarrow\pi_{m-1}(K_\varphi)$ is induced from the inclusion map. μ is an element which is independent of $\alpha(e_i)$ and the choice of β_i, $i=1, 2, \cdots, r$, and satisfies

$$j_*(\mu)=[\sigma_q^1,\ \iota_p^1]+\cdots+[\sigma_q^r,\ \iota_p^r]$$

if $q>2$ for $j_*: \pi_{m-1}(K_\varphi)\longrightarrow\pi_{m-1}(K_\varphi, \vee_{i=1}^r S_i^p)$ induced from the inclusion map. μ is of infinite order if $q\neq p+1$ and is called "the fundamental homotopy class".

Proof. As we have seen in §1, $\partial W=\tilde{Y}\cup(\cup_{j=1}^r D_j^q)\cup D^m$ by (2), and $\tilde{Y}$ has the homotopy type of $\vee_{i=1}^r S_i^p$ by $e: \vee_{i=1}^r S_i^p\longrightarrow\tilde{Y}$ with the homotopy inverse $d: \tilde{Y}\longrightarrow\vee_{i=1}^r S_i^p$. If we take e as an embedding, then $\tilde{Y}$ is deformed and retracted to $e(\vee_{i=1}^r S_i^p)$ by d. Let $\bar{d}: \tilde{Y}\cup(\cup_{j=1}^r D_j^q)\longrightarrow K_\varphi=(\vee_{i=1}^r S_i^p)\cup_{\{\varphi_j\}}(\cup_{j=1}^r D_j^q)$ be the canonical extension of d and let $\bar{\bar{d}}: (\tilde{Y}\cup(\cup_{j=1}^r D_j^q), \tilde{Y})\longrightarrow(K_\varphi, \vee_{i=1}^r S_i^p)$ be the relativization of $\bar{d}$. Then, we have the following commutative diagram :

$$\begin{array}{ccccc}
\pi_{m-1}(\#_{i=1}^r T_i^{q-1}\times T_i^p) & \xrightarrow{i_*} & \pi_{m-1}(\tilde{Q}) & \xrightarrow{j_*} & \pi_{m-1}(\tilde{Q}, \#_{i=1}^r T_i^{q-1}\times T_i^p) \\
\downarrow \ell_* & & \downarrow \tilde{\ell}_* & & \downarrow \tilde{\tilde{\ell}}_* \\
\pi_{m-1}(\tilde{Y}) & \xrightarrow{i_*} & \pi_{m-1}(\tilde{Y}\cup(\cup_{j=1}^r D_j^q)) & \xrightarrow{j_*} & \pi_{m-1}(\tilde{Y}\cup(\cup_{j=1}^r D_j^q), \tilde{Y}) \\
\downarrow d_* & & \downarrow \bar{d}_* & & \downarrow \bar{\bar{d}}_* \\
\pi_{m-1}(\vee_{i=1}^r S_i^p) & \xrightarrow{i_*} & \pi_{m-1}(K_\varphi) & \xrightarrow{j_*} & \pi_{m-1}(K_\varphi, \vee_{i=1}^r S_i^p)
\end{array}$$

Here, upper vertical homomorphisms are induced from inclusion maps.

By Lemma 4.1, we have $\chi=\varkappa+i_*(\tilde{\iota}_p^1\circ J\beta_1+\cdots+\tilde{\iota}_p^r\circ J\beta_r)$. So by Lemma 2.1,

$$\begin{aligned}\omega&=\bar{d}\circ\tilde{\ell}\circ\chi=(\bar{d}\circ\tilde{\ell})_*\chi\\&=(\bar{d}\circ\tilde{\ell})_*\varkappa+(\bar{d}\circ\tilde{\ell}\circ i)_*(\tilde{\iota}_p^1\circ J\beta_1+\cdots+\tilde{\iota}_p^r\circ J\beta_r)\\&=(\bar{d}\circ\tilde{\ell})_*\varkappa+(i\circ d\circ\ell)_*(\tilde{\iota}_p^1\circ J\beta_1+\cdots+\tilde{\iota}_p^r\circ J\beta_r)\\&=(\bar{d}\circ\tilde{\ell})_*\varkappa+i_*(\iota_p^1\circ J\beta_1+\cdots+\iota_p^r\circ J\beta_r)\end{aligned}$$

Let $\mu=(\bar{d}\circ\tilde{\ell})_*\varkappa$. Then, by Lemma 4.1, μ is independent of the values $\alpha(e_i)$, β_i, $i=1, 2, \cdots, r$. Since $j_*(\varkappa)=[\tilde{\sigma}_q^1, \tilde{\iota}_p^1]+\cdots+[\tilde{\sigma}_q^r, \tilde{\iota}_p^r]$ by Lemma 4.2, we have

$$j_*(\mu)=j_*\bar{d}_*\tilde{\ell}_*(\varkappa)=\bar{\bar{d}}_*\tilde{\tilde{\ell}}_*j_*(\varkappa)=[\sigma_q^1, \iota_p^1]+\cdots+[\sigma_q^r, \iota_p^r].$$

