Sci. Rep. Kanazawa Univ.
Vol. 29 No. 2 pp. 91-95
December 1984

On Generalized Siegel Domains of Tube Type

Akio KODAMA

Department of Mathematics, Faculty of Science, Kanazawa University

(Received September 25, 1984)

Abstract. This is a continuation of the previous paper [6], and the structures of qeneralized Siegel domains of tube type in the sense of Kaup, Matsushima and Ochiai [4] are studied.

1. Introduction.

Extending the notion of Siegel domains of the first and the second kind due to Pjateckii-Sapiro [8], Kaup, Matsushima and Ochiai introduced the following domain $\mathscr{D}$ in $C^n \times C^m$ [4]:

(1) $\mathscr{D}$ is biholomorphically equivalent to a bounded domain in C^{n+m} and $\mathscr{D} \cap (C^n \times \{O\}) \neq \phi$, where O denotes the origin of C^m,

(2) $\mathscr{D}$ is invariant by the following transformations of C^{n+m} of the following types:

i) $(z, w) \rightarrow (z+a, w)$ for all $a \in R^n$;

ii) $(z, w) \rightarrow (z, e^{\sqrt{-1}t} w)$ for all $t \in R$;

iii) $(z, w) \rightarrow (e^t z, e^{ct} w)$ for all $t \in R$,

where $(z, w) = (z_1, \cdots, z_n, w_1, \cdots, w_m)$ denotes a coordinate system in $C^n \times C^m$ and c is a fixed real number depending only on $\mathscr{D}$. Such a domain is called a *generalized Siegel domain with exponent c*. Now, in the previous paper [6] we investigated exclusively the structures of generalized Siegel domains in $C \times C^m$ with arbitrary exponents and obtained the following result: *Let $\mathscr{D}$ and $\mathscr{D}'$ be generalized Siegel domains in $C \times C^m$. Then $\mathscr{D}$ and $\mathscr{D}'$ are biholomorphically equivalent if and only if there exists a non-singular linear mapping $\mathscr{L}$: $C \times C^m \rightarrow C \times C^m$ such that $\mathscr{L}(\mathscr{D}) = \mathscr{D}'$.* As a continuation of this, we study in this note the structures of generalized Siegel domains in the case $m=0$, so that domains are contained in C^n.

Let $\mathscr{D}$ be a generalized Siegel domain C^n. Then $\mathscr{D}$ can be written in the form

(1. 1) $$\mathscr{D} = \{z \in C^n ; Im\ z \in \Omega\},$$

where Ω is an open, not necessarily convex, cone in R^n containing no straight line. So, if Ω is an open convex cone, $\mathscr{D}$ is nothing but a Siegel domain of the first kind by Pjateckii-Sapiro. In this note we often write $\mathscr{D}(\Omega)$ instead of the domain $\mathscr{D}$ as in (1. 1), and we refer to it as a *generalized Siegel domain of tube type.*

Our main purpose is to prove the following theorems:

THEOREM 1. *Let $\mathscr{D}(\Omega)$ and $\mathscr{D}(\Omega')$ be generalized Siegel domains in C^n of tube type. Then $\mathscr{D}(\Omega)$ and $\mathscr{D}(\Omega')$ are biholomorphically equivalent if and only if there exists a non-singular linear mapping $\mathscr{L}: C^n \to C^n$ such that $\mathscr{L}(\mathscr{D}(\Omega)) = \mathscr{D}(\Omega')$.*

THEOREM 2. *Any generalized Siegel domain $\mathscr{D}(\Omega)$ of tube type can be imbedded in a complex projective space $P_n(C)$ as an open subset of an algebraic subvariety V of $P_n(C)$ such that every holomorphic automorphism of $\mathscr{D}(\Omega)$ is induced by a projective transformation leaving V and $\mathscr{D}(\Omega)$ invariant.*

In the special case where Ω and Ω' are open convex cones in the above theorems, these are well-known results by Kaup, Matsushima and Ochiai [4] .

In the next section, using a characterization theorem of Kobayashi hyperbolicity among geometrically convex domains in C^n due to Barth [1] , we first prove that the envelope of holomorphy of any generalized Siegel domain in C^n of tube type is a Siegel domain of the first kind. Then we can apply the results by Kaup, Matsushima and Ochiai [4] to our proofs.

