Sci. Rep. Kanazawa Univ.
Vol. 33 No. 2 pp. 37~43
June 1989

Examples of complete minimal surfaces in $\mathbf{R}^m$ whose Gauss maps omit m(m+1)/2 hyperplanes in general position

By
Hirotaka FUJIMOTO

Department of Mathematics, Faculty of Science, Kanazawa University
(Received October 30, 1988)

Abstract Recently, the author has shown that, for a complete minimal surface M in $\mathbf{R}^m$, if the Gauss map of M is nondegenerate, then G can omit at most $m(m+1)/2$ hyperplanes in general position. We give some examples of minimal surfaces which show that the number $m(m+1)/2$ of the above result is best-possible for arbitrary odd numbers m.

§1. Introduction

Let $x: M \to \mathbf{R}^m$ be a (connected oriented) minimal surface immersed in $\mathbf{R}^m (m \geqq 3)$. Consider the set Π of all oriented 2-planes in $\mathbf{R}^m$. As is well-known, Π is canonically identified with the quadric

$$Q_{m-2}(\mathbf{C}) := \{(w_1 : \cdots : w_m)\,;\, w_1^2 + \cdots + w_m^2 = 0\}$$

in $P^{m-1}(\mathbf{C})$. By definition, the Gauss map of M is the map which maps each point p to the point in Π, or $Q_{m-2}(\mathbf{C})$, corresponding to the oriented tangent plane of M at p. For the case $m=3$, the space $Q_1(\mathbf{C})$ may be identified with the Riemann sphere $P^1(\mathbf{C})$ and the Gauss map of M may be considered as a map into $P^1(\mathbf{C})$. The author has shown that the Gauss map of a complete nonflat minimal surface in $\mathbf{R}^m$ can omit at most four points in $P^1(\mathbf{C})$ ([4]). Moreover, in the previous paper [6] he gave the following theorem.

THEOREM 1. *Let M be a complete minimal surface in $\mathbf{R}^m$ and assume that the Gauss map G is nondegenerate, namely, the image of G is not included in any hyperplane in $P^{m-1}(\mathbf{C})$. Then G can omit at most $m(m+1)/2$ hyperplanes in general position.*

The purpose of this note is to show that, for an arbitrary odd number m, the number $m(m+1)/2$ of Theorem 1 is best-possible, namely, there exist some complete minimal surfaces in $\mathbf{R}^m$ whose Gauss maps are non-degenerate and omit $m(m+1)/2$ hyperplanes in general position. We shall give also such examples for some particular even numbers m.

§2. Preliminaries on minimal surfaces in $\mathbf{R}^m$

Consider a surface M in $\mathbf{R}^m$ immersed by a map $x=(x_1, \dots, x_m): M \to \mathbf{R}^m$, where a surface means a connected and oriented 2-dimensional differentiable manifold. We may consider M as a Riemannian manifold with the metric ds^2 induced from the standard metric of $\mathbf{R}^m$. With each system of positive isothermal local coordinates (u, v) associating a holomorphic local coordinate $z=u+iv$, M may be considered as a Riemann surface with a conformal metric ds^2. The fact that (u, v) are isothermal local coordinates means that they satisfy the condition that

$$\Sigma_{i=1}^k \left(\frac{\partial x_i}{\partial u}\right)^2 = \Sigma_{i=1}^k \left(\frac{\partial x_i}{\partial v}\right)^2, \qquad \Sigma_{i=1}^k \frac{\partial x_i}{\partial u}\frac{\partial x_i}{\partial v}=0.$$

Set $f_j=\partial x_j/\partial z(=(\partial x_j/\partial u - i\ \partial x_j/\partial v)/2)$. The above condition is rewritten as

$$f_1^2+f_2^2+\dots+f_m^2=0. \tag{2. 1}$$

As is well-known, M is a minimal surface in $\mathbf{R}^m$ if and only if each x_i is a harmonic function on M, namely,

$$\frac{\partial^2 x_i}{\partial z \partial \bar{z}}=0, \qquad i=1, 2, \dots, m$$

for an arbitrary holomorphic local coordinate $z=u+iv$. This is equivalent to the condition that f_i is holomorphic on M. To construct minimal surfaces in $\mathbf{R}^m$, the following Proposition is useful.