Here, $j_*(\mu)$, and hence μ, is of infinite order if $q\neq p+1$ by Theorem II of Blakers-Massey [1]. In fact, the homomorphism defined by the relative Whitehead product

$$\pi_q(K_\varphi, \vee_{i=1}^r S_i^p)\otimes\pi_p(\vee_{i=1}^r S_i^p)\longrightarrow\pi_{p+q-1}(K_\varphi, \vee_{i=1}^r S_i^p)$$

is injective if $q>p+1$. If $q\leq p$, then $K_\varphi\simeq(\vee_{i=1}^r S_i^p)\vee(\vee_{j=1}^r S_j^q)$ since $\varphi=\{\varphi_j\}$ is trivial. Hence, the above homomorphism is clearly injective. □

Remark. If $p\geq q>1$, then ϕ is trivial and so ∂W is a connected sum of p-sphere bundles over q-spheres which admit cross-sections. Hence, the cellular decomposition of ∂W is known by Lemma 1.1 of [6, I] and just

$$\mu=[\iota_q^1, \iota_p^1]+\cdots+[\iota_q^r, \iota_p^r].$$

Theorem 4.3 includes this fact for $p\geq q>2$.

§5. Comparison of fundamental homotopy classes

In this section, we discuss the difference of fundamental homotopy classes $f_*(\mu)-\mu'$ for an orientation preserving homotopy equivalence $f:\partial W\longrightarrow\partial W'$, $W, W'\in\mathscr{H}(m+1, r, q)$ $(m=p+q, 2p>q>1)$.

Let $(H; \phi, \alpha)$, $(H'; \phi', \alpha')$ be the invariant systems of $W=D^{m+1}\cup_{\{f_i\}}(\cup_{i=1}^r D_i^q\times D_i^{p+1})$, $W'=D^{m+1}\cup_{\{f_i'\}}(\cup_{i=1}^r D_i^q\times D_i^{p+1})$ respectively, and assume that $\phi(x, x)=\phi'(x', x')=0$ for all $x\in H$, $x'\in H'$. Let $\{e_1, \cdots, e_r\}$, $\{e_1', \cdots, e_r'\}$ be the bases of H, H' respectively which correspond to the handles of W, W'.

LEMMA 5.1. *If there exists an isomorphism $h: H\longrightarrow H'$ such that $\phi=\phi'\circ(h\times h)$ and $h(e_i)=e_i'$, $i=1, 2, \cdots, r$, then the cellular decompositions of ∂W, $\partial W'$ by Theorem 4.3 satisfy $K_\varphi=K_{\varphi'}$ and $\mu=\mu'$ for those fundamental homotopy classes.*

Proof. Let φ_{ij}, $\varphi_{ij}'(i, j=1, 2, \cdots, r)$ be the linking elements of the links $f_1(S_1^{q-1}\times o)\cup$

$\cdots \cup f_r(S_r^{q-1}\times o)$, $f'_1(S_1^{q-1}\times o)\cup\cdots\cup f'_r(S_r^{q-1}\times o)$ in S^m respectively. By assumption, we have $\phi(e_i, e_j)=\phi'(h(e_i), h(e_j))=\phi'(e'_i, e'_j)$, $i, j=1, 2, \cdots, r$. So, by Lemma 1.1, the cellular decompositions of ∂W, $\partial W'$, $\partial W \simeq K_\varphi \cup_\omega D^m$, $\partial W' \simeq K_{\varphi'} \cup_{\omega'} D^m$ associated to the bases $\{e_1, \cdots, e_r\}$, $\{e'_1, \cdots, e'_r\}$ satisfy $K_\varphi = K_{\varphi'}$, where $K_\varphi = (\vee_{i=1}^r S_i^p)\cup_{\{\varphi_i\}}(\cup_{j=1}^r D_j^q)$, $K_{\varphi'} = (\vee_{i=1}^r S_i^p)\cup_{\{\varphi'_j\}}(\cup_{j=1}^r D_j^q)$. Since $\varphi_j=\varphi'_j$, $j=1, 2, \cdots, r$, the above two links are isotopic by Haefliger [2]. So, we may assume that $f_i \mid S^{q-1}\times o = f'_i \mid S_i^{q-1}\times o$, $i=1, 2, \cdots, r$. Hence, W is different from W' only at the twists of attached handles. But, by Lemma 4.1 and Theorem 4.3, the fundamental homotpy classes are independent of the values $\alpha(e_i)$, $\alpha(e'_i)$, $i=1, 2, \cdots, r$. Hence, we have $\mu=\mu'$. □

Henceforth we assume that there exists an isomorphism $h: H \longrightarrow H'$ such that $h(e_i)=e'_i$, $i=1, 2, \cdots, r$, and $\phi=\phi'\circ(h\times h)$. Then, by Lemma 5.1, there exist cellular decompositions $\partial W \simeq K_\varphi \cup_\omega D^m$, $\partial W' \simeq K_{\varphi'} \cup_{\omega'} D'^m$ such that $K_{\varphi'}$ is just a copy of K_φ. By Theorem 4.3, we have

$$\omega=\mu+i_*(\iota_p^1\circ J\beta_1+\cdots+\iota_p^r\circ J\beta_r)$$
$$\omega'=\mu'+i'_*(\iota'^1_p\circ J\beta'_1+\cdots+\iota'^r_p\circ J\beta'_r),$$

where $\mu=\mu'$ by Lemma 5.1 and β_i, β'_i can be arbitrary elements such that $S\beta_i=\alpha(e_i)$, $S\beta'_i=\alpha'(e'_i)$, $i=1, 2, \cdots, r$.

