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On Absolute Convergence of Vilenkin Fourier Series

Yoshikazu Uno

Department of Mathematics, Faculty of Science, Kanazawa University

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Abstract We shall show that some results on the absolute convergence of Vilenkin Fourier series cannot be improved in some sense on certain groups.

1. Introduction. Under the notaions in §2 the following theorems are known ([1], [2], [3], [4], [6]).

THEOREM A. *If a function f on G satisfies*

$$\sum_{k=0}^{\infty} m_{k+1}^{1/2}\omega(f,k) <\infty,$$

then the Vilenkin Fourier series of f is absolutely convergent.

THEOREM B. *If a function f on G satisfies*

$$\sum_{k=0}^{\infty} p_{k+1}^{1/2}v(f,k) <\infty,$$

then the Vilenkin Fourier series of f is absolutely convergent.

THEOREM C. *If a integrable function f on G satisfies*

$$\sum_{k=0}^{\infty} m_{k+1}^{1/2}d(f,k) <\infty,$$

then the Vilenkin Fourier series of f is absolutely convergent.

In this note we shall show that Theorems A, B and C cannot be improved in some sense on G with $p_k=p(k=1,2,3,\cdots)$.

2. Notations and Definitions. Let G be a Vilenkin group, that is, a compact, metrizable, zero-dimensional, abelian group. Then the dual group X of G is a discrete, countable, torsion, abelian group. N. Ja. Vilenkin([5]) proved that there exist an increasing sequence $\{X_k\}_{k=0}^{\infty}$ of finite subgroups of X and a sequence $\{\varphi_k\}_{k=0}^{\infty}$ of characters in X such that

(1) $X_0=\{\chi_0\}$, where $\chi_0(x)=1$ for all $x \in G$,

(2) X_k/X_{k-1} is of prime order p_k for every $k \geqq 1$,

(3) $X = \bigcup_{k=0}^{\infty} X_k$,

(4) $\varphi_k \in X_{k+1}/X_k$ for every $k \geqq 0$,

(5) $\varphi_k^{p_{k+1}} \in X_k$ for every $k \geqq 0$.

Let $m_0 = 1$ and $m_k = p_1 \cdots p_k$ for every $k \geqq 1$. If $n \geqq 1$ and if $n = \sum_{j=0}^{s} a_j m_j$ with $0 \leqq a_j < p_{j+1}$ for $0 \leqq j \leqq s$, then we put $\chi_n = \varphi_0^{a_0} \cdots \varphi_s^{a_s}$. Then $X_k = \{\chi_n : 0 \leqq n < m_k\}$. The annihilator of X_k is denoted by G_k for every $k \geqq 0$. Then it is clear that $G = G_0 \supset G_1 \supset G_2 \cdots$, $\bigcap_{k=0}^{\infty} G_k = \{0\}$ and that the $\{G_k\}_{k=0}^{\infty}$ is a fundamental system of neighborhoods of zero in G. Furthermore, for each $k \geqq 0$ there exists an $x_k \in G_k/G_{k+1}$ such that $\chi_{m_k}(x_k) = \exp(2\pi i/p_{k+1})$ and each $x \in G$ can be represented uniquely by $x = \sum_{j=0}^{\infty} b_j x_j$ with $0 \leqq b_j < p_{j+1}$ for all $j \geqq 0$. Then we have

$$G_k = \left\{ \sum_{j=k}^{\infty} b_j x_j : 0 \leqq b_j < p_{j+1} \text{ for all } j \geqq 0 \right\}.$$

For every $\{b_j\}_{j=0}^{k-1}$, $0 \leqq b_j < p_{j+1}$ $(0 \leqq j \leqq k-1)$, we put

$$z_{q,k} = \sum_{j=0}^{k-1} b_j x_j \quad \text{where } q = \sum_{j=0}^{k-1} b_j m_k / m_{j+1}.$$

Then the cosets of G_k in G are $z_{q,k} + G_k$ for $0 \leqq q \leqq m_k - 1$.