PROPOSITION 2. *Let M be a simply connected open Riemann surface and let $f_1, f_2, \dots, f_m$ be holomorphic functions on M which have no common zero and satisfy the identity* (2. 1). *Set*

$$x_i(z)=\mathrm{Re}\int_{z_0}^{z} f_i dz, \tag{2. 2}$$

where z_0 is an arbitrarily fixed point of M and the right hand side means the real part of the integral along an arbitrarily chosen continuous curve in M joining z_0 and z. Then, the surface $x=(x_1, \dots, x_m): M \to \mathbf{R}^m$ is a minimal surface in $\mathbf{R}^m$. The induced metric is locally given by

$$ds^2=2(|f_1|^2+\dots+|f_m|^2)|dz|^2. \tag{2. 3}$$

Proof. Since M is simply connected, x_i are well-defined single-valued functions on M and it is easily seen that $\partial x_i/\partial z=f_i (1 \leqq i \leqq m)$. By the assumption of holomorphy of f_i we have

$$(\partial^2/\partial z \partial \bar{z})x_i=(\partial/\partial \bar{z})f_i=0$$

and by the assumption (2. 1) the induced metric is conformal with respect to the complex structure of M. Moreover, the metric is given by

$$ds^2=\Sigma_{i=1}^k \left(\frac{\partial x_i}{\partial u}\right)^2 du^2+\Sigma_{i=1}^k \left(\frac{\partial x_i}{\partial v}\right)^2 dv^2$$

$$=2(|f_1|^2+\ldots+|f_m|^2)|dz|^2.$$

This completes the proof of Proposition 2.

§3. Constructions of minimal surfaces

We shall give the following proposition.

THEOREM 3. *For an arbitrarily given odd number m ($\geqq 3$) there is a complete minimal surface in $\mathbf{R}^m$ whose Gauss map is nondegenerate and omits $m(m+1)/2$ hyperplanes in $P^{m-1}(\mathbf{C})$ located in general position.*

For a given odd number m we set $n:=m-1$ and $k:=n/2$. We first recall an algebraic lemma which was given in the previous paper [6].

LEMMA 4. *Consider $m(m+1)/2$ polynomials*

$$g_i(u):=(u-a_0)^{m-i} \qquad (1\leqq i\leqq m)$$

$$g_{m+i}(u):=(u-a_1)^{m-i}(u-b_1)^{i-1} \qquad (1\leqq i\leqq m)$$

$$\cdots\cdots$$

$$g_{km+i}(u):=(u-a_k)^{m-i}(u-b_k)^{i-1} \qquad (1\leqq i\leqq m),$$

where a_σ, b_τ are mutually distinct complex numbers. These are in general position, namely, arbitrarily chosen m polynomials among them are linearly independent for suitably chosen a_σ and b_τ.

For the proof, see [6], §6.

To prove Theorem 3 we define m entire functions

$$h_{2\ell+1}(z)=e^{\ell z}+e^{(2k-\ell)z} \qquad (0\leqq \ell\leqq k-1)$$

$$h_{2\ell+2}(z)=i(e^{\ell z}-e^{(2k-\ell)z}) \qquad (0\leqq \ell\leqq k-1)$$

and

$$h_{2k+1}=2\sqrt{-k}\ e^{kz}.$$

Next we take suitable constants a_σ and b_τ such that the polynomials $g_i(1\leqq i\leqq q:=m(m+1)/2)$ are in general position. By changing the variable u suitably if necessary, we may assume that $a_0=0$. Set

$$M^*=\mathbf{C}-\{z\,;\,e^z=a_i \text{ or } e^z=b_i \text{ for some } i=1,\ldots,k\}$$

and consider the universal covering surface $\pi: M\to M^*$. Set

$$\psi(z)=\frac{1}{(e^z-a_1)(e^z-b_1)\ldots(e^z-a_k)(e^z-b_k)}$$

and define m holomorphic functions

$$f_i=\psi h_i\ (1\leqq i\leqq m)$$

on M^*. Then we see easily

$$f_1^2+f_2^2+\ldots+f_m^2=0.$$

Without permission, we denote the functions $f_i \cdot \pi$ by the abbreviated notation f_i in the following.

We consider the functions x_i defined by (2. 2) for the above functions f_i. By Proposition 2, the surface $x=(x_1, x_2, \dots, x_m): M \to \mathbf{R}^m$ is a minimal surface. The metric induced from the standard metric on $\mathbf{R}^m$ is given by (2. 3) and the Gauss map of M is equal to the map $f=(f_1 : f_2 : \dots : f_m): M \to P^{m-1}(\mathbf{C})$ and therefore to the map $h=(h_1 : \dots : h_m)$. As is easily seen, a polynomial $P(u)$ vanishes identically if and only if $P(e^z)$ vanishes identically. Since the polynomials

$$P_{2\ell+1}(u)=u^{\ell}+u^{2k-\ell} \qquad (0 \leqq \ell \leqq k-1)$$

$$P_{2\ell+2}(u)=i(u^{\ell}-u^{2k-\ell}) \qquad (0 \leqq \ell \leqq k-1)$$

and

$$P_{2k+1}(u)=2\sqrt{-k}\ u^k.$$

are linearly independent over $\mathbf{C}$, the Gauss map of M is nondegenerate. Moreover, since $P_1, \dots, P_m$ give a basis of the vector space of all polynomials of degree $\leqq m-1$, we can find some constants c_{ij} such that