Let $g_i : \partial D_i^q\times D_i^{p+1}\longrightarrow S^m$, $i=1, 2, \cdots, r$, be imbeddings such that $g_i \mid \partial D_i^q\times o = f_i \mid \partial D_i^q\times o$, $i=1, 2, \cdots, r$, and have no twists as attaching maps of handles. Then, we have a handlebody of $\mathscr{H}(m+1, r, q)$

$$W_0=D^{m+1}\cup_{\{g_i\}}(\cup_{i=1}^r D_i^q\times D_i^{p+1})$$

with the invariant system $(H; \phi, \alpha_0)$, where we note that $(H; \phi)$ is common to that of W and $\alpha_0(e_1)=\alpha_0(e_2)=\cdots=\alpha_0(e_r)=0$. By Theorem 4.3, ∂W_0 has a cellular decomposition $\partial W_0 \simeq K_\varphi \cup_\mu D^m$. Let $W'_0=D^{m+1}\cup_{\{g'_i\}}(\cup_{i=1}^r D'^q_i\times D'^{p+1}_i)$ be the handlebody constructed similarly to W_0 for W'. Since it may be assumed that $f_i \mid S_i^{q-1}\times o = f'_i \mid S_i^{q-1}\times o$, $i=1, 2, \cdots, r$, as is seen in the proof of Lemma 5.1, we have $g_i=g'_i$, $i=1, 2, \cdots, r$. So, W'_0 is a copy of W_0. $\partial W'_0$ has a cellular decomposition $\partial W'_0 \simeq K_{\varphi'} \cup_{\mu'} D'^m$ by Theorem 4.3, which is also a copy of $\partial W_0 \simeq K_\varphi \cup_\mu D^m$.

Let $\mathscr{W}$ be $W_0 \cup W'_0$ with D^{m+1} in common, that is,

$$\mathscr{W}=D^{m+1}\cup_{\{g_i\cup g'_i\}}\{\cup_{i=1}^r(D_i^q\times D_i^{p+1}\cup D'^q_i\times D'^{p+1}_i)\},$$

and let

$$\partial\mathscr{W}=Y\cup_{\{\bar{g}_i\cup\bar{g}'_i\}}\{\cup_{i=1}^r(D_i^q\times S_i^p\cup D'^q_i\times S'^p_i)\},$$

where $Y=S^m-\cup_{i=1}^r \mathrm{Int}\, g_i(S_i^{q-1}\times D_i^{p+1})$ and $\bar{g}_i=g_i \mid S_i^{q-1}\times S_i^p$, $\bar{g}'_i=g'_i \mid S'^{q-1}_i\times S'^p_i$, $i=1, 2, \cdots, r$. Then, $\mathscr{W}$ has doubly attached handles for each i which join at $g_i(S_i^{q-1}\times D_i^{p+1})=g'_i(S'^{q-1}_i\times D'^{p+1}_i)$, and the aspect is similar for $\partial\mathscr{W}$ (cf. Fig. 3).

Now we define a map $\delta : \#_{i=1}^r(S_i^q\times S_i^p)\longrightarrow\partial\mathscr{W}$ as follows: Let $S_i^q=D_i^q\cup(-D_i^q)$, the double of D_i^q attached by the identity map along the bouondary. We consider each $S_i^q\times$

S_i^p as $S_i^q \times S_i^p = D_i^q \times S_i^p \cup (-D_i^q) \times S_i^p$. Define $\delta_i : S_i^q \times S_i^p \longrightarrow D_i^q \times S_i^p \cup D'^q_i \times S'^p_i \subset \partial \mathscr{W}$ by $\delta_i = 1 \cup 1$. That is, δ_i maps $D_i^q \times S_i^p$, $(-D_i^q) \times S_i^p$ identically on $D_i^q \times S_i^p$, $D'^q_i \times S'^p_i$ of $\partial \mathscr{W}$ respectively. In $\#_{i=1}^r (S_i^q \times S_i^p)$, crush each cylinder $C_i = S^{m-1} \times [0, 1]$ connecting $S_i^q \times S_i^p$, $S_{i+1}^q \times S_{i+1}^p$ *to* $I_i = [0, 1]$. Here, we understand that every $S_i^q \times S_i^p$ and $(m-1)$-spheres in it where connected sums with $S_{i-1}^q \times S_{i-1}^p$ and $S_{i+1}^q \times S_{i+1}^p$ are performed have a common base point by joining the base points with arcs and then collapsing those to one point. Then, we have a map

