Sci. Rep. Kanazawa Univ.
Vol. 29 No. 2 pp. 97-102
December 1984

Absolute Convergence of Vilenkin Fourier Series

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(Received October 30, 1984)

Abstract. We shall give a theorem on the absolute convergence of Vilenkin Fourier series by using the notion of λ-bounded p-fluctuation.

1. Introduction

M. Shiba ([4]) generalized a theorem of M. and S. Izumi for the absolute convergence of trigonometric Fourier series by using the notion of λ-bounded variation. M. Schramm and D. Waterman ([3]) extended the result of M. Shiba. In this paper, we consider analogue of these results to Vilenkin Fourier series.

2. Notations and Definitions

Through this paper, G denotes a Vilenkin group, that is, an infinite, compact, metrizable, zero-dimensional abelian group. Then the dual group X of G is an infinite, discrete, countable, torsion abelian group. N. Ja. Vilenkin proved that there exist an increasing sequence $\{X_n\}_{n=0}^{\infty}$ of finite subgroups of X and a sequence $\{p_n\}_{n=1}^{\infty}$ of prime numbers with the following properties:

(i) $X_0=\{\chi_0\}$, $\chi_0=1$ on G,

(ii) the quotient group X_n/X_{n-1} is of order p_n for every $n\geqq 1$,

(iii) $X=\bigcup_{n=0}^{\infty} X_n$.

Let G_n be the annihilator of X_n for every $n\geqq 0$. Then $\{G_n\}_{n=0}^{\infty}$ is a decreasing sequence of open subgroups of G and is a fundamental basis of neighbourhoods of 0 in G.

Let $m_0=1$ and $m_n=p_1\cdots p_n$ for every $n\geqq 1$. We take $\varphi_n\in X_{n+1}\backslash X_n$ for every $n\geqq 0$. For an integer $k\geqq 1$, we can represent k as

$$k=\sum_{j=0}^{s} a_j m_j,\ 0\leqq a_j<p_{j+1}\ (0\leqq j\leqq s),\ a_s\neq 0.$$

Then we put $\chi_k=\prod_{j=0}^{s} \varphi_j^{a_j}$. We have $X_n=\{\chi_k : 0\leqq k<m_n\}$ for every $n\geqq 0$.

For $f\in L^1(G)$, we denote the k-th Fourier coefficient of f by

$$\hat{f}(k)=\int_G f(x)\,\overline{\chi_k(x)}\,dx$$

for every $k\geqq 0$ and we define the Vilenkin Fourier series of f by

$$S[f]=\sum_{k=0}^{\infty} \hat{f}(k)\,\chi_k(x)\ (x\in G).$$

DEFINITION 1. For $f\in L^p(G)$, $0<p\leqq\infty$ and an integer $N\geqq 0$, the integral modulus of continuity of order N for f is defined by

$$\omega_p(f\,;N)=\sup\{\|f_y-f\|_p : y\in G_N\}$$

where $f_y(x)=f(x-y)$ for every $x, y\in G$.

DEFINITION 2. For a function f on G and a subset H of G, the oscillation of f on H is defined by

$$\mathrm{osc}(f\,;H)=\sup\{|f(x)-f(y)| : x, y\in H\}.$$

For every $n\geqq 0$, $\{z_{q,n}\}_{q=0}^{m_n-1}$ is a sequence of elements in G such that $\{z_{q,n}+G_n\}_{q=0}^{m_n-1}$ is all cosets of G_n in G.

Let Λ be the set of increasing sequences of positive numbers.

DEFINTION 3. For $\lambda=\{\lambda_n\}_{n=1}^{\infty}\in\Lambda$, $0<p<\infty$ and a function f on G, we define

$$V_\lambda^{(p)}(f)=\sup_{N\geqq 0}\ \sup_{\{q_j\}_{j=0}^{m_N-1}} \sum_{j=0}^{m_N-1} \frac{\{\mathrm{osc}(f, z_{q_j,N}+G_N)\}^p}{\lambda_{j+1}}$$

where $\{q_j\}_{j=0}^{m_N-1}$ runs over all permutations of $\{0, 1, 2, \cdots, m_N-1\}$.

If $V_\lambda^{(p)}(f)<\infty$, we call f a function of generalized λ-bounded p-fluctuation. We denote by $G\lambda BF^{(p)}(G)$ the set of all functions of generalized λ-bounded p-fluctuation.