DEFINITION 1. If f is a function on G and if $H \subset G$, then

$$\operatorname{osc}(f,H) = \sup \{ |f(x) - f(y)| : x,y \in H \}.$$

DEFINITION 2. If f is a function on G, then the k-th modulus of continuity of f is

$$\omega(f,k) = \sup \{ |f(x) - f(y)| : x,y \in G,\ x - y \in G_k \}.$$

DEFINITION 3, Let f be a function on G. Then we define

$$v(f,k) = \left[\sum_{q=0}^{m_k - 1} \{\operatorname{osc}(f, z_{q,k} + G_k)\}^2 \right]^{1/2} \quad (k = 0,1,2,\cdots).$$

DEFINITION 4. The Dirichlet kernels are denoted by $\{D_n\}_{n=1}^{\infty}$. Then we define

$$d(f,k) = \| (D_{m_{k+1}} - D_{m_k}) * f \|_2 \quad (k = 0,1,2,\cdots).$$

$C(G)$ and $L^p(G)$ $(p \geqq 1)$ denote the usual spaces.

Let Ω be the set of all sequences $\{\omega(k)\}_{k=0}^{\infty}$ which are monotonically decreasing to 0. For $\omega=\{\omega(k)\}_{k=0}^{\infty}$ in Ω, we define the set

$$\Lambda[\omega]=\{f\in C(G): \omega(f,k)=O(\omega(k)) \text{ as } k\to\infty\}.$$

Let $\mathfrak{V}$ be the set of all sequences $\{v(k)\}_{k=0}^{\infty}$ such that $\{m_k^{-1/2}v(k)\}_{k=0}^{\infty}$ monotonically decreases to 0. For $v=\{v(k)\}_{k=0}^{\infty}$ in $\mathfrak{V}$ we define

$$V[v]=\{f\in C(G): v(f,k)=O(v(k)) \text{ as } k\to\infty\}.$$

Moreover let $\mathfrak{D}$ be the set of all square summable sequences of nonnegative numbers. For $d=\{d(k)\}_{k=0}^{\infty}$ in $\mathfrak{D}$, we define

$$D[d]=\{f\in C(G): d(f,k)=O(d(k)) \text{ as } k\to\infty\}.$$

We denote by $A(G)$ the space of functions with absolutely convergent Vilenkin Fourier series.

3. At first we consider the relation of Theorems A, B and C.

LEMMA 1. *For $k\geqq 0$ and $j\geqq m_k$, we have*
$$\int_{G_k}|\chi_j(x)-1|^2dx=2/m_k.$$

PROOF. It is well known that $m_k^{-1}D_{m_k}$ is the characteristic function of the set G_k. Thus we have

$$\int_{G_k}|\chi_j(x)-1|^2dx=2\int_{G_k}\{1-\operatorname{Re}\chi_j(x)\}\,dx$$

$$=2\{1/m_k-(1/m_k)\operatorname{Re}\sum_{r=0}^{m_k-1}\int_G\chi_r(x)\chi_j(x)dx\}$$

$$=2/m_k.$$ q.e.d.

THEOREM 1. *Let $f\in C(G)$. Then*

$$2^{1/2}d(f,k)\leqq m_k^{-1/2}v(f,k)\leqq\omega(f,k)\qquad(k=0,1,2,\cdots).$$

PROOF. Since the second inequality is trivial, we show the first inequality. For $k\geqq 0$ and $x\in G_k$, we have

$$\sum_{j=m_k}^{\infty}|\hat{f}(j)|^2|\chi_j(x)-1|^2=\sum_{j=0}^{\infty}|\hat{f}(j)|^2|\chi_j(x)-1|^2$$

$$=\int_G|f(y+x)-f(y)|^2dy=\sum_{q=0}^{m_k-1}\int_{z_{q,k}+G_k}|f(y+x)-f(y)|^2dy$$

$$\leqq \sum_{q=0}^{m_k-1} m_k^{-1}\mathrm{osc}(f, z_{q,k}+G_k)=m_k^{-1}v(f,k).$$

Therefore we have, by Lemma 1,

$$2\, d(f,k)^2=2\sum_{j=m_k}^{m_{k+1}-1}|\hat{f}(j)|^2\leqq m_k\sum_{j=m_k}^{\infty}|\hat{f}(j)|^2\int_{G_k}|\chi_j(x)-1|^2dx$$

$$=\int_{G_k}v(f,k)^2dx=m_k^{-1}v(f,k)^2.$$

q.e.d.