$$\gamma : \#_{i=1}^r S_i^q \times S_i^p \longrightarrow S = S_1^q \times S_1^p \cup I_1 \cup S_2^q \times S_2^p \cup \cdots \cup I_{r-1} \cup S_r^q \times S_r^p,$$

where the base points of $S_i^q \times S_i^p$, $i=1, 2, \cdots, r$, are tied by segments I_i, $i=1, 2, \cdots, r-1$. Let $\delta' : S \longrightarrow \partial \mathscr{W}$ be the map which maps each $S_i^q \times S_i^p$ by δ_i and maps I_i to the path going from $\bar{f}_i(x_i, y_i)$ to $\bar{f}_{i+1}(x_{i+1}, y_{i+1})$ through the boundary of the band b_i which connects the handles $D_i^q \times S_i^p$, $D_{i+1}^q \times S_{i+1}^p$ of ∂W_0 (*cf.* §1). Here, note that $\bar{f}_i(x_i, y_i) = \bar{g}_i(x_i, y_i)$, $i=1, 2, \cdots, r$. Thus, we have $\delta = \delta' \circ \gamma : \#_{i=1}^r (S_i^q \times S_i^p) \longrightarrow \partial \mathscr{W}$.

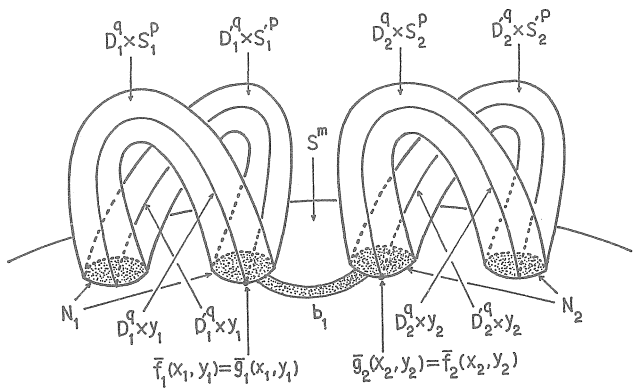

Fig. 3 (r=2)

Let
$$B = Y \cup \{\cup_{i=1}^r (D_i^q \times y_i \cup D'^q_i \times y_i)\},$$
$$\tilde{B} = \tilde{Y} \cup \{\cup_{i=1}^r (D_i^q \times y_i \cup D'^q_i \times y_i)\}.$$

Since b_i, $i=1, 2, \cdots, r$, connect m-celles D'^m_i, $i=1, 2, \cdots, r$, of $\partial W_0'$ also (cf. §1), $\partial \mathscr{W}$ is the union of $\tilde{B}$ and the two m-celles $D^m = D_1^m \natural \cdots \natural D_r^m$, $D'^m = D'^m_1 \natural \cdots \natural D'^m_r$. But, note that this does not provide a cellular decomposition of $\partial \mathscr{W}$ since D^m, D'^m include the bands in common. Let $\mathscr{K}$ be the union $K_\varphi \cup K_{\varphi'}$ with the identification $S_i^p = S'^p_i$, $i=1, 2, \cdots, r$. Hence $\vee_{i=1}^r S_i^p$ of K_φ is identified with $\vee_{i=1}^r S'^p_i$ of $K_{\varphi'}$ in $\mathscr{K}$. Then, $\tilde{B}$ has the homotopy type of $\mathscr{K}$. On the other hand, $\#_{i=1}^r (S_i^q \times S_i^p)$ has the cellular decomposition

$$\#_{i=1}^r (S_i^q \times S_i^p) \simeq \vee_{i=1}^r (S_i^q \vee S_i^p) \cup_\Sigma D^m,$$

where $\Sigma = [\iota_q^1, \iota_p^1] + \cdots + [\iota_q^r, \iota_p^r]$.

Let A be the subspace of S defined by $A = (S_1^q \vee S_1^p) \cup I_1 \cup \cdots \cup I_{r-1} \cup (S_r^q \vee S_r^p)$, and let $\tilde{A}$ be the subspace of $\#_{i=1}^r (S_i^q \times S_i^p)$ defined similarly by $S_i^q \vee S_i^p$, $i=1, 2, \cdots, r$, and $(r-1)$ pathes

which tie them through C_i, $i=1, 2, \cdots, r-1$. Then, $\delta(\tilde{A})\subset\tilde{B}$ since $\gamma(\tilde{A})=A$ and $\delta'(A)\subset\tilde{B}$. Since $\tilde{A}$, $\tilde{B}$ have the homotopy type of $\vee_{i=1}^{r}(S_i^q\vee S_i^p)$, $\mathscr{K}$ respectively, we may assume that $\delta(\vee_{i=1}^{r}(S_i^q\vee S_i^p))\subset\mathscr{K}$ up to homotopy.