2. Proofs of the theorems

Let $\mathscr{D} = \mathscr{D}(\Omega)$ be a generalized Siegel domain in C^n of tube type. Let Aut($\mathscr{D}$) be the Lie group of all holomorphic automorphisms of $\mathscr{D}$ and let $\mathfrak{g}(\mathscr{D})$ be the real Lie algebra consisting of all complete holomorphic vector fields on $\mathscr{D}$. From the definition of generalized Siegel domains, we see that

$$E = \sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k} \in \mathfrak{g}(\mathscr{D}).$$

Let us set, for any $\lambda \in R$,

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g}(\mathscr{D}) ; [E, X] = \lambda X\} .$$

Then it is known [4 ; Theorem 4] that $\mathfrak{g}(\mathscr{D})$ has the following graded structure:

$$(2.\ 1) \qquad \mathfrak{g}(\mathscr{D}) = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1, \ [\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}.$$

Now we study the envelope of holomorphy of the tube domain $\mathscr{D}(\Omega)$. Let $\hat{\Omega}$ be the convex envelope of Ω in R^n. Then the envelope of holomorphy of $\mathscr{D}(\Omega)$ coincides with the tube domain $\mathscr{D}(\hat{\Omega}) = \{z \in C^n ; Im\ z \in \hat{\Omega}\}$ [7 ; Chap. 7] . We need the following basic fact:

LEMMA 1. (See [7; Chap. 6, Proposition 2]). *Let* $H : \mathscr{D}(\hat{\Omega}) \to C$ *be the unique holomorphic extension of a holomorphic function* $h : \mathscr{D}(\Omega) \to C$. *Then* $H(\mathscr{D}(\hat{\Omega})) = h(\mathscr{D}(\Omega))$. The following lemma is an essential part of the proofs.

LEMMA 2. *The envelope of holomorphy* $\mathscr{D}(\hat{\Omega})$ *of a generalized Siegel domain* $\mathscr{D}(\Omega)$ *of tube type is a Siegel domain of the first kind.*

Proof. We first notice that every holomorphic automorphim of a given domain M can be extended uniquely to a holomorphic automorphism of its envelope of holomorphy $\hat{M}$ [7]. Consequently, $\mathscr{D}(\hat{\Omega})$ is invariant by the following transformations of C^n: (i) $z \to z + a (a \in R^n)$ and (ii) $z \to e^t z (t \in R)$, since the restrictions of these transformations to $\mathscr{D}(\Omega)$ are holomorphic automorphisms of $\mathscr{D}(\Omega)$. Also it is obvious that $\hat{\Omega}$ is an open convex cone in R^n. Thus we have only to show that $\mathscr{D}(\hat{\Omega})$ is biholomorphically equivalent to a bounded domain in C^n. Since $\mathscr{D}(\hat{\Omega})$ is a geometrically convex domain in C^n, by a result of Barth [1] it is enough to prove that $\mathscr{D}(\hat{\Omega})$ contains no complex affine line in C^n. Recall that $\mathscr{D}(\Omega)$ is biholomorphically equivalent to some bounded domain $\mathscr{B}$ in C^n. Let us now fix a biholomorphic mapping $f = (f_1, \cdots, f_n) : \mathscr{D}(\Omega) \to \mathscr{B}$. We may assume that

$$|f_i(z)| < 1, \ z \in \mathscr{D}(\Omega) \text{ for } i = 1, \cdots, n.$$

Extending each component f_i to a holomorphic function F_i on $\mathscr{D}(\hat{\Omega})$, we obtain a holomorphic mapping $F = (F_1, \cdots, F_n) : \mathscr{D}(\hat{\Omega}) \to C^n$. Let J_f (resp. J_F) be the complex Jacobian determinant of f (resp. of F). Then $J_f(z) \neq 0$ for all $z \in \mathscr{D}(\Omega)$, and $J_F : \mathscr{D}(\hat{\Omega}) \to C$ is the unique holomorphic extension of $J_f : \mathscr{D}(\Omega) \to C$. Hence it follows from Lemma 1 that

$$J_F(\mathscr{D}(\hat{\Omega})) = J_f(\mathscr{D}(\Omega)) \not\ni 0 \text{ and } F(\mathscr{D}(\hat{\Omega})) \subset \Delta^n,$$

where Δ^n denotes the unit polydisk in C^n. This implies that $F : \mathscr{D}(\hat{\Omega}) \to C^n$ gives rise to a local biholomorphic mapping from $\mathscr{D}(\hat{\Omega})$ into Δ^n. We are now prepared to complete the proof. Assume that $\mathscr{D}(\hat{\Omega})$ contains a complex affine line L in C^n. Then the restriction $F|_L : L \to C^n$ is holomorphic mapping from $L \cong C$ into the unit polydisk Δ^n, and hence it reduces to a constant mapping [5]. Clearly this is a contradiction, because we know that $F : \mathscr{D}(\hat{\Omega}) \to C^n$ is a local biholomorphic mapping. Thus $\mathscr{D}(\hat{\Omega})$ cannot contain any complex affine line in C^n, as desired. Q. E. D.