It follows that Theorem A $\subset$ Theorem B $\subset$ Theorem C.

4. Now we shall show that a function with a given modulus of continuity exists.

THEOREM 2. *For every* $\omega=\{\omega(k)\}_{k=1}^{\infty}$ *in* Ω, *there exists a continuous function* f *such that*

(6) $\mathrm{osc}(f, z_{q,k}+G_k)=\omega(k)$ $(0\leqq q<m_k,\ k=0,1,2,\cdots)$.

PROOF. Let $x\in G$. Then x has the unique representation

$$x=\sum_{j=0}^{\infty}b_jx_j,\quad 0\leqq b_j<p_{j+1}\ (j=0,1,2,\cdots).$$

We put $N(x)=\{j\geqq 0 : b_j=0\}$. If $N(x)\neq\phi$, then $N(x)$ is written as

$$N(x)=\{j_1,\ j_2,\ j_3,\ \cdots\},\ j_1<j_2<j_3<\cdots$$

(possibly $N(x)$ may be finite). Then we define

$$f(x)=\sum_r(-1)^{r-1}\omega(j_r).$$

If $N(x)=\phi$, then we define $f(x)=0$.

We shall show that f satisfies (6). Fix $k\geqq 0$ and $0\leqq q<m_k$. Suppose that $N(z_{q,k})\cap\{1,2,\cdots,k-1\}=\phi$. Since

$$f(z_{q,k}+\sum_{j=k+1}^{\infty}x_j)=\omega(k),$$

$$f(z_{q,k}+\sum_{j=k}^{\infty}x_j)=0,$$

we have

$$\operatorname{osc}(f, z_{q,k}+G_k) \geqq \omega(k).$$

Let $x \in z_{q,k}+G_k$. If $N(x)=\phi$, then $f(x)=0$. If $N(x)\neq\phi$, then

$$0 \leqq f(x) \leqq \omega(\min N(x)) \leqq \omega(k).$$

Thus we have

$$\operatorname{osc}(f, z_{q,k}+G_k) \leqq \omega(k).$$

Therfore we have

$$\operatorname{osc}(f, z_{q,k}+G_k) = \omega(k)$$

if $N(z_{q,k}) \cap \{1,2,\cdots,k-1\}=\phi$.

Suppose that $N(z_{q,k}) \cap \{1,2,\cdots,k-1\}\neq\phi$. We write

$$N(z_{q,k}) \cap \{1,2,\cdots,k-1\} = \{j_1, \cdots, j_N\}, \quad j_1<\cdots<j_N.$$

Since

$$f(z_{q,k}+\sum_{j=k+1}^{\infty} x_j) = \sum_{r=1}^{N}(-1)^{r-1}\omega(j_r)+(-1)^N\omega(k),$$

$$f(z_{q,k}+\sum_{j=k}^{\infty} x_j) = \sum_{r=1}^{N}(-1)^{r-1}\omega(j_r),$$

we have

$$\operatorname{osc}(f, z_{q,k}+G_k) \geqq \omega(k).$$

Now assume that N is even. Let $x \in z_{q,k}+G_k$. We write

$$N(x) = \{j_1, j_2, \cdots\}, \quad j_1<j_2<\cdots.$$

Then $j_{N+1}\geqq k$. Thus we have

$$f(x) = \sum_{r=1}^{N}(-1)^{r-1}\omega(j_r) + \sum_{r\geqq N+1}(-1)^{r-1}\omega(j_r).$$

Since N is even, we have

$$0 \leqq \sum_{r\geqq N+1}(-1)^{r-1}\omega(j_r) \leqq \omega(j_{N+1}) \leqq \omega(k).$$

Thus

$$\operatorname{osc}(f, z_{q,k}+G_k) \leqq \omega(k).$$

Therefore we have

(7) $$\operatorname{osc}(f, z_{q,k}+G_k) = \omega(k).$$

if N is even. Similarly (7) holds if N is odd. Thus we obtain (6).

q.e.d.

COROLLARY 1. (i) *If f is a continuous function on G, then* $\{\omega(f,k)\}_{k=0}^{\infty} \in \Omega$.