LEMMA 5.2. *Let $n: K_\varphi\longrightarrow\mathscr{K}$, $n': K_{\varphi'}\longrightarrow\mathscr{K}$ be inclusion maps and let $\bar{\mu}=n_*\mu$, $\bar{\mu}'=n'_*\mu'$. Then,*

$$\bar{\mu}-\bar{\mu}'=\delta_*([\iota_q^1, \iota_p^1]+\cdots+[\iota_q^r, \iota_p^r])$$

for $\delta_*: \pi_{m-1}(\vee_{i=1}^{r}(S_i^q\vee S_i^p))\longrightarrow\pi_{m-1}(\mathscr{K})$, $m=p+q$. Here, we put the assumption that there exists an isomorphism $h: H\longrightarrow H'$ such that $\phi=\phi'\circ(h\times h)$ and *$h(e_i)\doteq e'_i$, $i=1, 2, \cdots, r$.*

Proof. $(\#_{i=1}^{r}(S_i^q\times S_i^p), \tilde{A})$ has the homotopy type of $(\vee_{i=1}^{r}(S_i^q\vee S_i^p)\cup_\Sigma D^m, \vee_{i=1}^{r}(S_i^q\vee S_i^p))$. Let ν be the orientation generator of $H_m(\#_{i=1}^{r}(S_i^q\times S_i^p), \tilde{A})\cong Z$. (S, A) has the homotopy type of $(\vee_{i=1}^{r}(S_i^q\times S_i^p), \vee_{i=1}^{r}(S_i^q\vee S_i^p))$. Let $\nu_1, \cdots, \nu_r$ be the orientation generators of $H_m(S, A)\cong Z+\cdots+Z$. Then, we have $\bar{\gamma}_*(\nu)=\nu_1\oplus\cdots\oplus\nu_r$, where $\bar{\gamma}: (\#_{i=1}^{r}(S_i^q\times S_i^p), \tilde{A})\longrightarrow(S, A)$ is the relativization of γ.

Let B_0, $\tilde{B}_0$ be the subspaces of $\partial W_0=Y\cup_{\{\bar{g}_i\}}(\cup_{i=1}^{r}D_i^q\times S_i^p)$ defined by $B_0=Y\cup(\cup_{i=1}^{r}D_i^q\times y_i)$, $\tilde{B}_0=\tilde{Y}\cup(\cup_{i=1}^{r}D_i^q\times y_i)$. Similarly, we define B'_0, $\tilde{B}'_0$ for $\partial W'_0$. $(\partial W_0, \tilde{B}_0)$, $(\partial W'_0, \tilde{B}'_0)$ have the homotopy types of $(K_\varphi\cup_\mu D^m, K_\varphi)$, $(K_{\varphi'}\cup_{\mu'}D'^m, K_{\varphi'})$ respectively. We denote the orientation generators of $H_m(\partial W_0, \tilde{B}_0)\cong Z$, $H_m(\partial W'_0, \tilde{B}'_0)\cong Z$ by τ, τ' respectively, and denote the orientation generators of $H_m(\partial W_0, B_0)\cong Z\oplus\cdots\oplus Z$ by $\tau_1, \cdots, \tau_r$ and similarly by $\tau'_1, \cdots, \tau'_r$, for $H_m(\partial W'_0, B'_0)\cong Z\oplus\cdots\oplus Z$. Let $\bar{\tau}$, $\bar{\tau}'$ be the images of τ, τ' by the homomorphisms induced from inclusions $(\partial W_0, \tilde{B}_0)\subset(\partial\mathscr{W}, \tilde{B})$, $(\partial W'_0, \tilde{B}'_0)\subset(\partial W, \tilde{B})$ respectively, and define $\bar{\tau}_i$, $\bar{\tau}'_i$, $i=1, 2, \cdots, r$, similarly by inclusions $(\partial W_0, B_0)\subset(\partial\mathscr{W}, B)$, $(\partial W'_0, B'_0)\subset(\partial\mathscr{W}, B)$ respectively.

We have the commutative diagram

$$\begin{tikzcd}
 & H_m(\partial\mathscr{W}, B) \\
H_m(S, A) \arrow[ur, "\bar{\delta}'_*"] \arrow[dr, "\tilde{\delta}'_*"'] & \\
 & H_m(\partial\mathscr{W}, \tilde{B}) \arrow[uu, "\bar{j}_*"']
\end{tikzcd}$$

where $\bar{\delta}'$ $\tilde{\delta}'$ are relativizations of δ' and $\bar{j}$ is the inclusion map. We know that $\bar{\delta}'_*(\nu_1\oplus\cdots\oplus\nu_r)=(\bar{\tau}_1-\bar{\tau}'_1)\oplus\cdots\oplus(\bar{\tau}_r-\bar{\tau}'_r)$ from the definition of δ' and that $\bar{j}_*(\bar{\tau}-\bar{\tau}')=(\bar{\tau}_1-\bar{\tau}'_1)\oplus\cdots\oplus(\bar{\tau}_r-\bar{\tau}'_r)$ since $\bar{j}_*(\bar{\tau})=\bar{\tau}_1\oplus\cdots\oplus\bar{\tau}_r$, $\bar{j}_*(\bar{\tau}')=\bar{\tau}'_1\oplus\cdots\oplus\bar{\tau}'_r$. So, from the diagram, we have $\tilde{\delta}'_*(\nu_1\oplus\cdots\oplus\nu_r)=\bar{\tau}-\bar{\tau}'$ since $\bar{j}_*$ is injective. In fact, in the homology exact sequence of the triad $(\partial\mathscr{W}, B, \tilde{B})$, we have