PROOF OF THEOREM 1: It is trivial that $\mathscr{D}(\Omega)$ and $\mathscr{D}(\Omega')$ are biholomorphically equivalent if there exists a non-singular linear mapping $\mathscr{L} : C^n \to C^n$ such that $\mathscr{L}(\mathscr{D}(\Omega)) = \mathscr{D}(\Omega')$. Thus we have only to prove the converse.

Assume that there exists a biholomorphic mapping $\Phi : \mathscr{D}(\Omega) \to \mathscr{D}(\Omega')$. Denoting by $\mathscr{D}(\hat{\Omega})$ (resp. $\mathscr{D}(\hat{\Omega}')$) the envelope of holomorphy of $\mathscr{D}(\Omega)$ (resp. of $\mathscr{D}(\Omega')$), we now have a biholomorphic mapping $\hat{\Phi} : \mathscr{D}(\hat{\Omega}) \to \mathscr{D}(\hat{\Omega}')$, which is of course the holomorphic extension of Φ. Since both $\mathscr{D}(\hat{\Omega})$ and $\mathscr{D}(\hat{\Omega}')$ are Siegel domains of the first kind by Lemma 2, it follows then from [4; Theorem 11] that there exists a non-singular linear mapping $\mathscr{L} : C^n \to C^n$ such that $\mathscr{L}(\mathscr{D}(\hat{\Omega})) = \mathscr{D}(\hat{\Omega}')$. Here we want to prove that this linear mapping $\mathscr{L}$ can be chosen in such a way that $\mathscr{L}(\mathscr{D}(\Omega))=\mathscr{D}(\Omega')$.

Let $\mathfrak{g}(\mathscr{D}(\Omega))=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$ be the decomposition of $\mathfrak{g}(\mathscr{D}(\Omega))$ as in (2. 1). Then we can check along the same line as in [4; Lemma 8.3] that there is a principal maximal solvable subalgebra $\mathfrak{m}$ of $\mathfrak{g}(\mathscr{D}(\Omega))$ such that

(2. 2) $\mathfrak{m}$ contains the vector field $E=\sum_{k=1}^{n} z_k\frac{\partial}{\partial z_k}$, and

(2. 3) $\mathfrak{m}=\mathfrak{g}_{-1}+\mathfrak{m}\cap\mathfrak{g}_0$.

Let $\mathfrak{m}'$ be a principal maximal solvable subalgebra of $\mathfrak{g}(\mathscr{D}(\Omega'))$ satisfying the similar conditions as in (2. 2) and (2. 3). Once the following two assertions are verified, our proof can be done with exactly the same arguments as in the proof of [4; Theorem 11]:

(2. 4) $\Phi:\mathscr{D}(\Omega)\rightarrow\mathscr{D}(\Omega')$ is a birational holomorphic mapping;

(2. 5) $\Phi_*(\mathfrak{m})=\mathfrak{m}'$,

where $\Phi_*:\mathfrak{g}(\mathscr{D}(\Omega))\rightarrow\mathfrak{g}(\mathscr{D}(\Omega'))$ is the Lie algebra isomorphism induced by the differential of Φ. First, since $\Phi_*(\mathfrak{m})$ is a principal maximal solvable subalgebra of $\mathfrak{g}(\mathscr{D}(\Omega'))$ and since any two such subalgebras of $\mathfrak{g}(\mathscr{D}(\Omega'))$ are conjugate under an inner automorphism, there exists a holomorphic automorphism Ψ of $\mathscr{D}(\Omega')$ such that $\Psi_*(\Phi_*(\mathfrak{m}))=\mathfrak{m}'$ [4]. So, taking $\Psi\circ\Phi$ instead of Φ if necessary, we may assume that (2. 5) is satisfied. Secondly, since both $\mathscr{D}(\hat{\Omega})$ and $\mathscr{D}(\hat{\Omega}')$ are Siegel domains by Lemma 2, we know by [4; Theorem 10] that the biholomorphic mapping $\hat{\Phi}:\mathscr{D}(\hat{\Omega})\rightarrow\mathscr{D}(\hat{\Omega}')$ is birational. Hence, being its restriction to $\mathscr{D}(\Omega)$, $\Phi:\mathscr{D}(\Omega)\rightarrow\mathscr{D}(\Omega')$ is necessarily birational. Therefore, the condition (2. 4) is also satisfied, as desired. Q. E. D.

