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On Theorems of A. Zygmund Type on Absolutely Convergent Vilenkin Fourier Series

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Abstract. We shall compare two theorems of A. Zygmund type on absolutely convergent Vilenkin Fourier series.

1. Introduction

C. W. Onneweer and D. Waterman ([1]) gave two concepts of bounded fluctuation(BF) and generalized bounded fluctuation(GBF) on Vilenkin groups. Making use of GBF, T. S. Quek and L. Y. H. Yap proved([2]) a theorem of A. Zygmund type on absolutely convergent Vilenkin Fourier series. We shall compare this theorem with a theorem which is obtained by exchanging GBF for BF.

2. Notations and Definitions

Let G be a Vilenkin group, that is, a compact, metrizable, zero-dimensional, abelian group. Then the dual group X of G is a discrete, countable, torsion, abelian group. N. Ja. Vilenkin([3]) proved that there exist an increasing sequence $\{X_n\}_{n=0}^{\infty}$ of finite subgroups of X and a sequence $\{\varphi_n\}_{n=0}^{\infty}$ of characters in X such that

(1) $X_0 = \{\chi_0\}$, where $\chi_0(x)=1$ for all $x \in G$,

(2) X_n/X_{n-1} is of prime order p_n for every $n \geqq 1$,

(3) $X = \bigcup_{n=0}^{\infty} X_n$,

(4) $\varphi_n \in X_{n+1}/X_n$ for every $n \geqq 0$,

(5) $\varphi_n^{p_{n+1}} \in X_n$ for every $n \geqq 0$.

Let $m_0 = 1$ and $m_n = p_1 \cdots p_n$ for every $n \geqq 1$. If $k \geqq 1$ and if $k = \sum_{j=0}^{s} a_j m_j$ with $0 \leqq a_j < p_{j+1}$ for $0 \leqq j \leqq s$, then we put $\chi_k = \varphi_0^{a_0} \cdots \varphi_s^{a_s}$. Then $X_n = \{\chi_k : 0 \leqq k < m_n\}$. The annihilator of X_n

is denoted by G_n for every $n \geqq 0$. Then it is clear that $G=G_0 \supset G_1 \supset G_2 \cdots$, $\bigcap_{n=0}^{\infty} G_n = \{0\}$ and that the $\{G_n\}_{n=0}^{\infty}$ is a fundamental system of neighborhoods of zero in G. Furthermore, for each $n \geqq 0$ there exists an $x_n \in G_n/G_{n+1}$ such that $\chi_{m_n}(x_n) = \exp(2\pi i/p_{n+1})$ and each $x \in G$ can be represented uniquely by $x=\sum_{j=0}^{\infty} b_j x_j$ with $0 \leqq b_j < p_{j+1}$ for all $j \geqq 0$. Then we have

$$G_n = \left\{ \sum_{j=n}^{\infty} b_j x_j : \ 0 \leqq b_j < p_{j+1} \text{ for all } j \geqq 0 \right\}.$$

For every $\{b_j\}_{j=0}^{n-1}$, $0 \leqq b_j < p_{j+1}$ $(0 \leqq j \leqq n-1)$, we put

$$z_{q,n} = \sum_{j=0}^{n-1} b_j x_j \qquad \text{where } q = \sum_{j=0}^{n-1} b_j m_n / m_{j+1}.$$

Then the cosets of G_n in G are $z_{q,n} + G_n$ for $0 \leqq q \leqq m_n - 1$. We put

$$\mathfrak{C} = \{ z_{q,n} + G_n : \ 0 \leqq q \leqq m_n - 1, \ n = 0,1,2,\cdots \}.$$

DEFINITION 1. If f is a function on G and if $H \subset G$, then

$$\operatorname{osc}(f,H) = \sup \{ |f(x) - f(y)| : x, y \in H \}.$$

DEFINITION 2. A function f on G is of bounded fluctuation(BF) if there exists a constant M such that

$$\sum_{n=1}^{\infty} \operatorname{osc}(f, C_n) \leqq M$$

for every sequence $\{C_n\}_{n=1}^{\infty}$ of pairwise disjoint subsets in $\mathfrak{C}$.

