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An Asymptotic Behavior of $\{f(n_k t)\}$

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Abstract Let $f(t) \in \mathrm{Lip}\ \delta$ $(\delta > 1/2)$ and $f(t+1) = f(t)$. Then if $\{n_k\}$ satisfies $n_{k+1}/n_k > 1 + ck^{-\alpha}$ $(c > 0$ and $0 \leqq \alpha < 1/2)$, the law of the iterated logarithms for $\{f(n_k t)\}$ is studied.

1. Introduction Let $f(t)$ be a real valued Lebesgue measurable function on $(-\infty, +\infty)$ satisfying the conditions

$$f(t+1) = f(t), \quad \int_0^1 f(t)dt = 0 \quad \text{and} \quad \int_0^1 f^2(t)dt < +\infty,$$

and $\{n_k\}$ be an increasing sequence of positive integers. Then it is well known that the sequence of functions $\{f(n_k t)\}$, although themselves not independent, exhibits the properties of independent random variables.

In [3] we proved that if $f \in \mathrm{Lip}\ \delta$ $(\delta > 0)$ and

(1. 1) $$n_{k+1}/n_k > 1 + c \quad (c > 0 \quad \text{and} \quad k \geqq 1),$$

then we have

(1. 2) $$\overline{\lim_{N\to\infty}} (N \log\log N)^{-1/2} \sum_{k=1}^{N} f(n_k t) \leqq C, \quad \text{a. e. } t,$$

where C is a constant depending on f and c in (1. 1).

Recently, Dhompongsa [1] showed that if $f \in \mathrm{Lip}\ \delta$ $(\delta > 1/2)$ and $\{n_k\}$ satisfies the gap condition

(1. 3) $$\begin{cases} n_{k+1}/n_k > 1 + ck^{-\alpha}, \ k \geqq 1, \\ \text{for some } c > 0 \text{ and } 0 < \alpha < 1/2, \end{cases}$$

then (1. 2) holds for some constant $C > 0$.

The purpose of the present note is to prove the

THEOREM. *If* $f \in \mathrm{Lip}\ \delta$ $(\delta > 1/2)$ *and* $\{n_k\}$ *satisfies* (1. 3), then we have

$$\overline{\lim_{N\to\infty}} (N \log\log N)^{-1/2} \sum_{k=1}^{N} f(n_k t) \leqq \| f \|, \quad a. e. \ t,$$

where $f\sim\sum_{h=1}^{\infty} a_h\cos 2\pi h(t+\alpha_h)$, $a_h\geqq 0$, *and* $\|f\|=\sum_{h=1}^{\infty} a_h$.

2. Some Lemmas From now on let us assume that f belongs to the class Lip δ ($\delta>1/2$) and $\{n_k\}$ satisfies the gap condition (1. 3).

i. If $f(t)\sim\sum_{h=1}^{\infty} a_h\cos 2\pi h(t+\alpha_h)$, $a_h\geqq 0$, then the following facts are well known (c f. [5] , Vol. 1);

(2. 1) $$\sum_{h=1}^{\infty} a_h\log(h+1)<+\infty$$

and

(2. 2) $$|f(t)-\sum_{h=1}^{n} a_h\cos 2\pi h(t+\alpha_h)|=O(n^{-\delta}\log n)$$
$$=o(n^{-1/2}),\qquad \text{uniformly in } t,\quad \text{as } n\to+\infty.$$

For simplicity of writing the formulas we consider only cosine series. The general case follows the same lines.

ii. Let us put

$$p(0)=0 \quad\text{and}\quad p(k)=\max\{m\,;\, n_m<2^k\} \qquad \text{for } k\geqq 1.$$

If $p(k)+1<p(k+1)$, then we have

$$2 > n_{p(k+1)}/n_{p(k)+1} > \prod_{m=p(k)+1}^{p(k+1)-1}(1+cm^{-\alpha})$$
$$> 1+c\{p(k+1)-p(k)-1\}p^{-\alpha}(k+1).$$

Therefore, we have

(2. 3) $p(k+1)-p(k)=O(p^{\alpha}(k))$, as $k\to\infty$.

Further, the following lemmas are proved (cf. [4]).