(ii) *If* $\{\omega(k)\}_{k=0}^{\infty} \in \Omega$, *then there exists a continuous function f such that* $\omega(f,k) = \omega(k)$ $(k=0,1,2,\cdots)$.

COROLLARY 2. (i) *If f is a continuous function on G, then* $\{v(f,k)\}_{k=0}^{\infty} \in \mathfrak{V}$.

(ii) *If* $\{v(k)\}_{k=0}^{\infty} \in \mathfrak{V}$, *then there exists a continuous function f such that* $v(f,k) = v(k)$ $(k=0,1,2,\cdots)$.

PROOF. (i) Since f is continuous, we have immediately

$$v(f,k) \to 0 \qquad \text{as } k \to \infty.$$

We show that $\{m_k^{-1/2} v(f,k)\}_{k=0}^{\infty}$ is monotone decreasing.
Fix $k \geqq 0$. For every r, $0 \leqq r < m_k$, we put

$$Q_r = \{q : 0 \leqq q < m_{k+1},\ z_{q,k+1}+G_{k+1} \subset z_{r,k}+G_k\}.$$

Since Card $Q_r = p_{k+1}$, we have

$$v(f,k+1) = \Big[\sum_{q=0}^{m_{k+1}-1} \{\operatorname{osc}(f, z_{q,k+1}+G_{k+1})\}^2\Big]^{1/2}$$

$$= \Big[\sum_{r=0}^{m_k-1} \sum_{q \in Q_r} \{\operatorname{osc}(f, z_{q,k+1}+G_{k+1})\}^2\Big]^{1/2}$$

$$\leqq \Big[\sum_{r=0}^{m_k-1} \sum_{q \in Q_r} \{\operatorname{osc}(f, z_{r,k}+G_k)\}^2\Big]^{1/2}$$

$$= \Big[p_{k+1} \sum_{r=0}^{m_k-1} \{\operatorname{osc}(f, z_{r,k}+G_k)\}^2\Big]^{1/2}$$

$$= p_{k+1}^{1/2} v(f,k).$$

Therefore we have

$$m_{k+1}^{-1/2} v(f,k+1) \leqq p_{k+1}^{-1/2} m_k^{-1/2} v(f,k+1) \leqq m_k^{-1/2} v(f,k).$$

(ii) Let $\omega(k)=m_k^{-1/2}v(k)$, $k=0,1,2,\cdots$. Since $\{\omega(k)\}_{k=0}^{\infty}\in\Omega$, there exists a continuous function f which satisfies (6) by Theorem 2. Then it is clear that $v(f,k)=v(k)$ for every $k\geqq 0$.

q.e.d.

5. We shall show that Theorems A, B and C cannot be improved in some sense on G with $p_k=p$ for all $k\geqq 1$.

LEMMA 2([7]). *Let* $\{a_k\}_{k=0}^{\infty}$ *be a bounded sequence of positive numbers which satisfies*

$$\sum_{k=0}^{\infty} a_k=\infty.$$

Then for any δ, $0<\delta<1$, *there exists a subsequence* $\{a_{k_q}\}_{q=1}^{\infty}$ *of* $\{a_k\}_{k=0}^{\infty}$ *such that*

$$\sum_{q=1}^{\infty} a_{k_q}=\infty,$$

$$(1-\delta)^{k_{q+1}-k_q}\leqq a_{k_q}/a_{k_{q+1}}\leqq(1+\delta)^{k_{q+1}-k_q}.$$

LEMMA 3([1]). *Suppose that* $p_k=p$ *for all* $k\geqq 1$. *Then there exists a sequence* $\{Q_k\}_{k=0}^{\infty}$ *of trigonometric polynomials on G such that, for every* $k\geqq 0$,

$$Q_k(x)=\sum_{j=m_k}^{m_{k+1}-1}\hat{Q}_k(j)\chi_j(x) \qquad (x\in G),$$

$$\| Q_k \|_{\infty}\leqq Cm_k^{1/2},$$

$$\sum_{j=m_k}^{m_{k+1}-1} |\hat{Q}_k(j)|\geqq p^k.$$

THEOREM 3. *Suppose that* $p_k=p$ *for all* $k\geqq 1$. *If a sequence* $\omega=\{\omega(k)\}_{k=0}^{\infty}\in\Omega$ *satisfies* $\sum_{k=0}^{\infty} m_{k+1}^{1/2}\omega(k)=\infty$, *there exists a function f in* $\Lambda[\omega]$ *whose Vilenkin Fourier series is not absolutely convergent.*

