Sci. Rep. Kanazawa Univ.
Vol. 31, No. 1, 2, pp. 1—4
December 1986

A Note on Prime Ideals in a Formal Power Series Ring

Hideyasu KAWAI*

Department of Mathematics, Faculty of Science, Kanazawa University

(Received September 17, 1986)

Abstract. Let V be a valuation ring. We study relations between the prime ideals of V and those of V $[[X]]$.

Introduction.

Let V be a valuation ring and let V $[[X]]$ be the formal power series ring over V. If P is a prime ideal of V, we denote by $P\cdot V$ $[[X]]$ the ideal generated by P in V $[[X]]$. In this paper we consider the following question : When is $P\cdot V$ $[[X]]$ a prime ideal of V $[[X]]$? This question has answers in the following two cases. The first is the case where P is finitely generated. The second is the case where P cannot be generated by countably many elements. In both cases, $P\cdot V$ $[[X]]$ is prime because the ideal P $[[X]]=\{\sum_{i=0}^{\infty} a_i X^i \varepsilon V[[X]]\ ;\ a_i \varepsilon P,\ i=0,1,2,\dots\}$ is prime and the equality $P\cdot V[[X]]=P$ $[[X]]$ holds (Lemma 1. 1). Hence we will descuss only the case where P is countably generated i.e. P can be generated by at most countably many elements of V. Then our main theorem is : *If V is a valuation ring, and P is a prime ideal of V which is countably generated, then the following statements hold.*

(1) *If $P=P^2$, and there exists a prime ideal Q in V such that the height of P/Q is* 1, *then $P\cdot V$ $[[X]]$ is prime.*

(2) *If $P\neq P^2$, then $P\cdot V$ $[[X]]$ is prime if and only if P is principal.*

Standard Notation :

(i) All rings in this paper are assumed to be commutative with identity.

(ii) For a subset A of a ring R, (A) denotes the ideal generated by A in R.

(iii) We assume every prime ideal in a ring R doesn't contain the identity element of R.

(iv) If I is an ideal of a ring R, we denote by $I\cdot R$ $[[X]]$ the ideal generated by I in the formal power series ring R $[[X]]$, and by I $[[X]]$the set of elements $f(X)$ of R

* Present address : Ishikawa College of Technology, Tsubata, Ishikawa, Japan

$[[X]]$ whose coefficients are all in I.

(v) For an ideal I of a ring R and $a \varepsilon R$, $\bar{a}$ will denote the residue class of a in $\bar{R}=R/I$. And also for $f=\Sigma_{i=0}^{\infty} a_i X^i \varepsilon R[[X]]$, $\bar{f}$ will denote the power series $\Sigma_{i=0}^{\infty} \bar{a}_i X^i$ of $\bar{R}[[X]]$.

For undefined terminology, our general reference is [4] .

Preliminary results.

We recall the following lemmas which will be needed in the next section.

LEMMA 1.1. ([5, Proposition 1]) *Let R be a ring and let A be an ideal of R. Then the equality $A \cdot R[[X]]=A[[X]]$ holds if and only if A satisfies the following condition: For any countably generated ideal B contained in A, there exists an ideal C such that C is finitely generated and $B \subseteq C \subseteq A$.*

LEMMA 1.2. ([3, Lemma 4.2]) *Let V be a valuation ring and let P be a prime ideal of V. If $P=P^2$, then $P \cdot V[[X]]=(PV_p) \cdot V_p[[X]]$.*

LEMMA 1. 3. ([3, Lemma 2.1]) *Let V be a valuation ring and let P be a prime ideal of V. If $P \neq P^2$, then $(P[[X]])^2 \subseteq P \cdot V[[X]]$. In particular, $\sqrt{P \cdot V[[X]]}=P[[X]]$.*

LEMMA 1.4. ([2, Theorem 24]) *Let V be a valuation ring of rank 1 and let M be the maximal ideal of V. Then $M \cdot V[[X]]$ is a prime ideal of $V[[X]]$.*

LEMMA 1. 5. (c. f. [6, Corollary 11]) *If V is a valuation ring with only countably many prime ideals, then every ideal of V is countably generated.*

Main theorem.

First, we prove the following lemma.

LEMMA 2.1. *If V is a valuation ring and P is a prime ideal of height 1, then the following two statements hold:*

(1) *If $P=P^2$,* then *$P \cdot V[[X]]$ is a prime ideal of $V[[X]]$, and $P \cdot V[[X]] \neq P[[X]]$.*

(2) *If $P \neq P^2$, then $P \cdot V[[X]]$ is a prime ideal of $V[[X]]$ if and only if $P \cdot V[[X]]=P[[X]]$.*

PROOF. (1). By Lemma 1.2, $P \cdot V[[X]]=(PV_p) \cdot V_p[[X]]$. The right hand side of this equality is a prime ideal of the power series ring $V_p[[X]]$ by Lemma 1.4. Hence $P \cdot V[[X]]$ is a prime ideal of $V[[X]]$. By [1, Proposition 3.2], $P \cdot V[[X]] \neq P[[X]]$.
(2). This is immediate from Lemma 1.3.

We now state and prove our main theorem.