LEMMA 1. *For any given integers k, j, q and h satisfying*

$$p(j)+1<h\leqq p(j+1)<p(k)+1<q\leqq p(k+1),$$

the number of solutions (n_r, n_i) of the equation

$$n_q-n_r=n_h-n_i,$$

where $p(j)<i<h$ and $p(k)<r<q$, is at most $C2^{j-k}p^{\alpha}(k)$, where C is a positive constant independent of k, j, q and h.

LEMMA 2. *For any given integers k, j, q and h satisfying*

$$p(j+1)<h\leqq p(j+2)<p(k+1)<q\leqq p(k+2),$$

the number of solutions (n_r, n_i) of the equation

$$n_q-n_r=n_h-n_i,$$

where $p(j)<i\leqq p(j+1)$ *and* $p(k)<r\leqq p(k+1)$, *is at most* $C2^{j-k}p^{\alpha}(k)$, *where* C *is a positive constant independent of* k, j, q *and* h.

iii. Let β be a positive constant satisfying

(2. 4) $\quad 0 < 1/(1-\alpha) < \beta < 2,$

and $\{q(k)\}$ be a sequence of integers such that

(2. 5) $\quad p(q(k)-1) \leqq k^{\beta} < p(q(k)).$

Further, we put

$$(2.6)\qquad \begin{cases} \Delta_m(t)=\sum_{j=p(m)+1}^{p(m+1)} \cos 2\pi n_j t, \\ Q_k(t)=\sum_{m=q(k-1)}^{q(k)-2} \Delta_m(t). \end{cases}$$

Then we have, by (2. 4) and (2. 5),

$$(2.7)\qquad k^{\beta}-p(q(k)-1)\leqq p(q(k))-p(q(k)-1)$$
$$=O(p^{\alpha}(q(k))=O(k^{\alpha\beta})=o(k^{\beta-1}), \qquad \text{as } k\to+\infty.$$

LEMMA 3. *We have*

$$\left\| \sum_{k=1}^{N} (Q_k^2-\|Q_k\|_2^2)\right\|_2^2=O(N^{3\beta-2}), \qquad \text{as } N\to+\infty.$$

PROOF. We have

$$Q_k^2-\|Q_k\|_2^2=\sum_{m=q(k-1)}^{q(k)-2} (\Delta_m^2-\|\Delta_m\|_2^2)$$
$$+\ 2\sum_{m=q(k-1)+3}^{q(k)-2} \Delta_m \sum_{j=q(k-1)}^{m-2} \Delta_j \ +\ 2\sum_{m=q(k-1)+1}^{q(k)-2} \Delta_m\Delta_{m-1}.$$

Since for each r, $0\leqq r\leqq 2$, the functions $(\Delta_{3m+r}\sum_{j=q(k-1)}^{3m+r-2}\Delta_j)$, $q(k-1)+2<3m+r<q(k)-1$ and $k=1, 2, \cdots$, are orthogonal and by (2. 5) and (2. 7),

$$(2.8)\qquad \left\|\sum_{j=q(k-1)}^{q(k)-2} |\Delta_j| \right\|_{\infty}\leqq p(q(k)-1)-p(q(k-1))$$
$$=O(k^{\beta-1}), \qquad \text{as } k\to+\infty,$$

we have

$$\left\|\sum_{k=1}^{N}\sum_{m=q(k-1)+3}^{q(k)-2} \Delta_m \sum_{j=q(k-1)}^{m-2} \Delta_j \right\|_2^2$$
$$\leqq 3\sum_{k=1}^{N}\sum_{m=q(k-1)+3}^{q(k)-2} \left\|\Delta_m \sum_{j=q(k-1)}^{m-2}\Delta_j\right\|_2^2$$
$$=O(\sum_{k=1}^{N} k^{2\beta-2}\sum_{m=q(k-1)+3}^{q(k)-2} \|\Delta_m\|_2^2)$$

$$=O(\sum_{k=1}^{N} k^{2\beta-2} \{p(q(k)-1)-p(q(k-1))\})=O(N^{3\beta-2}), \qquad \text{as } N\to+\infty.$$