$$H_m(B, \tilde{B}) \cong H_m(\cup_{i=1}^{r-1} b_i, \cup_{i=1}^{r-1} \dot{b}_i) = 0$$

by excision, where $\dot{b}_i$ denotes the cylinder of the band b_i for each i. Thus, we have $\bar{\delta}_*(\nu) = \tilde{\delta}'_* \circ \bar{\gamma}_*(\nu) = \tilde{\delta}'_*(\nu_1 \oplus \cdots \oplus \nu_r) = \bar{\tau} - \bar{\tau}'$, where $\bar{\delta}: (\#_{i=1}^r (S_i^q \times S_i^p), \tilde{A}) \longrightarrow (\partial \mathscr{W}, \tilde{B})$ is the relativization of δ and satisfies $\bar{\delta} = \tilde{\delta}' \circ \bar{\gamma}$.

Since Y, $\tilde{Y}$ are simply connected, so are $\partial \mathscr{W}$, $\tilde{B}$ $(p, q > 1)$. We have $H_i(\partial \mathscr{W}, \tilde{B}) \cong H_i(\partial \mathscr{W}/\tilde{B})$ for $i > 0$ and the latter homology group vanishes for $0 < i < m$. Because, if $r = 2$ for example, $\partial \mathscr{W}/\tilde{B}$ has the homotopy type of $S_1^m \cup S_2^m$ with $S_1^m \cap S_2^m = S^{m-1}$. Hence, the Mayer-Vietoris sequence will show it. So, $(\partial \mathscr{W}, \tilde{B})$ is $(m-1)$-connected and $\pi_m(\partial \mathscr{W}, \tilde{B}) \cong H_m(\partial \mathscr{W}, \tilde{B})$ by the relative Hurewicz isomorphism theorem. We have the following commutative diagram:

$$\begin{array}{ccc}
H_m(\#_{i=1}^r (S_i^q \times S_i^p), \tilde{A}) & \xrightarrow{\bar{\delta}_*} & H_m(\partial \mathscr{W}, \tilde{B}) \\
\cong \uparrow \Xi & & \cong \uparrow \Xi \\
\pi_m(\#_{i=1}^r (S_i^q \times S_i^p), \tilde{A}) & \xrightarrow{\bar{\delta}_*} & \pi_m(\partial \mathscr{W}, \tilde{B}) \\
\downarrow \partial & & \downarrow \partial \\
\pi_{m-1}(\tilde{A}) = \pi_{m-1}(\vee_{i=1}^r (S_i^q \vee S_i^p)) & \xrightarrow{\delta_*} & \pi_{m-1}(\tilde{B}) = \pi_{m-1}(\mathscr{K})
\end{array}$$

Since $\bar{\delta}_*(\nu) = \bar{\tau} - \bar{\tau}'$ and $(\partial \circ \Xi^{-1})(\nu) = \Sigma$, we have $\delta_*(\Sigma) = \partial \circ \Xi^{-1}(\bar{\tau} - \bar{\tau}') = \bar{\mu} - \bar{\mu}'$ from the diagram. Thus, $\bar{\mu} - \bar{\mu}' = \delta_*([\iota_q^1, \iota_p^1] + \cdots + [\iota_q^r, \iota_p^r])$. □

If $p < q-1$ or $p > q$, then $H_p(\partial W)$ has the basis $\varepsilon_i = [x_i \times S_i^p]$ $(x_i \in \partial D_i^q)$, $i = 1, 2, \cdots, r$, and $H_q(\partial W)$ has the basis ξ_j, $j = 1, 2, \cdots, r$, which corresponds to $[D_j^q \times o] \in H_q(W, D^{m+1})$, $j = 1, 2, \cdots, r$, under the isomorphisms

$$H_q(\partial W) \cong H_q(W) \cong H_q(W, D^{m+1})$$

induced from inclusion maps. We call $\{\varepsilon_1, \cdots, \varepsilon_r\}$, $\{\xi_1, \cdots, \xi_r\}$ the *bases associated with the handles* of W. We have $k_*(\xi_j) = e_j$, $j = 1, 2, \cdots, r$, for the inclusion map $k: \partial W \longrightarrow W$. Let $\{\varepsilon_1', \cdots, \varepsilon_r'\}$, $\{\xi_1', \cdots, \xi_r'\}$ be the bases associated with the handles of W', and let $k': \partial W' \longrightarrow W'$ be the inclusion map.