PROOF OF THEOREM 2: Let $\mathscr{D}(\hat{\Omega})$ be the envelope of holomorphy of $\mathscr{D}(\Omega)$. Then $\mathscr{D}(\hat{\Omega})$ is a Siegel domain of the first kind by Lemma 2. So it follows from [4; Theorem 9] that $\mathscr{D}(\hat{\Omega})$ admits an Aut$(\mathscr{D}(\hat{\Omega}))$-equivariant holomorphic imbedding $j:\mathscr{D}(\hat{\Omega})\rightarrow P_n(\mathbf{C})$ satisfying all the conditions of Theorem 2. On the other hand, the automorphism group Aut$(\mathscr{D}(\Omega))$ can be identified with a subgroup of Aut$(\mathscr{D}(\hat{\Omega}))$ via the correspondence $\sigma\rightarrow\hat{\sigma}$, where $\hat{\sigma}\in$ Aut$(\mathscr{D}(\hat{\Omega}))$ denotes the unique holomorphic extension of $\sigma\in$ Aut$(\mathscr{D}(\Omega))$. Thus, by composing the natural inclusion mapping $i:\mathscr{D}(\Omega)\rightarrow\mathscr{D}(\hat{\Omega})$ and the mapping $j:\mathscr{D}(\hat{\Omega})\rightarrow P_N(\mathbf{C})$, we obtain a desired Aut$(\mathscr{D}(\Omega))$-equivariant holomorphic imbedding $j\circ i:\mathscr{D}(\Omega)\rightarrow P_N(\mathbf{C})$.

Q. E. D.

3. A conjecture.

We finish this note by the following:

CONJECTURE. *Let $\mathscr{D}$ and $\mathscr{D}'$ be generalized Siegel domains in $\mathbf{C}^n\times\mathbf{C}^m$ and $\mathbf{C}^{n'}\times\mathbf{C}^{m'}$ with exponents c and c', respectively, where $n+m=n'+m'=N$. Then $\mathscr{D}$ and $\mathscr{D}'$ are biholomorphically equivalent if and only if the following two conditions are satisfied:*

(3. 1) $(n, m)=(n', m')$ *and* $c=c'$;

(3. 2) *There exists a non-singular linear mapping* $\mathscr{L}$: $C^N \rightarrow C^N$ *such that* $\mathscr{L}(\mathscr{D})=\mathscr{D}'$.

In the following cases it is known that this conjecture is, in fact, true.

Case 1. $\mathscr{D}$ and $\mathscr{D}'$ are Siegel domains of the second kind (see [4 ; Theorem 11]).

Case 2. $n=n'=1$: In this case, $\mathscr{D}$ and $\mathscr{D}'$ are generalized Siegel domains in $C \times C^m$ with arbitrary exponents (see [6 ; Theorem 2]).

Case 3. $n=n'=0$: In this case, $\mathscr{D}$ and $\mathscr{D}'$ are bounded circular domains in C^m containing the origin 0 of C^m, and $c=c'=0$. Indeed, it is obvious that $\mathscr{D}$ and $\mathscr{D}'$ are circular domains containing the origin 0. Furthermore, since $\mathscr{D}$ and $\mathscr{D}'$ are biholomorphically equivalent to bounded domains, it follows from [3 ; Théorème V] that they themselves are bounded. Accordingly we have $c=c'=0$ (see [2 ; Theorem 1. 2] , [6 ; Theorem 3]).

Case 4. $m=m'=0$: In this case $\mathscr{D}$ and $\mathscr{D}'$ are generalized Siegel domains in C^n of tube type (see Theorem 1 in this note).

References

[1] T. J. Barth, Convex domains and Kobayashi hyperbolicity, Proc. Amer. Math. Soc., **79**(1980), 556-558.

[2] R. Braun, W. Kaup and H. Upmeier, On the automorphisms of circular and Reinhardt domains in complex Banach spaces, Manuscripta Math., **25**(1978), 97-133.

[3] H. Cartan, Les fonctions de deux variables complexes et le problème de la représentation analytique, J. Math. Pures Appl., **10**(1931), 1-114.

[4] W. Kaup, Y. Matsushima and T. Ochiai, On the automorphisms and equivalences of generalized Siegel domains, Amer. J. Math., **92**(1970), 475-498.

[5] S. Kobayashi, *Hyperbolic Manifolds and Holomorphic Mappings*, Marcel Dekker, New York, 1970.

[6] A. Kodama, On the equivalence problem for a certain class of generalized Siegel domains, III, Osaka J. Math., **18**(1981), 481-499.

[7] R. Narasimhan, *Several Complex Variables*, The University of Chicago Press, Chicago and London, 1971.

[8] I. I. Pjateckii-Sapiro, *Géométrie des domaines classiques et théorie des fonctions automorphes*, Fizmatgiz, Moscow, 1961 ; French translation, Dunod, Paris, 1966.