On the other hand we have

$$\Delta_k^2-\|\Delta_k\|_2^2=U_k+V_k,$$

where

$$U_k=\sum_{q=p(k)+1}^{p(k+1)} \{(\cos 4\pi n_q t)/2 + \sum_{r=p(k)+1}^{q-1} \cos 2\pi(n_q+n_r)t\} ,$$

$$V_k=\sum_{q=p(k)+2}^{p(k+1)} \sum_{r=p(k)+1}^{q-1} \cos 2\pi(n_q-n_r)t.$$

Since $\{U_m\}$ is orthogonal, we have, by the Minkowski inequality and (2. 3) and (2. 5),

$$\|\sum_{k=1}^{N} \sum_{m=q(k-1)}^{q(k)-2} U_m\|_2^2=\sum_{k=1}^{N} \sum_{m=q(k-1)}^{q(k)-2} \|U_m\|_2^2$$

$$\leqq\sum_{k=1}^{N} \sum_{m=q(k-1)}^{q(k)-2} \{p(m+1)-p(m)\}^3=O(p^{2\alpha+1}(q(N)))$$

$$=O(N^{(2\alpha+1)\beta})=o(N^{3\beta-2}), \qquad \text{as } N\to+\infty.$$

In the same way we have

$$\sum_{k=1}^{N} \sum_{m=q(k-1)}^{q(k)-2} \|V_m\|_2^2=o(N^{3\beta-2}), \qquad \text{as } N\to+\infty.$$

On the other hand we have, by Lemma 1,

$$\sum_{j=1}^{k-1} \left| \int_0^1 V_k(t)V_j(t)dt \right|$$

$$\leqq Cp^{\alpha}(k)\{p(k+1)-p(k)\} \sum_{j=1}^{k-1} 2^{j-k} \{p(j+1)-p(j)\}$$

$$\leqq Cp^{\alpha}(k)\{p(k+1)-p(k)\} \sum_{j=1}^{k-1} 2^{j-k}p^{\alpha}(j).$$

Since $p(j+1)/p(j)\to1$, as $j\to+\infty$, we have

$$\sum_{j=1}^{k-1} 2^{j-k}p^{\alpha}(j)=O(p^{\alpha}(k)), \qquad \text{as } k\to+\infty.$$

Hence, we have, by (2. 3) and (2. 7),

$$\sum_{k=1}^{q(N)} \sum_{j=1}^{k-1} \left| \int_0^1 V_k(t)V_j(t)dt \right| =O(\sum_{k=1}^{q(N)} p^{2\alpha}(k)\{p(k+1)-p(k)\})$$

$$=O(p^{2\alpha+1}(q(N)))=O(N^{(2\alpha+1)\beta})=o(N^{3\beta-2}), \qquad \text{as } N\to+\infty.$$

Therefore, we have

$$\| \sum_{k=1}^{N} \sum_{m=q(k-1)}^{q(k)-2} V_m \|_2^2 = o(N^{3\beta-2}), \qquad \text{as } N \to +\infty.$$

In the same way we have, by (2. 3) and Lemma 2,

$$\| \sum_{k=1}^{N} \sum_{m=q(k-1)+1}^{q(k)} \Delta_m \Delta_{m-1} \|_2^2 = O(N^{3\beta-2}), \qquad \text{as } N \to +\infty.$$

By the above relations we can complete the proof of the Lemma 3.

In the same way we can prove the following

LEMMA 4. *We have, for any M and N,*

$$\| \sum_{k=N}^{M} (\Delta_k^2 - \| \Delta_k \|_2^2) \|_2^2 \leqq Cp^{2\alpha}(M)\{p(M+1)-p(N)\},$$

where C is a positive constant independent of M and N.

iii. We have, by (2. 3) and (2. 7),

$$\text{(2. 9)} \qquad \| \sum_{k=1}^{N-1} \Delta_{q(k)-1} \|_2^2 \leqq \sum_{k=1}^{N-1} \{p(q(k))-p(q(k)-1)\}$$

$$= O(\sum_{k=1}^{N} p^{\alpha}(q(k))) = O(\sum_{k=1}^{N} k^{\alpha\beta}) = O(N^{\alpha\beta+1}), \qquad \text{as } N \to +\infty.$$

