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A Note on the Dirichlet Problem for a Class of Quasilinear Elliptic Equations

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Abstract. We study the behavior of weak solutions of the Dirichlet problem in the neighborhood of an irregular point of the boundary for a class of quasilinear elliptic equations.

1. Introduction.

The energy estimate with a weight of solutions of the Dirichlet problem for linear elliptic equations was obtained by Kondrat'ev(see Theorem 3.3 in [1]), where he considered in the neighborhood of an irregular point of the boundary and the weight is a potential function having its singularity at this point. His method is to make use of a perturbation technique for a Fredholm equation. A further developement was made by Kondrat'ev, Kopachek and Oleinik [2] from the view point of Saint-Venant's principle in elasticity. In [2] the pointwise estimate with a weight was proven.

When the boundary is smooth, there are many papers connected with the regularity up to the boundary. For example, Koshelev [3] proved the Hölder continuity up to the boundary for weak solutions of the Dirichlet problem for quasilinear elliptic systems. For this purpose he prepared an energy estimate with a weight.

When the boundary is not smooth and the equations are quasilinear, it seems to us that the behavior of weak solutions at the boundary is unknown. In this paper we consider the solution of quasilinear elliptic equations of Leray-Lions type (see [4] or [5]). And using other approaches, we give a L^2-energy estimate with a weight for the solution in the neighborhood of an irregular point. The weight used here is a potential function, which differs somewhat from the usual one.

2. Assumptions and result.

Let Ω be a bounded domain in the n-dimensional Euclidean space R^n with the boundary

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$\partial\Omega$ and the closure $\bar{\Omega}$. We denote by $(x_1, \ldots, x_n)$ the point x in R^n. It is assumed that Ω is in the upper half space $\{x_n>0\}$ and $\partial\Omega$ contains the origin O. There is no assumption on the smoothness of $\partial\Omega$. Thus O is an irregular point and Ω is convex at O.

Let $\eta \in R^1$ and $\xi=(\xi_1, \ldots, \xi_n) \in R_n$. We consider the $n+1$ functions $A_i(x, \eta, \xi)$ $(i=1, \ldots, n)$ and $B(x, \eta, \xi)$ defined in $\bar{\Omega}\times R^1\times R^n$. We assume that A_i, $B \in C^0(\bar{\Omega}\times R^1_\eta \times R^n_\xi)$.

Hereafter all constants are denoted simply by the letter C.

First we state our precise assumptions on A_i and B.

(A. 1) $\partial_{\xi_j} A_i \in C^0 (\bar{\Omega}\times R^1_\eta \times R^n_\xi)$ *for* $i, j=1, \ldots, n$.

There is a positive function $k(x)$ such that $k(x) \mid x \mid^{(1-n)/2} \in L^2(\Omega)$ and

$$\mid A_i(x, \eta, \xi) \mid \leqq C \mid \xi \mid + k(x),$$

$$\mid \partial_{\xi_j} A_i(x, \eta, \xi) \mid \leqq C$$

uniformly on $\bar{\Omega}\times R^1_\eta \times R^n_\xi$.

(A. 2) For every $\varepsilon>0$ there is a number $\delta>0$ with the property that for any $(\eta, \xi) \in R^{n+1}$

$$\mid \partial_{\xi_j} A_i (x, \eta, \xi)-\partial_{\xi_j} A_i (y, \eta, \xi) \mid <\varepsilon,$$

if $\mid x-y \mid <\delta$.

(A. 3) There is a positive constant c_0 such that

$$c_0 \mid \zeta \mid^2 \leqq \partial_{\xi_j} A_i (x, \eta, \xi)\, \zeta_i\, \zeta_j \text{ [2)]}$$

uniformly for x, η, ξ and $\zeta=(\zeta_1, \ldots, \zeta_n) \in R^n$.

(A. 4) We have

$$\mid B(x, \eta, \xi) \mid \leqq M_0 (\mid \eta \mid + \mid \xi \mid)+k(x),$$

$$\mid B(x, \eta, \xi)-B(x, \eta, \xi') \mid \leqq C \mid \xi-\xi' \mid$$

uniformly for x, η, ξ and ξ', where $k(x)$ is the function in (A. 1) and M_0 is a positive constant determined later depending only on C_0, $\{\partial_{\xi_j} A_i\}$ and the diameter of Ω.

(A. 5) Setting

$$a_{ij}(x, \eta, \xi, \xi')=\int_0^1 (\partial_{\xi_j} A_i)(x, \eta, \xi+t\xi')\, dt,$$

we can write

$$a_{ij} (O, \eta, \xi, \xi')=g(\eta, \xi, \xi')\, c_{ij},$$

where all c_{ij} are constants and the matrix (c_{ij}) is symmetric and positive definite.