PROOF. We may assume that $m_{k+1}^{1/2}\omega(k)\leqq 1$ for every $k\geqq 0$. We take α with $p^{-1/2}<\alpha<1$. By Lemma 2, there exists a subsequence $\{\omega(k_q)\}_{q=1}^{\infty}$ of ω such that

$$\sum_{q=1}^{\infty} m_{k_q+1}^{1/2}\omega(k_q)=\infty, \tag{8}$$

$$\alpha^{k_{q+1}-k_q}\leqq m_{k_q}^{1/2}\omega(k_q)/\{m_{k_{q+1}}^{1/2}\omega(k_{q+1})\}. \tag{9}$$

We define

$$f(x)=\sum_{q=1}^{\infty} m_{k_q}^{-1/2}\omega(k_q)Q_{k_q}(x) \qquad (x\in G)$$

where $\{Q_k\}_{k=0}^{\infty}$ is the trigonometric polynomials in Lemma 3.

Since

$$\sum_{q=1}^{\infty} m_{k_q}^{-1/2}\omega(k_q)\,\|\,Q_{k_q}\,\|_{\infty} \leqq C\sum_{q=1}^{\infty} m_{k_q}^{-1/2}\omega(k_q) m_{k_q}^{1/2} \leqq C\sum_{q=1}^{\infty} m_{k_q}^{-1/2} < \infty,$$

the function f is continuous on G.

On the other hand, we have, by (8),

$$\sum_{j=0}^{\infty} |\,\hat{f}(j)\,| \geqq \sum_{q=1}^{\infty} m_{k_q}^{-1/2}\omega(k_q)\,\|\,Q_{k_q}\,\|_{A(G)}$$

$$\geqq \sum_{q=1}^{\infty} m_{k_q}^{-1/2}\omega(k_q)p^{k_q} \geqq \sum_{q=1}^{\infty} m_{k_q}^{1/2}\omega(k_q) = \infty$$

and so the Vilenkin Fourier series of f is not absolutely convergent.

Let N be any positive integer. We take $s \geqq 1$ with $k_{s-1} < N \leqq k_s$. For any $x,y \in G$ with $x-y \in G_{N'}$ we have

$$|\,f(x) - f(y)\,| \leqq \sum_{q=s}^{\infty} m_{k_q}^{-1/2}\omega(k_q)\,|\,Q_{k_q}(x) - Q_{k_q}(y)\,|$$

$$\leqq 2\sum_{q=s}^{\infty} m_{k_q}^{-1/2}\omega(k_q)\,\|\,Q_{k_q}\,\|_{\infty} \leqq C\sum_{q=s}^{\infty}\omega(k_q).$$

By (9) we have

$$\omega(k_q)/\omega(k_{q+1}) \geqq (\alpha p^{1/2})^{k_{q+1}-k_q} \qquad (q \geqq 1)$$

and so

$$\omega(k_q)/\omega(k_s) \leqq (\alpha p^{1/2})^{k_s-k_q} \qquad (q \geqq s).$$

Therefore

$$\sum_{q=s}^{\infty}\omega(k_q) \leqq \omega(k_s)\sum_{q=s}^{\infty}(\alpha p^{1/2})^{k_s-k_q} = O(\omega(k_s)).$$

Since ω is monotone decreasing, we have

$$\omega(f,N) = O(\omega(N)).$$

q.e.d.

COROLLARY 3. *Suppose that* $p_k = p$ *for all* $k \geqq 1$. *For* $\omega = \{\omega(k)\}_{k=0}^{\infty} \in \Omega$, $\Lambda[\omega] \subset A(G)$ *if and only if* $\sum_{k=0}^{\infty} m_{k+1}^{1/2}\omega(k)$ *is convergent.*

COROLLARY 4. *Suppose that* $p_k = p$ *for all* $k \geqq 1$. *For* $v = \{v(k)\}_{k=0}^{\infty} \in \mathfrak{V}$, $V[v] \subset$

$A(G)$ if and only if $\sum_{k=0}^{\infty} p_{k+1}^{1/2} v(k) < \infty$.