For an orientation preserving homotopy equivalence $f: \partial W \longrightarrow \partial W'$, let $\xi_j'' = f_*(\xi_j)$, $e_j'' = k'_*(\xi_j'')$, $j = 1, 2, \cdots, r$. Then, $\{e_1'', \cdots, e_r''\}$ is a basis of H' and we can take another representation of W' to a handlebody using $\{e_1'', \cdots, e_r''\}$ so that e_i'', $i = 1, 2, \cdots, r$, correspond

to the new handles of W' by the basis theorem of handlebodies (cf. [11], Theorem 7.6). Since k'_* is isomorphic and f_* preserves intersection numbers, we have the following lemma, by which we can assume that f_* preserves the bases associated with the the handles of W, W'. (cf. [6, II], Lemma 2.1).

LEMMA 5.3. *Let* $1<p<q-1$ or $p>q>1$ *and let* $f: \partial W \longrightarrow \partial W'$ *be an orientation preserving homotopy equivalence. Then, there exists a representation of* W' *to a handlebody such that* $f_*(\varepsilon_i)=\varepsilon''_i$, $f_*(\xi_j)=\xi''_j$, $i, j=1, 2, \cdots, r$, *where* $\{\varepsilon''_1, \cdots, \varepsilon''_r\}$, $\{\xi''_1, \cdots, \xi''_r\}$ *are the bases associated with the new handles of* W'.

PROPOSITION 5.4. *Let* $1<p<q-1$ *and let* $f: \partial W \longrightarrow \partial W'$ *be an orientation preserving homotopy equivalence which satisfies* $f_*(\varepsilon_i)=\varepsilon'_i$, $f_*(\xi_j)=\xi'_j$, $i, j=1, 2, \cdots, r$. *Then, under cellular decompositions* $\partial W \simeq K_\varphi \cup_\omega D^m$, $\partial W' \simeq K_{\varphi'} \cup_{\omega'} D'^m$ *by Theorem* 4.3, *we have*

$$f_*\mu-\mu'=i'_*([\theta'_1, \iota'^1_p]+\cdots+[\theta'_r, \iota'^r_p])$$

for certain elements $\theta'_j \in \pi_q(\vee_{i=1}^r S'^p_i)$, $j=1, 2, \cdots, r$, *where* $i': \vee_{i=1}^r S'^p_i \longrightarrow K_{\varphi'}$ *is the inclusion map and it is understood that* $f(K_\varphi) \subset K_{\varphi'}$.

Proof. We consider f as a map between the cellular decompositions. It may be assumed that $f(K_\varphi) \subset K_{\varphi'}$ and f is identical on $\vee_{i=1}^r S^p_i$ since $f_*(\varepsilon_i)=\varepsilon'_i$, $i=1, 2, \cdots, r$. Let $\rho': \mathscr{K} \longrightarrow K_{\varphi'}$ be the retraction defined by $\rho' \mid K_\varphi = f$, $\rho' \mid K_{\varphi'} =$ identity. Then,

$$\rho'_*(\bar{\mu}-\bar{\mu}')=\rho'_*(n_*\mu-n'_*\mu')=f_*\mu-\mu'$$

Let $h: H \longrightarrow H'$ be the isomorphism defined by $h=k'_* \circ f_* \circ k_*^{-1}$. Then, we have $\phi=\phi' \circ (h \times h)$ by Theorem 1.2 and $h(e_i)=e'_i$, $i=1, 2, \cdots, r$, since $k_*^{-1}(e_i)=\xi_i$, $k'_*(\xi'_i)=e'_i$, $i=1, 2, \cdots, r$. So, by Lemma 5.2,

$$f_*\mu-\mu'=(\rho' \circ \delta)_*([\iota^1_q, \iota^1_p]+\cdots+[\iota^r_q, \iota^r_p]). \tag{1}$$

we have the following commutative diagram:

$$\begin{tikzcd}
\pi_q(\vee_{i=1}^r S^p_i) \arrow[r, "i_*"] \arrow[dd, "\mathbb{1}"] & \pi_q(K_\varphi) \arrow[r, "j_*"] \arrow[dd, "f_*"] \arrow[dr, "n_*"] & \pi_q(K_\varphi, \vee_{i=1}^r S^p_i) \arrow[dd, "\bar{f}_*"] \arrow[dr, "\bar{n}_*"] & \\
 & & \pi_q(\mathscr{K}) \arrow[rr, "\ell_*"] \arrow[dl, shift right, "\rho'_*"'] \arrow[ul, shift right, "n'_*"'] & & \pi_q(\mathscr{K}, \vee_{i=1}^r S^p_i) \arrow[dl, shift right, "\bar{\rho}'_*"'] \arrow[ul, shift right, "\bar{n}'_*"'] \\
\pi_q(\vee_{i=1}^r S'^p_i) \arrow[r, "i'_*"] & \pi_q(K_{\varphi'}) \arrow[r, "j'_*"] & \pi_q(K_{\varphi'}, \vee_{i=1}^r S'^p_i) &
\end{tikzcd}$$

Here, horizontal maps are all induced from inclusions and $\bar{n}$, $\bar{n}'$, $\bar{\rho}'$, and $\bar{f}$ are relativizations of n, n', ρ', and f respectively.