DEFINITION 3. A function f on G is of generalized bounded fluctuation(GBF) if

$$\sup_{n \geqq 0} \sum_{q=0}^{m_n - 1} \operatorname{osc}(f, z_{q,n} + G_n) < \infty.$$

DEFINITION 4. If f is a function on G, then the n-th modulus of continuity of f is

$$\omega_n(f) = \sup \{ |f(x) - f(y)| : x, y \in G, \ x - y \in G_n).$$

3. Results

T. S. Quek and L. Y. H. Yap ([2]) showed the following theorem of A. Zygmund type on absolutely convergent Vilenkin Fourier series.

THEOREM A. *If a function f on G is of* GBF *and if*

(6) $$\sum_{n=0}^{\infty} \{p_{n+1}\omega_n(f)\}^{1/2} < \infty,$$

then the Vilenkin Fourier series of f is absolutely convergent.

Since BF⊂GBF, Theorem A implies the following :

THEOREM B. *If a function f on G is of* BF *and if* (6) *holds, then the Vilenkin Fourier series of f is absolutely convergent.*

In this note we consider whether Theorem A includes properly Theorem B.

THEOREM 1. *Suppose that there exists $\eta>0$ such that $p_{n+1} \geqq \eta m_n (n=0,1,2,\cdots)$. If f is a function on G and if $\{p_{n+1}\omega_n(f)\}_{n=0}^{\infty}$ is bounded, then f is of* BF.

PROOF. Let $\{C_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint elements in $\mathfrak{C}$. Then $\{C_n : n=1,2,3,\cdots\}$ is written as

$$\{z_{q_j,n} + G_n : 1 \leqq j \leqq k_n,\ n=0,1,2,\cdots\}.$$

Then

$$\sum_{n=0}^{\infty} k_n/m_n = \sum_{n=0}^{\infty}\sum_{j=1}^{k_n} | z_{q_j,n} + G_n | \leqq | G | = 1.$$

But there exists $M \geqq 0$ such that

$$p_{n+1}\omega_n(f) \leqq M \ (n=0,1,2,\cdots),$$

and hence we have

$$\sum_{n=0}^{\infty} \mathrm{osc}(f, C_n) = \sum_{n=0}^{\infty}\sum_{j=1}^{k_n} \mathrm{osc}(f, z_{q_j,n} + G_n) \leqq \sum_{n=0}^{\infty} k_n \omega_n(f)$$
$$\leqq M \sum_{n=0}^{\infty} k_n/p_{n+1} \leqq M\eta^{-1} \sum_{n=0}^{\infty} k_n/m_n \leqq M\eta^{-1}.$$

Thus f is of BF.

q.e.d.

COROLLARY 1. *Suppose that there exists $\eta > 0$ such that $p_{n+1} \geqq \eta m_n$ $(n=0,1,2,\cdots)$. Then Theorem A is equivalent to Theorem* B.

THEOREM 2. *Suppose that there exist a monotone decreasing sequence $\{\alpha_n\}_{n=0}^{\infty}$ of positive numbers and a sequence $\{k_n\}_{n=0}^{\infty}$ of nonnegative integers which satisfy*

(7) $$\sum_{n=0}^{\infty} (p_{n+1}\alpha_n)^{1/2} < \infty,$$

(8) $$\sum_{n=0}^{\infty} k_n/m_n \leqq 1,$$

(9) $\sum_{n=0}^{\infty} k_n \alpha_n = \infty,$

(10) $\sup_{n \geqq 0} \{k_n \alpha_n + \sum_{r=n+1}^{\infty} k_r \alpha_r / (p_{n+1} \cdots p_r)\} < \infty.$

Then there exists a continuous function f *on* G *such that* f *is of* GBF *but not of* BF *and* (6) *holds.*

Proof. Let $q_0 = 0$ and

$$q_n = m_n \sum_{r=0}^{n-1} k_r / m_r \quad (n=1,2,3,\cdots).$$

Then $0 \leqq q_n \leqq q_n + k_n \leqq m_n$ for every $n \geqq 1$ by (8). We put

$$G'_n = \bigcup_{q=q_n}^{q_n+k_n-1} (z_{q,n} + G_n) \quad (n=1,2,3,\cdots),$$

where $G'_n = \phi$ if $k_n = 0$. Then $G'_n, n=1,2,3,\cdots$, are pairwise disjoint. We define

$$f(x) = \begin{cases} \alpha_n & \text{if } x \in \bigcup_{q=q_n}^{q_n+k_n-1} (z_{q,n} + G_{n+1}),\ n=0,1,2,\cdots, \\ 0 & \text{otherwise.} \end{cases}$$

We show that this function f satisfies the conclusion of Theorem 2.