3. Analogue of a theorem of M. Shiba and a theorem of M. Schramn and D. Waterman.

LEMMA 1 ([3]). *If* $c_1\geqq c_2\geqq\cdots\geqq c_N>0$, $\sum_{k=1}^{N} c_k=1$ *and* $a_1\geqq a_2\geqq\cdots\geqq a_N$, *then*

$$\frac{1}{N}\sum_{k=1}^{N} a_k\leqq \sum_{k=1}^{N} c_k a_k.$$

LEMMA 2. *Let* $N\geqq 0$ *and* $n\geqq m_N$. *Then*

$$\int_{G_N} |\chi_n(y)-1|^2 dy=2|G_N|.$$

PROOF. Since $D_{m_N}(x)=\sum_{k=0}^{m_N-1}\chi_k(x)=|G_N|^{-1}I_{G_N}(x)$, we have

$$\int_{G_N}|\chi_n(y)-1|^2dy=|G_N|\int_G D_{m_N}(y)|\chi_n(y)-1|^2dy=2|G_N|,$$

where I_{G_N} is the characteristic function of the set G_N.

THEOREM (Analogue of a theorem of M. Schramm and D. Waterman) *Let* $1\leqq r<\infty$, $1/r+1/s=1$, $1\leqq p<2r$ *and* $\lambda=\{\lambda_n\}_{n=1}^{\infty}\in\Lambda$. *If a Borel function* $f\in \mathrm{G}\lambda\mathrm{BF}^{(p)}(G)$ *satisfies*

$$\sum_{N=0}^{\infty}(m_{N+1})^{1/2}\,\omega_{(2-p)s+p}(f,N)^{1-p/2r}\Big(\sum_{k=1}^{m_N}\frac{1}{\lambda_k}\Big)^{-1/2r}<\infty,$$

then the Vilenkin Fourier series of f *converges absolutely.*

PROOF. For $N\geqq 0$, we set

$$\Omega_N=\omega_{(2-p)s+p}(f,N)^{2r-p}$$

and

$$\mu_N=\sum_{k=1}^{m_N}\frac{1}{\lambda_k}.$$

For $y\in G_N$, we have

$$\int_G|f(x+y)-f(x)|^2\,dx$$

$$\leqq\Big\{\int_G|f(x+y)-f(x)|^{(2-p/r)s}dx\Big\}^{1/s}\Big\{\int_G|f(x+y)-f(x)|^pdx\Big\}^{1/r}$$

$$\leqq\Omega_N^{1/r}\Big\{\int_G|f(x+y)-f(x)|^p\,dx\Big\}^{1/r}.$$

Thus we obtain, for every $0\leqq q\leqq m_N-1$ and $y\in G_N$,

$$\int_G|f(x+z_{q,N}+y)-f(x+z_{q,N})|^2\,dx$$

$$\leqq\Omega_N^{1/r}\Big\{\int_G|f(x+z_{q,N}+y)-f(x+z_{q,N})|^pdx\Big\}^{1/r}$$

and so

$$\sum_{q=0}^{m_N-1}\Big\{\int_G|f(x+z_{q,N}+y)-f(x+z_{q,N})|^2\,dx\Big\}^r$$

$$\leqq\Omega_N\int_G\sum_{q=0}^{m_N-1}|f(x+z_{q,N}+y)-f(x+z_{q,N})|^p\,dx.$$

For every $x\in G$ and every $y\in G_N$, we take $\{q_j\}_{j=0}^{m_N-1}$ such that the sequence

$$\{|f(x+z_{q_j,N}+y)-f(x+z_{q_j,N})|\}_{j=0}^{m_N-1}$$

is monotone decreasing. Lemma 1 implies that

$$\sum_{j=0}^{m_N-1} |f(x+z_{q_j,N}+y)-f(x+z_{q_j,N})|^p$$

$$\leqq m_N \sum_{j=0}^{m_N-1} \frac{|f(x+z_{q_j,N}+y)-f(x+z_{q_j,N})|^p}{\mu_N \ \lambda_{j+1}}$$

$$\leqq \frac{m_N}{\mu_N} \sum_{j=0}^{m_N-1} \frac{\{\mathrm{osc}(f,\ x+z_{q_j,N}+G_N)\}^p}{\lambda_{j+1}}$$

$$\leqq m_N \ \mu_N^{-1} \ V_\lambda^{(p)}(f).$$

It follows that

$$\sum_{q=0}^{m_N-1} \{\int_G |f(x+z_{q,N}+y)-f(x+z_{q,N})|^2 dx\}^r$$

$$\leqq \Omega_N \ m_N \ \mu_N^{-1} \ V_\lambda^{(p)}(f).$$

But we have, for $y \in G_N$ and $0 \leqq q \leqq m_N - 1$,

$$\int_G |f(x+z_{q,N}+y)-f(x+z_{q,N})|^2 dx = \int_G |f(x+y)-f(x)|^2 dx$$

$$= \sum_{n=0}^{\infty} |\hat{f}(x)|^2 |\chi_n(y)-1|^2 = \sum_{n=m_N}^{\infty} |\hat{f}(n)|^2 |\chi_n(y)-1|^2.$$