We know that

$$(\rho'\circ\delta)_*(\iota_p^j)=\rho'_*(\delta_*\iota_p^j)=i'_*\iota'^j_p, \tag{2}$$

$$\ell_*(\delta_*\iota_q^j)=\bar{n}_*\sigma_q^j-\bar{n}'_*\sigma'^j_q$$

from the construction of δ, where $\sigma_q^j\in\pi_q(K_\varphi, \vee_{i=1}^r S_i^p)$, $\sigma'^j_q\in\pi_q(K_{\varphi'}, \vee_{i=1}^r S'^p_i)$ denote the elements represented by D_j^q, D'^q_j in K_φ, $K_{\varphi'}$ respectively. So, we have

$$\begin{aligned} j'_*\circ(\rho'\circ\delta)_*(\iota_q^j)&=(j'_*\circ\rho'_*)(\delta_*\iota_q^j)=(\bar{\rho}'_*\circ\ell_*)(\delta_*\iota_q^j)\\ &=\bar{\rho}'_*(\bar{n}_*\sigma_q^j-\bar{n}'_*\sigma'^j_q)=\bar{f}_*\sigma_q^j-\sigma'^j_q, \end{aligned}$$

from the diagram.

On the other hand, since $f_*(\xi_j)=\xi'_j$, $j=1, 2, \cdots, r$, from the assumption, we know that $\bar{f}_*\sigma_q^j=\sigma'^j_q$, $j=1, 2, \cdots, r$. For, σ_q^j, σ'^j_q correspond to ξ_j, ξ'_j respectively in the following commutative diagram:

$$\begin{array}{ccc}
\pi_q(K_\varphi, \vee_{i=1}^r S_i^p) & \xrightarrow{\bar{f}_*} & \pi_q(K_{\varphi'}, \vee_{i=1}^r S'^p_i)\\
\downarrow\cong & & \downarrow\cong\\
H_q(K_\varphi, \vee_{i=1}^r S_i^p) & \xrightarrow{\bar{f}_*} & H_q(K_{\varphi'}, \vee_{i=1}^r S'^p_i)\\
\uparrow\cong & & \uparrow\cong\\
H_q(K_\varphi) & \xrightarrow{f_*} & H_q(K_{\varphi'})\\
\downarrow\cong & & \downarrow\cong\\
H_q(K_\varphi\cup_\omega D^m) & \xrightarrow{f_*} & H_q(K_{\varphi'}\cup_{\omega'} D'^m)
\end{array}$$

Here, the top isomorphisms are based on the relative Hurewicz isomorphism theorem and the other vertical isomorphisms are induced from inclusions.

Thus, we have $j'_*((\rho'\circ\delta)_*\iota_q^j)=0$, $j=1, 2, \cdots, r$. So, there exist $\theta'_j\in\pi_q(\vee_{i=1}^r S'^p_i)$, $j=1, 2, \cdots, r$, such that

$$(\rho'\circ\delta)_*\iota_q^j=i'_*\theta'_j, \quad j=1, 2, \cdots, r. \tag{3}$$

Hence, by (1), (2), and (3), we have

$$f_*\mu-\mu'=i'_*([\theta'_1\,,\,\iota'^1_p]+\cdots+[\theta'_r\,,\,\iota'^r_p])$$

for cetain elements $\theta'_j \in \pi_q(\vee_{i=1}^r S'^p_i)$, $j=1, 2, \cdots, r$. □

Remark. In Proposition 5.4, if $p>q>1$, then $\phi=\phi'=0$ and so $K_\varphi \simeq (\vee_{i=1}^r S^p_i)\vee(\vee_{j=1}^r S^q_j)$, $K_{\varphi'} \simeq (\vee_{i=1}^r S'^p_i)\vee(\vee_{j=1}^r S'^q_j)$. We have $\mu=\sum_{i=1}^r[\iota^i_q\,,\,\iota^i_p]$, $\mu'=\sum_{i=1}^r[\iota'^i_q\,,\,\iota'^i_p]$ by Theorem 4.3 and the remark. Since $f_*(\varepsilon_i)=\varepsilon'_i$, $f_*(\xi_j)=\xi'_j$, $i, j=1, 2, \cdots, r$, it is easily seen that $f_*(\iota^i_q)=\iota'^i_q$ and there exists $\theta'_i \in \pi_p(\vee_{j=1}^r S'^q_j)$ such that $f_*\iota^i_p=\iota'^i_p+i'_*\theta'_i$, $i=1, 2, \cdots, r$, i': $\vee_{j=1}^r S'^q_j \subset K_{\varphi'}$ (cf. Lemma 2.2 of [6, I]). Therefore, we have a similar formula

$$f_*\mu-\mu'=i'_*([\iota'^i_q,\ \theta'_1]+\cdots+[\iota'^r_q,\ \theta'_r])$$

at once.

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