It is clear that

$$\mathrm{osc}(f, z_{q,n} + G_n) = 0 \quad (0 \leqq q \leqq q_n - 1),$$

$$\mathrm{osc}(f, z_{q,n} + G_n) = \alpha_n \quad (q_n \leqq q \leqq q_n + k_n - 1).$$

If we put

$$N(n) = \min \{r \geqq n+1 : k_r \neq 0\},$$

then

$$\mathrm{osc}(f, z_{q,n} + G_n) \leqq \alpha_{N(n)} \quad (q_n + k_n \leqq q \leqq m_n - 1).$$

Therefore we have

$$\omega_n(f) = \begin{cases} \alpha_n & \text{if } k_n \neq 0, \\ \alpha_{N(n)} & \text{if } k_n = 0. \end{cases}$$

Thus f is a continuous function on G and (6) holds by the assumption (7). Since

$$\sum_{n=0}^{\infty} \sum_{q=q_n}^{q_n+k_n-1} \mathrm{osc}(f, z_{q,n} + G_n) = \sum_{n=0}^{\infty} k_n \alpha_n = \infty,$$

by (9), f is not of BF.

Now we shall show that f is of GBF. Fix $n \geqq 1$. For every $s \geqq 1$, we choose the integer r_s such that

$$r_s - 1 < (q_{n+s} + k_{n+s})\ (p_{n+1} \cdots P_{n+s})^{-1} \leqq r_s.$$

Then we have

$$q_n + k_n \leqq r_1 \leqq r_2 \leqq r_3 \leqq \cdots \leqq m_n\ ,$$

(11) $$\bigcup_{t=1}^{s} G'_{n+t} \subset \bigcup_{q=q_n+k_n}^{r_s-1} (z_{q,n} + G_n) \quad (s = 1,2,3,\cdots).$$

Let $r_0 = q_n + k_n$. Then we have

$$\sum_{q=0}^{m_n-1} \operatorname{osc}(f, z_{q,n} + G_n)$$

$$= \sum_{q=q_n}^{q_n+k_n-1} \operatorname{osc}(f, z_{q,n} + G_n) + \sum_{q=q_n+k_n}^{m_n-1} \operatorname{osc}(f, z_{q,n} + G_n)$$

$$= k_n \alpha_n + \sum_{s=1}^{\infty} \sum_{q=r_{s-1}}^{r_s-1} \operatorname{osc}(f, z_{q,n} + G_n).$$

Since $\operatorname{osc}(f, z_{q,n} + G_n) \leqq \alpha_{n+s}$ for $q \geqq r_{s-1}$ by (12), we have

$$\sum_{q=0}^{m_n-1} \operatorname{osc}(f, z_{q,n} + G_n) \leqq k_n \alpha_n + \sum_{s=1}^{\infty} (r_s - r_{s-1}) \alpha_{n+s}$$

$$\leqq k_n \alpha_n + \sum_{r=n+1}^{\infty} k_r \alpha_r (p_{n+1} \cdots p_r)^{-1} + \sum_{s=1}^{\infty} \alpha_s.$$

Therefore f is of GBF by the assumptions (7) and (10).

q.e.d.

THEOREM 3. *Suppose that there exists* $\sigma (0 < \sigma < 1)$ *such that* $p_{n+1} = O(m_n^{\sigma})$ $(n \to \infty)$. *Then there exist* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{k_n\}_{n=0}^{\infty}$ *which satisfy conditions* (7)~(10).

PROOF. We choose τ with $\sigma < \tau < 1$ and put

$$M = \sum_{n=0}^{\infty} m_n^{\tau - 1},$$

$$\alpha_n = m_n^{-\tau} \qquad (n = 0,1,2,\cdots),$$

$$k_n = [m_n^{\tau}/M] \quad (n=0,1,2,\cdots)$$

where $[x]$ denotes the integral part of x. Then it is easy to see that these sequences satisfy the conditions (7)~(10).

q.e.d.

COROLLARY 2. *Suppose that there exists* σ $(0<\sigma<1)$ *such that* $p_{n+1}=O(m_n^{\sigma})$ $(n\to\infty)$. *Then Theorem* A *includes properly Theorem* B.

References

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