$$g_i=\Sigma_{j=1}^{m} c_{ij}P_j \qquad (1 \leqq i \leqq q).$$

Now, consider q hyperplanes

$$H_i:\ c_{i1}w_1+\dots+c_{im}w_m=0 \qquad (1 \leqq i \leqq q),$$

which are located in general position because g_i are in general position. Then, the functions

$$g_i(e^z)=\Sigma_{j=1}^{m} c_{ij}P_j(e^z)$$

$$=\Sigma_{j=1}^{m} c_{ij}h_j(z)$$

for $i=1, \dots, q$. Obviously, each $g_i(e^z)$ vanishes nowhere on M. This shows that the Gauss map h of M omits q hyperplanes H_i located in general position. In the next section, we shall prove that the Riemann surface M with the induced metric ds^2 is complete. This will complete the proof of Theorem 3.

§4. The proof of completeness

The purpose of this section is to prove that the minimal surface M in $\mathbf{R}^m$ constructed in the previous section is complete. We use the same notation as in §3.

In our case, the induced metric is induced from the metric

$$ds^2=\frac{\Sigma_{\ell=0}^{k-1}(|e^{\ell z}+e^{(2k-\ell)z}|^2+|e^{\ell z}-e^{(2k-\ell)z}|^2)+4k|e^{kz}|^2}{|(e^z-a_1)(e^z-b_1)\dots(e^z-a_k)(e^z-b_k)|^2}|dz|^2$$

$$=\frac{2\Sigma_{\ell=0}^{k-1}(|e^{\ell z}|^2+|e^{(2k-\ell)z}|^2)+4k|e^{kz}|^2}{|(e^z-a_1)(e^z-b_1)\dots(e^z-a_k)(e^z-b_k)|^2}|dz|^2.$$

on M^* by the projection map of M onto M^*. If M^* is complete, then M is also complete. It suffices to prove that M^* is complete. For the simplicity of notation, we denote the surface M^* by M. We now take a piecewise smooth curve $\gamma(t)$ $(0 \leqq t < 1)$ which tends to the boundary of M, namely, satisfies the condition that, for each compact set K in M, $\gamma(t)$ is not contained in K if t is sufficiently near 1. Our purpose is to show that the length of γ is infinite. The proof is given by reduction to absurdity. Assume that the length of γ is finite.

We first consider the case where there exists a sequence $\{t_i\}$ with $\lim_{i\to\infty} t_i = 1$ such that $\{\gamma(t_i)\}$ has an accumulation point z_0 in $\mathbf{C}$. If $\gamma(t)$ does not tend to z_0 as t tends to 1, then γ is obviously of infinite length. By the assumption, we see $\lim_{t\to 1} \gamma(t) = z_0$. Then, by the assumption we have necessarily $e^{z_0} = a_i$ or $= b_i$ for some i. Then we can write

$$e^z - e^{z_0} = (z - z_0)k(z)$$

with a holomorphic function k on a neighborhood of z_0 with $k(z_0) \neq 0$. Therefore, we can conclude

$$ds^2 \geqq C^2 \frac{1}{|z - z_0|^2} |dz|^2$$

for a positive constant C. This leads to an absurd conclusion

$$\text{the length of } \gamma = \int_\gamma ds \geqq C \int_{\overline{z_1 z_0}} \frac{1}{|z - z_0|} |dz| = \infty,$$

where z_1 is a point sufficiently near z_0 and $\overline{z_1 z_0}$ denotes the line segment between z_1 and z_0. This contradicts the assumption.

Accordingly, we have only to study the case that $\gamma(t)$ tends to ∞ as t tends to 1. Firstly, assume that $\{e^{\gamma(t)}\}$ is bounded. Then there is a positive constant C' such that

$$|(e^z - a_1)(e^z - b_1) \ldots (e^z - b_k)| \leqq C'$$

on the curve γ and so

$$\text{the length of } \gamma = \int_\gamma ds \geqq \frac{1}{C'} \int_\gamma |dz| = \infty,$$

which is impossible by the assumption. Otherwise, there exists a sequence $\{t_i\}$ which tends to 1 such that $\{e^{\gamma(t_i)}\}$ tends to ∞. Set $w := e^z$. Then $|dw| = |w|\,|dz|$ and the metric is given by

$$\begin{aligned} ds^2 &= \frac{2\Sigma_{\ell=0}^{k-1}(|w|^{2\ell} + |w|^{2(2k-\ell)}) + 4k|w|^{2k}}{|(w - a_1)(w - b_1) \ldots (w - a_k)(w - b_k)|^2} \frac{|dw|^2}{|w|^2} \\ &\geqq \frac{4k}{|(1 - a_1 w^{-1})(1 - b_1 w^{-1}) \ldots (1 - a_k w^{-1})(1 - b_k w^{-1})|^2} \frac{|dw|^2}{|w|^2}. \end{aligned}$$

Consider the curve

$$\gamma' : w(t) = e^{\gamma(t)}.$$

We have

$$\int_\gamma ds \geqq C_o \int_{\gamma'} \frac{|dw|}{|w|} = \infty.$$

for a positive constant C_o. Thus the proof of Theorem 3 is completed.