3. The Estimations of Probabilities.

i. If x is real and $|x| < 1/3$, then we have

$$\text{(3. 1)} \qquad \exp\{x-(x^2+|x|^3)/2\} \leqq (1+x).$$

Hence, if $|\lambda| \max_{m \leqq N} \| Q_m \|_\infty < 1/3$, then

$$\exp(\sum_{m=1}^{N} \{\lambda Q_m(t) - (\lambda^2 Q_m^2(t) + |\lambda Q_m(t)|^3)/2\})$$

$$\leqq \prod_{m=1}^{N} \{1+\lambda Q_m(t)\}.$$

Since $\{Q_m(t)\}$ is multiplicatively orthogonal, that is, if $0 \leqq m_1 < m_2 < \cdots < m_n$, then

$$\int_0^1 \prod_{j=1}^{n} Q_{m_j}(t)dt = 0,$$

we have

$$\int_0^1 \exp(\sum_{m=1}^{N} \{\lambda Q_m(t) - (\lambda^2 Q_m^2(t) + |\lambda Q_m(t)|^3)/2\})dt \leqq 1.$$

If we put, for $x > 0$ and $\lambda > 0$,

$$E(\lambda, N, x) = [t ; t \in [0, 1], \sum_{m=1}^{N} Q_m(t) \geqq x + \sum_{m=1}^{N} \{\lambda Q_m^2(t) + \lambda^2 |Q_m(t)|^3\}/2],$$

then we have, by Tchebyschev's inequality,

(3. 2) $$|E(\lambda, N, x)| \leqq \exp(-\lambda x),$$

where $|E|$ denotes the Lebesgue measure of the set E.

ii. For any fixed $\theta > 1$, we take an integer $M(k)$ and real numbers $\lambda(h, k)$ and $x(h, k)$, $1 \leqq h \leqq \theta^k$, as follows:

(3. 3) $$\begin{cases} M(k)^\beta \leqq \theta^k < (M(k)+1)^\beta, \\ \lambda(h, k) = 2\ [\{\log(h+1) + \log\log \theta^k\}/\theta^k]^{1/2}, \\ x(h, k) = (1+\eta)\ \{\log(h+1) + \log\log \theta^k\}/\lambda(h, k), \end{cases}$$

where η is any given positive number.

Then we have, by (2. 4) and (2. 8),

(3. 4) $$\max\ \{\lambda(h, k) \| Q_m \|_\infty;\ m \leqq M(k),\ h \leqq \theta^k\}$$
$$= O(k\theta^{k(2^{-1}-\beta^{-1})}) = o(1), \qquad \text{as } k \to +\infty.$$

Hence, we have, by (3. 2),

$$|E(\lambda(h, k), M(k), x(h, k))| \leqq \exp\{-(1+\eta)\ (\log(h+1) + \log\log \theta^k)\},$$

and this implies that

$$\sum_{k=1}^{\infty} \sum_{h \leqq \theta^k} |E(\lambda(h, k), \mathrm{M}(k), x(h, k))| < +\infty.$$

Therefore, by Borel-Cantelli's lemma, for a. e. t, there exists an integer $k_o(t)$ such that $k \geqq k_o(t)$ implies

$$\Big| \sum_{h \leqq \theta^k} a_h \sum_{m=1}^{M(k)} Q_m(ht) \Big|$$
$$\leqq \sum_{h \leqq \theta^k} a_h\ [x(h, k) + \lambda(h, k) \sum_{m=1}^{M(k)} \{Q_m^2(ht) + \lambda(h, k) |Q_m(ht)|^3\}/2].$$

On the other hand we have, by (2. 1) and (3. 3),

$$\overline{\lim_{k \to \infty}} \sum_{h \leqq \theta^k} a_h x(h, k)/(\theta^k \log\log \theta^k)^{1/2} \leqq (1+\eta) \sum_{h=1}^{\infty} a_h/2.$$

By (2. 1) and Lemma 3, we have

$$\Big\| \sum_{h \leqq \theta^k} a_h \lambda(h, k) \sum_{m=1}^{M(k)} \{Q_m^2(h\cdot) - \|Q_m\|_2^2\} \Big\|_2$$
$$\leqq \sum_{h \leqq \theta^k} a_h \lambda(h, k) \Big\| \sum_{m=1}^{M(k)} (Q_m^2 - \|\mathrm{Q}_\mathrm{m}\|_2^2) \Big\|_2$$
$$= O(kM(k)^{3\beta-2}\ \theta^{-k})^{1/2} = O(k\theta^{2k(1-\beta^{-1})})^{1/2}, \qquad \text{as } k \to +\infty.$$