PROOF. Nessecity is trivial by Theorem B. Let $\sum_{k=0}^{\infty} p_{k+1}^{1/2} v(k) = \infty$. We put $\omega(k) = m_k^{-1/2} v(k)$. Then $\omega = \{\omega(k)\}_{k=0}^{\infty} \in \Omega$ and $\sum_{k=0}^{\infty} m_{k+1}^{1/2} \omega(k) = \infty$. By Theorem 3, there exists a continuous function f such that $\omega(f,k) \leqq \omega(k)$ for all $k \geqq 0$ and the Vilenkin Fourier series of f is not absolutely convergent. Then it is clear that $v(f,k) \leqq v(k)$ for every $k \geqq 0$. Thus sufficiency is proved.

q.e.d.

THEOREM 4. *Suppose that $p_k = p$ for all $k \geqq 1$. If $d = \{d(k)\}_{k=0}^{\infty} \in \mathfrak{D}$ and $\sum_{k=0}^{\infty} m_{k+1}^{1/2} d(k) = \infty$, then there exists function f in $D[d]$ whose Vilenkin Fourier series is not absolutely convergent.*

PROOF. For $k = 0, 1, 2, \cdots$, we put

$$\lambda_k = \sum_{j=0}^{k} m_{j+1}^{1/2} d(j),$$

$$c(k) = d(k)/\lambda_k.$$

Then

$$\sum_{k=0}^{\infty} c(k) = \sum_{k=0}^{\infty} d(k)/\lambda_k \leqq \sum_{k=0}^{\infty} m_{k+1}^{-1/2} < \infty,$$

$$c(k) = O(d(k)) \qquad \text{as } k \to \infty,$$

$$\sum_{k=0}^{\infty} m_{k+1}^{1/2} c(k) = \sum_{k=0}^{\infty} m_{k+1}^{1/2} d(k)/\lambda_k = \infty.$$

We define the function

$$f(x) = \sum_{k=0}^{\infty} m_{k+1}^{-1/2} c(k) Q_k(x) \qquad (x \in G)$$

where $\{Q_k\}_{k=1}^{\infty}$ is as in Lemma 2.

Then we have

$$\sum_{k=0}^{\infty} m_{k+1}^{-1/2} c(k) \| Q_k \|_{\infty} \leqq C \sum_{k=0}^{\infty} c(k) < \infty$$

and so f is a continuous function on G.

On the other hand, we have

$$\sum_{j=0}^{\infty} |\hat{f}(j)| = \sum_{k=0}^{\infty} m_{k+1}^{-1/2} c(k)(m_{k+1} - m_k) \geqq 2^{-1} \sum_{k=0}^{\infty} m_{k+1}^{1/2} c(k) = \infty$$

and then the Vilenkin Fourier series of f is not absolutely convergent. We have, for every $k \geqq 0$,

$$d(f,k) = \{\sum_{j=m_k}^{m_{k+1}-1} |\hat{f}(j)|^2\}^{1/2} = [\sum_{j=m_k}^{m_{k+1}-1} \{m_{k+1}^{1/2} c(k)\}^2]^{1/2} \leqq c(k)$$

and thus $d(f,k) = O(c(k)) = O(d(k))$ as $k \to \infty$.

q.e.d.

COROLLARY 5. *Suppose that $p_k = p$ for all $k \geqq 1$. Then, for $d = \{d(k)\}_{k=0}^{\infty}$ in $\mathfrak{D}$, $D[d] \subset A(G)$ if and only if $\sum_{k=0}^{\infty} m_{k+1}^{1/2} d(k)$ converges.*

6. We shall show that $V[v] \neq A(G)$ for every v in $\mathfrak{V}$ on G with $p_k = p$ for all $k \geqq 1$.

LEMMA 4. *Let $\{v(k)\}_{k=0}^{\infty} \in \mathfrak{V}$ and $v(k) > 0$ for all $k \geqq 0$. If $\sum_{k=0}^{\infty} v(k) < \infty$,* then there exists a sequence $\{\lambda_k\}_{k=0}^{\infty}$ of positive numbers such that

(10) $\limsup_{k \to \infty} \lambda_k = \infty,$

(11) $\sum_{k=0}^{\infty} v(k)\lambda_k < \infty,$

(12) $m_k^{-1/2} v(k)\lambda_k \downarrow 0$ *as* $k \to \infty$.