§5. Concluding remarks

In case that the dimension m is even, we can conclude the same conclusion of Theorem 3 for some particular cases. For an arbitrary even number m set $k := m/2$. In this case we use entire functions

$$h_{2\ell+1} = e^{\ell z} + e^{(2k-\ell-1)z} \qquad (0 \leqq \ell \leqq k-1)$$

and

$$h_{2\ell+2} = i(e^{\ell z} - e^{(2k-\ell-1)z}) \qquad (0 \leqq \ell \leqq k-1).$$

Instead of Lemma 4 we use the following conjecture, which was not yet proved for general cases but for $m \leqq 16$([6], §6).

CONJECTURE. *Set $k := m/2$ for an arbitrarily given even number m. Then $3k$ polynomials*

$$g_i(u) := u^{i-1} \qquad (1 \leqq i \leqq k)$$

$$g_i(u) := (u-1)^{i-1} \qquad (k+1 \leqq i \leqq 2k)$$

$$g_i(u) := u^{i-k-1}(u-1)^{m-i+k} \qquad (2k+1 \leqq i \leqq 3k)$$

are in general position.

If the above conjecture is true for an even number m, then we can show that there exist m distinct constants $a_1 := 0$, $b_1 := 1$, $a_2, b_2, \ldots, a_k, b_k$ such that, for further polynomials

$$g_{3k+i}(u) := (u-a_2)^{m-i}(u-b_2)^{i-1} \qquad (1 \leqq i \leqq m)$$

$$g_{3k+2k(k-2)+i}(u) := (u-a_k)^{m-i}(u-b_k)^{i-1} \qquad (1 \leqq i \leqq m),$$

$g_1, g_2, \ldots, g_q$ are in general position.

As in the previous section, taking constants a_σ and b_τ satisfying the above condition, we consider the universal covering surface M of the set

$$M^* = \mathbf{C} - \{z \,;\, e^z = a_i \text{ or } e^z = b_i \text{ for some } i = 1, \ldots, k\}$$

and, using the function

$$\psi = \frac{1}{(e^z-1)(e^z-a_2)(e^z-b_2)\ldots(e^z-a_k)(e^z-b_k)}$$

we define m holomorphic functions

$$f_i = \psi h_i \quad (1 \leqq i \leqq m)$$

on M^*. Then, by the similar manner as in the previous sections we can prove that for the functions x_i defined by (2. 2) the surface $x=(x_1, x_2, \ldots, x_m): M \to \mathbf{R}^m$ is a complete minimal surface whose Gauss map omits $m(m+1)/2$ hyperplanes in general position.

Concludingly, if m ($\geqq 3$) is odd or the above conjecture is valid for an even number m, then the number $m(m+1)/2$ of Theorem 1 is best-possible.

References

[1] S. S. Chern and R. Osserman, Complete minimal surfaces in euclidean n-spece, J. Analyse Math., **19**(1967), 15-34.

[2] H. Fujimoto, On the Gauss map of a complete minimal surface in $\mathbf{R}^m$, J. Math. Soc. Japan, **35**(1983), 279-288.

[3] H. Fujimoto, Value distribution of the Gauss map of complete minimal surfaces in $\mathbf{R}^m$, J. Math. Soc. Japan, **35**(1983), 663-681.

[4] H. Fujimoto, On the number of exceptional values of the Gauss map of minimal surfaces, Math. Soc. Japan **40**(1988), 235-247.

[5] H. Fujimoto, Modified defect relations for the Gauss map of minimal surfaces, to appear in J. Diff. Geometry.

[6] H. Fujimoto, Modified defect relations for the Gauss map of minimal surfaces, II, to appear in J. Diff. Georetry.

[7] R. Osserman, Minimal surfaces in the large, Comm. Math. Helv., **35**(1961), 65-76.

[8] R. Osserman, Global properties of minimal surfaces in E^3 and E^n, Ann. of Math., **80**(1964), 340-364.

[9] R. Osserman, A survey of minimal surfaces, 2nd edition, Dover, New York 1986.

[10] F. Xavier, The Gauss map of a complete non-flat minimal surface cannot omit 7 points of the sphere, Ann. of Math., **113**(1981), 211-214.