Hence, we have

$$\sum_{k=1}^{\infty} \Big\| \sum_{h \leqq \theta^k} a_h \lambda(h, k) \sum_{m=1}^{M(k)} \{Q_m^2(h\cdot) - \|Q_m\|_2^2\} \Big\|_2^2/(\theta^k \log\log \theta^k) < +\infty.$$

By (2. 1), (2. 9) and (3. 3), this shows that, for a. e. t,

$$\sum_{h\leqq\theta^{k'}} a_h\lambda(h, k)\sum_{m=1}^{M(k)} Q_m^2(ht)/(\theta^k\log\log\theta^k)^{1/2}$$

$$\sim\sum_{h\leqq\theta^k} a_h\lambda(h, k)\sum_{m=1}^{M(k)} \| Q_m \|_2^2/(\theta^k\log\log\theta^k)^{1/2}$$

$$\sim\sum_{h\leqq\theta^k} a_h, \qquad \text{as } k\to+\infty,$$

and thus, by (3. 4),

$$\sum_{h\leqq\theta^k} a_h\lambda^2(h, k)\sum_{m=1}^{M(k)} | Q_m^3(ht) | / \{\theta^k\log\log\theta^k\}^{1/2}\to 0, \qquad \text{as } k\to+\infty.$$

Further, we have, by (2. 9), (3. 3) and (2.4),

$$\sum_{k=1}^{\infty} \| \sum_{h\leqq\theta^k} a_h \sum_{m=1}^{M(k)} \Delta_{q(m)-1}(h\cdot) \|_2^2/(\theta^k\log\log\theta^k)<+\infty.$$

The above relations show that

$$\overline{\lim_{k\to\infty}}\sum_{h\leqq\theta^k} a_h \sum_{m=1}^{M(k)^\beta} \cos 2\pi n_m ht/(\theta^k\log\log\theta^k)^{1/2}$$

$$\leqq (1+\eta)\sum_{h=1}^{\infty} a_h, \qquad \text{a. e. } t.$$

Since η is arbitrary, we have, by (2. 2), (2. 5) and (3. 3),

$$\overline{\lim_{k\to\infty}}\sum_{m\leqq\theta^k} f(n_m t)/(\theta^k\log\log\theta^k)^{1/2}\leqq\sum_{h=1}^{\infty} a_h, \qquad \text{a. e. } t. \tag{3. 5}$$

4. The Maximal Theorems. By (2. 3) we can take, for large k, an increasing sequence of integers $\{m_{j,k}\}$, $0\leqq j\leqq k$, as follows:

$$p(m_{j,k})\leqq\theta^k+\{(\theta^{k+1}-\theta^k)j/k\}<p(m_{j,k}+1). \tag{4. 1}$$

Then we have, for some constant C,

$$\max_{0\leqq j\leqq k}\{p(m_{j,k}+1)-p(m_{j,k})\} \tag{4. 2}$$

$$=C(p^\alpha(m_{j,k}))=O(\theta^{k\alpha}), \text{ as } k\to+\infty.$$

i. If we put, for $1\leqq h\leqq\theta^k$ and $k\geqq 1$,

$$\begin{cases}\eta(h, k)=[\{\log(h+1)+\log\log\theta^k\}/(\theta^{k+1}-\theta^k)]^{1/2},\\ y(h, k)=3\{\log(h+1)+\log\log\theta^k\}/\eta(h, k).\end{cases} \tag{4. 3}$$

Then we have, by (4. 2) and (4. 3),

$$\max\{\eta(h, k)\| \Delta_m \|_\infty;\ 1\leqq h\leqq\theta^k,\ m_{o,k}\leqq m\leqq m_{k,k}\}$$

$$= O(k^{1/2}\theta^{k(\alpha-1/2)}) = o(1), \qquad \text{as } k \to +\infty$$

By (3. 1), we have, for large k,

$$\exp\{\eta(h, k)\sum_{m=m_{o,k}}^{m_{j,k}} \Delta_m - \eta^2(h, k)\sum_{m=m_{o,k}}^{m_{h,k}} \Delta_m^2\}$$

$$\leqq \exp\{\eta(h, k)\sum_{m=m_{o,k}}^{m_{j,k}} (\Delta_m - \eta(h, k)\Delta_m^2 - 2 \mid \eta^2(h, k)\Delta_m^3 \mid)\}$$