PROOF. By $\sum_{k=0}^{\infty} v(k) < \infty$, there exists a sequence $\{\mu_k\}_{k=0}^{\infty}$ of positive numbers such that

$$\sum_{k=0}^{\infty} v(k)\mu_k < \infty, \quad \mu_k \uparrow \infty \text{ as } k \to \infty.$$

For $k = 0,1,2,\cdots$, we put

$$a_k = m_k^{-1/2} v(k)$$

$$\nu_k = \min\{\mu_k, a_k^{1/2}\}.$$

Then ν_k, k=0,1,2,$\cdots$, are positive and

(13) $\nu_k \uparrow \infty$ as $k\to\infty$,

(14) $a_k \nu_k \to 0$ as $k\to\infty$.

Let $k_0=0$. Suppose that $k_0, \cdots, k_{n-1}$ have been defined.

We put

$$K_n = \{k>k_{n-1} : a_k\nu_k \leqq a_{k_{n-1}}\nu_{k_{n-1}}\} .$$

By (14) we have $K_n \neq \phi$. We define

$$k_n = \min K_n.$$

Thus $\{k_n\}_{n=0}^{\infty}$ is defined inductively. We put

$$\lambda_k = a_k^{-1} a_{k_n} \nu_{k_n} \text{ for } k_n \leqq k \leqq k_{n+1} \quad (n=0,1,2,\cdots).$$

Since $\lambda_{k_n} = \nu_{k_n}$ for all $n \geqq 0$, (10) holds by (13).

If $k_n \leqq k < k_{n+1}$, then

$$\lambda_k a_k = a_{k_n} \nu_{k_n} \leqq a_k \nu_k$$

from the definition of k_{n+1}. Thus

$$\lambda_k \leqq \nu_k \quad (k=0,1,2,\cdots).$$

Therefore

$$\sum_{k=0}^{\infty} v(k)\lambda_k \leqq \sum_{k=0}^{\infty} v(k)\nu_k < \infty.$$

Since $a_{k_n}\nu_{k_n} \geqq a_{k_{n+1}}\nu_{k_{n+1}}$, we have

$$a_{k_{n+1}-1}\lambda_{k_{n+1}-1} = a_{k_n}\nu_{k_n} \geqq a_{k_{n+1}}\nu_{k_{n+1}} = a_{k_{n+1}}\lambda_{k_{n+1}}.$$

Therefore $\{a_k\lambda_k\}_{k=0}^{\infty}$ is monotone decreasing. But since

$$a_{k_n}\lambda_{k_n} = a_{k_n}\nu_{k_n} \to 0 \text{ as } n\to\infty,$$

we have $a_k\lambda_k \to 0$ as $k\to\infty$.

q.e.d.

THEOREM 5. *Suppose that $p_k = p$, $k = 1,2,3,\cdots$. Then there is not v in $\mathfrak{V}$ such that $V[v] = A(G)$.*

PROOF. Suppose that $V[v] = A(G)$ for some $v = \{v(k)\}_{k=0}^{\infty} \in \mathfrak{V}$. Then we obtain $\sum_{k=0}^{\infty} v(k) < \infty$ by Corollary 4 and $v(k) > 0$ for all $k \geqq 0$. There exists a sequence $\{\lambda_k\}_{k=0}^{\infty}$ of positive numbers which satisfies (10), (11) and (12) by Lemma 4. Since $\{v(k)\lambda_k\}_{k=0}^{\infty} \in \mathfrak{V}$ by (12), we can find a continuous function f which satisfies $v(f,k) = v(k)\lambda_k$ by Corollary 2(ii). It is clear that f does not belong to $V[v]$ by (10). Since $\sum_{k=0}^{\infty} v(f,k) < \infty$ by (11), $f \in A(G)$ by Theorem A. This contradiction completes the proof.

q.e.d.

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