Thus we obtain, for $y \in G_N$,

$$m_N \ \{\sum_{n=m_N}^{\infty} |\hat{f}(n)|^2 |\chi_n(y)-1|^2\}^r$$

$$= \sum_{q=0}^{m_N-1} \{\int_G |f(x+z_{q,N}+y)-f(x+z_{q,N})|^2 dx\}^r$$

$$\leqq V_\lambda^{(p)}(f) \ \Omega_N \ m_N \ \mu_N^{-1}$$

and hence

$$\sum_{n=m_N}^{\infty} |\hat{f}(n)|^2 |\chi_n(y)-1|^2 \leqq V_\lambda^{(p)}(f)^{1/r} \ \Omega_N^{1/r} \ \mu_N^{-1/r}.$$

Thus by Lemma 3 we have

$$2 \sum_{n=m_N}^{\infty} |\hat{f}(n)|^2$$

$$= |G_N|^{-1} \int_{G_N} \sum_{n=m_N}^{\infty} |\hat{f}(n)|^2 |\chi_n(y)-1|^2 dy$$

$$\leqq V_\lambda^{(p)}(f)^{1/r} \ \Omega_N^{1/r} \ \mu_N^{-1/r}.$$

Therefore we obtain

$$\sum_{n=1}^{\infty} |\hat{f}(n)| = \sum_{N=0}^{\infty} \sum_{n=m_N}^{m_{N+1}-1} |\hat{f}(n)|$$

$$\leqq \sum_{N=0}^{\infty} (m_{N+1}-m_N)^{1/2} \{ \sum_{n=m_N}^{m_{N+1}-1} |\hat{f}(n)|^2\}^{1/2}$$

$$\leqq (1/\sqrt{2}) V_{\lambda}^{(p)}(f)^{1/2r} \sum_{N=0}^{\infty} m_{N+1}^{1/2} \Omega_N^{1/2r} \mu_N^{-1/2r} < \infty.$$

q. e. d.

COROLLARY 1(Analoque of a theorem of M. Shiba). *Let* $1 \leqq r < \infty$, $1/r+1/s=1$, $1 \leqq p < 2r$ *and* $\lambda = \{\lambda_n\}_{n=1}^{\infty} \in \Lambda$. *If a Borel function f in* $G\lambda BF^{(p)}(G)$ *satisfies*

$$\sum_{N=0}^{\infty} p_{N+1}^{1/2} m_N^{1/2-1/2r} \lambda_{m_N}^{1/2r} \omega_{(2-p)s+p}(f, N)^{1-p/2r} < \infty,$$

then the Vilenkin Fourier series of f is converges absolutely.

PROOF. Since $\{\lambda_n\}_{n=1}^{\infty}$ is monotone increasing, we have

$$\sum_{k=1}^{m_N} \frac{1}{\lambda_k} \geqq \frac{m_N}{\lambda_N}$$

and then Corollary 1 follows from Theorem. q. e. d.

COROLLARY 2 (Analogue of a theorem of A. Zygmund [2]). *If* $f \in GBF(G) = G\{1\} BF^{(1)}(G)$ *and*

$$\sum_{N=0}^{\infty} \{p_{N+1}\, \omega_{\infty}(f\,;\,N)\}^{1/2} < \infty,$$

then the Vilenkin Fourier series of f converges absolutely.

This Corollary is the case of $\lambda_n=1$, $p=1$, $r=1$ in Corollary 1.

COROLLARY 3 ([1]). *Let G be bounded, i. e.,* $\sup p_n < \infty$. *Let* $0 < \beta < \alpha \leqq 1$, $\lambda = \{\lambda_n\}_{n=1}^{\infty} \in \Lambda$ *and* $f \in \mathrm{Lip}\,\alpha \cap G\lambda BF^{(1)}(G)$. *If* $\lambda_{m_N} = O(m_N^{\beta})$ *as* $N \to \infty$, *then the Vilenkin Fourier series of f converges absolutely.*

PROOF. Since

$$\sum_{N=0}^{\infty} p_{N+1}^{1/2} \lambda_{m_N}^{1/2} \omega_{\infty}(f\,;\,N)^{1/2} \leqq \text{const.} \sum_{N=0}^{\infty} m_N^{(\beta-\alpha)/2} < \infty,$$

this Corollary follows by applying Corollary 1 as $p=r=1$.

q. e. d.

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