$$\leqq \sum_{m=m_{o,k}}^{m_{j,k}} \{1+2\eta(h, k)\Delta_m(t)\}^{1/2}.$$

Since both sequences $\{\Delta_{2m}(t)\}$ and $\{\Delta_{2m+1}(t)\}$ are multiplicatively orthogonal on the interval (0, 1), we have

$$\int_0^1 \exp\{\eta(h, k)\sum_{m=m_{o,k}}^{m_{j,k}} \Delta_m - \eta^2(h, k)\sum_{m=m_{o,k}}^{m_{h,k}} \Delta_m^2\}\, dt$$

$$\leqq \int_0^1 \prod_{m=m_{o,k}}^{m_{j,k}} \{1+2\eta(h, k)\Delta_m(t)\}^{1/2} dt$$

$$\leqq [\int_0^1 \Pi_1\{1+2\eta(h, k)\Delta_{2m}(t)\}\, dt \int_0^1 \Pi_2\{1+2\eta(h, k)\Delta_{2m+1}(t)\}\, dt]^{1/2} = 1,$$

where Π_1 (or Π_2) is the product over all m such that

$$m_{o,k} \leqq 2m \leqq m_{j,k} \qquad (\text{or } m_{o,k} \leqq 2m+1 \leqq m_{j,k}).$$

If we put

$$E'(h, k, j)$$

$$= \{t\,;\, t \in [0, 1]\,,\, \sum_{m=m_{o,k}}^{m_{j,k}} \Delta_m(ht) \geqq y(h, k) + \eta(h, k)\sum_{m=m_{o,k}}^{m_{h,k}} \Delta_m^2(ht)\}\,.$$

Then we have, for any j,

$$\mid E'(h, k, j) \mid \leqq \exp\{-3(\log(h+1) + \log\log\theta^k)\}$$
$$\leqq C\{(h+1)k\}^{-3}, \qquad \text{for some } C > 0.$$

Hence, we have

$$(4.\ 4) \qquad \sum_{k=1}^{\infty}\sum_{j=o}^{k}\sum_{h \leqq \theta^k} \mid E'(h, k, j) \mid < +\infty.$$

On the other hand we have, by Lemma 4 and (2. 1),

$$\sum_{k=1}^{\infty} \| \sum_{h \leqq \theta^k} a_h \eta(h, k) \sum_{m=m_{o,k}}^{m_{h,k}} \{\Delta_m^2(h\,\cdot) - \| \Delta_m \|_2^2\} \|_2 / (\theta^k \log\log\theta^k)^{1/2}$$

$$\leqq \sum_{k=1}^{\infty}\sum_{h \leqq \theta^k} a_h \eta(h, k) \| \sum_{m=m_{o,k}}^{m_{h,k}} (\Delta_m^2 - \| \Delta_m \|_2^2) \|_2 / (\theta^k \log\log\theta^k)^{1/2} < +\infty.$$

Thus we have, by (2. 1) and (4. 1), for a. e. t,

$$2\sum_{h\leqq\theta^k} a_h\eta(h, k) \sum_{m=m_{o,k}}^{m_{k,k}} \Delta_m^2(ht)/(\theta^k \log\log\theta^k)^{1/2}$$

$$\sim 2\sum_{h\leqq\theta^k} a_h\eta(h, k) \sum_{m=m_{o,k}}^{m_{k,k}} \|\Delta_m\|_2^2/(\theta^k \log\log\theta^k)^{1/2}$$

$$\sim\sqrt{\theta-1}\sum_{h=1}^{\infty} a_h, \qquad \text{as } k\to+\infty.$$

Further, we have, by (2. 1),

$$\overline{\lim_{k\to\infty}} \sum_{h\leqq\theta^k} a_h y(h, k)/(\theta^k\log\log\theta^k)^{1/2} \leqq 3\sqrt{\theta-1}\sum_{h=1}^{\infty} a_h.$$