THEOREM 2.2. *Let V be a valuation ring and let P be a prime ideal of V which is countably generated. Then, the following statements hold:*

(1) *If $P=P^2$, and there exists a prime ideal Q in V such that the height of P/Q is* 1, *then $P\cdot V$ $[[X]]$ is prime.*

(2) *If $P\neq P^2$, then $P\cdot V$ $[[X]]$ is a prime ideal if and only if P is principal.*

PROOF. First, we prove (2). Let P be principal. Then $P\cdot V$ $[[X]]=P$ $[[X]]$ by Lemma 1.1. So, $P\cdot V$ $[[X]]$ is prime. If $P\cdot V$ $[[X]]$ is prime, then $P\cdot V$ $[[X]]=P$ $[[X]]$ by Lemma 1.3. Hence P is principal by Lemma 1.1. Secondly we turn to the proof of (1). Let $\bar{V}=V/Q$ and let $\bar{P}=P/Q$. Then $\bar{P}$ is a prime ideal of the valuation ring $\bar{V}$ such that $\bar{P}=\bar{P}^2$ and the height of $\bar{P}$ is 1. So, $\bar{P}\cdot\bar{V}$ $[[X]]$ is a prime ideal of $\bar{V}$ $[[X]]$ by Lemma 2.1. If $\bar{f}\varepsilon\bar{P}\cdot\bar{V}$ $[[X]]$, then $\bar{f}=\bar{p}\cdot\bar{g}$ for some element p of P and some g of V $[[X]]$. Let $f=\sum_{i=0}^{\infty}a_iX^i\varepsilon V$ $[[X]]$. *Then* $a_i\varepsilon(p)+Q$ for every i. If $p\not\varepsilon Q$, then $a_i\varepsilon(p)$ for every i. Hence $f\varepsilon P\cdot V$ $[[X]]$. If $p\varepsilon Q$, then $a_i\varepsilon Q$ for every i. But $Q\subsetneq(p')$ for some element p' of P. So we have $a_i\varepsilon(p')$ for every i. Hence $f\varepsilon P\cdot V$ $[[X]]$. Thus we have seen that $f\varepsilon P\cdot V$ $[[X]]$ if $\bar{f}\varepsilon\bar{P}\cdot\bar{V}$ $[[X]]$. Hence $P\cdot V$ $[[X]]$ is a prime ideal of V $[[X]]$. This completes the proof of the theorem.

COROLLARY 2.3. *Let V be a valuation ring of finite rank. Let P be a prime ideal of V Then the following conditions are equivalent.*

(1) $P\cdot V$ $[[X]]$ is a prime ideal of V $[[X]]$.

(2) $P=P^2$ or P is principal.

PROOF. By Lemma 1.5, P is countably generated. First, we assume that (1) holds. If $P\neq P^2$, then P is principal by Theorem 2.2. So (1) implies (2). Now we assume (2) holds. If P is principal, then (1) holds by Lemma 1.1. On the other hand, if $P=P^2$, and $P\neq(0)$, then (1) holds by Theorem 2.2.

Example.

We note the following lemma before showing an example of a valuation ring V with a nonprime $P\cdot V$ $[[X]]$.

LEMMA 3.1. *Let G be a totally ordered abelian group. Let v be a valuation on a field K with value group G and let V be the valuation ring of v. Let H_1 and H_2 be consecutive convex subgroups of G such that $H_1\subsetneq H_2$, and let P_1 and P_2 be the prime ideals of V corresponding to H_1 and H_2, respectively (i. e. $P_i=\{x\varepsilon V;v(x)\not\varepsilon H_i\}$ for each i.). The following conditions are equivalent.*

(1) $P_1\neq P_1^2$

(2) The ordered factor group H_2/H_1 and the additive group of integers $\mathbf{Z}$ are isomorphic as ordered groups.

PROOF. This can be deduced from [4,§17, Exercises 22 and 31].

Example. Let $G_1 = \mathbf{Z}$, the additive group of integers, and let $G_2 = \mathbf{Q}$, the additive group of rational numbers. Let $G = G_1 \oplus G_2$, the lexicographic sum, and let V be a valuation ring with value group G. Then, the rank of V is 2. If P is the height one prime ideal of V and M is the maximal ideal of V, then P and M correspond to the proper convex subgroups $\mathbf{Q}$ and $\{0\}$ of G, respectively. Thus neither P nor M is principal. Moreover, by Lemma 3.1, $P \neq P^2$, $M = M^2$. So, by Corollary 2.3, $P \cdot V[[X]]$ is not prime, but $M \cdot V[[X]]$ is prime.

References

[1] J. T. Arnold, R. Gilmer and W. Heinzer, Some countability conditions in a commutative ring, Ill. J. Math., **21** (1977), 648-665.

[2] J. W. Brewer, *Power series over commutative rings,* Marcel Dekker, New York, 1981.

[3] D. E. Fields, Dimension theory in power series rings, Pac. J. Math., **35** (1970), 601-611.

[4] R. Gilmer, *Multiplicative Ideal Theory,* Marcel Dekker, New York, 1972.

[5] R. Gilmer and W. Heinzer, Rings of formal power series over a Krull domain, Math. Zeitschr., **106** (1968), 379-387.

[6] R. Gilmer and W. Heinzer, Cardinality of generating sets for ideals of a commutative ring, Indiana Univ. Math. J., **26** (1977), 791-798.