From the above two relations and (4. 4), we obtain

$$\overline{\lim_{k\to\infty}} \sup_{o\leqq j\leqq k} \sum_{h\leqq\theta^k} a_h \sum_{m=m_{o,k}}^{m_{j,k}} \Delta_m(ht)/(\theta^k\log\log\theta^k)^{1/2}$$

$$\leqq 4\sqrt{\theta-1}\sum_{h=1}^{\infty} a_h, \qquad \text{a. e. } t.$$

Therefore, we have, by (2. 2),

$$(4.\ 5)\quad \overline{\lim_{k\to\infty}} \sup_{o\leqq j\leqq k} \sum_{m=p(m_{o,k})}^{p(m_{j,k})} f(n_m t)/(\theta^k\log\log\theta^k)^{1/2}$$

$$\leqq 4\sqrt{\theta-1}\sum_{h=1}^{\infty} a_h, \qquad \text{a. e. } t.$$

ii. If we put

$$A(j, k, t) = \sum_{h\leqq\theta^k} a_h \sup_{m_{j,k}\leqq m\leqq m_{j+1,k}} \sum_{r=m_{j,k}}^{m} \Delta_r(ht),$$

then we have, by (2. 1)

$$\|A(j, k. \cdot)\|_4 \leqq \Big(\sum_{h\leqq\theta^k} a_h\Big) \Big\|\sup_{m_{j,k}\leqq m\leqq m_{j+1,k}} \sum_{r=m_{j,k}}^{m} \Delta_r\Big\|_4.$$

By the theorems of trigonometric series (c f. [5] vol. II (4. 4) p. 231 and (4. 24) p. 233) and Lemma 4, we have, for some constants c_1, c_2 and c_3.

$$\Big\|\sup_{m_{j,k}\leqq m\leqq m_{j+1,k}} \sum_{r=m_{j,k}}^{m} \Delta_r\Big\|_4^4 \leqq c_1 \Big\|\sum_{r=m_{j,k}}^{m_{j+1,k}} \Delta_r\Big\|_4^4$$

$$\leqq c_2\Big\|\sum_{r=m_{j,k}}^{m_{j+1,k}} \Delta_r^2\Big\|_2^2 \leqq 2c_2\Big(\Big\|\sum_{r=m_{j,k}}^{m_{j+1,k}} (\Delta_r^2 - \|\Delta_r\|_2^2)\Big\|_2^2 + \Big(\sum_{r=m_{j,k}}^{m_{j+1,k}} \|\Delta_r\|_2^2\Big)^2\Big)$$

$$\leqq c_3(\theta^{2k}k^{-2}), \qquad \text{for } j=0, 1, \cdots, k, \text{ as } k\to+\infty.$$

Therefore, we have

$$\sum_{k=1}^{\infty} \int_0^1 \max_{0 \leq j \leq k} | A(j, k, t) |^4 / (\theta^k \log \log \theta^k)^2 dt$$

$$\leq \sum_{k=1}^{\infty} \sum_{j=0}^{k} \int_0^1 | A(j, k, t) |^4 / (\theta^k \log \log \theta^k)^2 dt < +\infty,$$

and this shows that

$$\lim_{k \to \infty} \max_{0 \leq j \leq k} | A(j, k, t) | / (\theta^k \log \log \theta^k)^{1/2} = 0, \qquad \text{a. e. } t.$$

By (2. 2), we have

$$\lim_{k \to \infty} \max_{0 \leq j \leq k} \sup_{m_{j,k} \leq m \leq m_{j+1,k}} \sum_{r=p(m_{j,k})}^{p(m)} f(n_r t) / (\theta^k \log \log \theta^k)^{1/2} = 0, \qquad \text{a. e. } t.$$

By the above relation and (4. 5) and (2. 3), we have

$$\text{(4. 6)} \qquad \overline{\lim_{k \to \infty}} \sup_{\theta^k \leq m \leq \theta^{k+1}} \sum_{r=\theta^k}^{m} f(n_r t) / (\theta^k \log \log \theta^k)^{1/2}$$

$$\leq 4\sqrt{\theta - 1} \sum_{h=1}^{\infty} a_h, \qquad \text{a. e. } t.$$

Since we can choose θ as close as 1, by (3. 5) and (4. 6), we can complete the proof of the theorem.

As a result in the opposite direction P. Erdös [2] proved that there exist a sequence $\{n_k\}$ of positive integers with *Hadamard's gap* and a function $f \in L^2(0, 1)$ such that

$$\overline{\lim_{N \to \infty}} N^{-1} \sum_{k=1}^{N} f(n_k t) = +\infty, \qquad \text{a. e. } t.$$

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