By the mean value theorem we have

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2) The notation of the sum is abbreviated.

$$A_i(x, \eta, \xi)-A_i(x, \eta, \xi')$$
$$=(\xi_j-\xi'_j)\, a_{ij}(x, \eta, \xi', \xi-\xi').$$

Writing $\mathbf{A}=(A_1, \ldots, A_n)$, we obtain the following inequalities from (A. 1) and (A. 3):

(2. 1) $$(\mathbf{A}(x, \eta, \xi)-\mathbf{A}(x, \eta, \xi'))\cdot(\xi-\xi') \geqq c_0 |\xi-\xi'|^2,$$

(2. 2) $$\mathbf{A}(x, \eta, \xi)\cdot\xi \geqq c_0 |\xi|^2-\sqrt{n}\ k(x)|\xi|.$$

The norm and the inner product in $L^2(\Omega)$ are simply denoted by $\| \ \|$ and $(\ ,\ )$ respectively. We define $\|u\|_1=\|\nabla u\|$. The completion of $C_0^1(\Omega)$ with respect to the norm $\| \ \|_1$ is denoted by $H_0^1(\Omega)$, where $C_0^1(\Omega)$ is the set of all functions in $C^1(\Omega)$ with compact support in Ω. The dual space of $H_0^1(\Omega)$ is written by $H^{-1}(\Omega)$. We define

$$A(u)=-\nabla\cdot\mathbf{A}(x, u, \nabla u)+B(x, u, \nabla u).$$

Then A is an operator from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$.

From (2. 2) and (A 4) we have for $u\in H_0^1(\Omega)$

$$(A(u), u)\geqq c_0\|\nabla u\|^2-\sqrt{n}\int_\Omega k(x)|\nabla u|\,dx$$
$$-\int_\Omega [M_0(|u|+|\nabla u|)+k(x)]|u|\,dx$$

(by Cauchy's inequality)

$$\geqq\frac{1}{2}c_0\|u\|_1^2-C\|k\|^2.$$

Hence we obtain

$$(A(u), u)/\|u\|_1\to\infty, \text{ if } \|u\|_1\to\infty.$$

That is, A is coercive in $H_0^1(\Omega)$. Therefore there exsits u in $H_0^1(\Omega)$ of the equation

(2.3) $$A(u)=f$$

for any given $f\in H^{-1}(\Omega)$ (see e. g. [5]).

For α with $0<\alpha<1$ we set

(2. 4) $$\phi(x)=(1+x_n|x|^{-n})^\alpha.$$

For u, $v\in C_0^1(\Omega)$ we define

$$((u, v))=(\phi\nabla u, \nabla v)+((-\nabla\phi)u, v)$$

and

$$|||u|||=\sqrt{((u, u))}.$$

Later we shall see that ϕ is positive in Ω. Let $V_\alpha(\Omega)$ be the completion of $C_0^1(\Omega)$ with the norm $|||\ |||$. It is natural that $V_\alpha(\Omega)$ is a Hilbert space containing $H_0^1(\Omega)$ properly. The dual space of $V_\alpha(\Omega)$ is denoted by $V^{-1}(\Omega)$.

Our object is to prove

THEOREM. *Suppose that the equation* (2. 3) *has no more than one solution in* $H_0^1(\Omega)$. *Let* α *satisfy the inequality* (4. 1) *in Section* 4. *Let* $f \in H^{-1}(\Omega)$ *and* $\phi f \in V^{-1}(\Omega)$. *Then the solution* u *of* (2. 3) *is in* $V_\alpha(\Omega)$.

Remark. By Lemma 2 in the next section, the assumption $\phi f \in V^{-1}(\Omega)$ means that $f \in H^{-1}(\Omega)$.

3. Lemmas.

The following lemma holds for the function ϕ defined in (2. 4):

LEMMA 1. *We have for* $x_n \geqq 0$

$$|\nabla(x_n |x|^{-n})| \geqq |x|^{-n},$$

$$\triangle\phi = \alpha(\alpha-1)\phi^{(\alpha-2)/\alpha} |\nabla(x_n |x|^{-n})|^2$$

and

$$|\nabla\phi| = \sqrt{\frac{\alpha}{1-\alpha}}\ \phi^{1/2}(-\nabla\phi)^{1/2}.$$

Proof. The first inequality is clear. The remained inequalities are deduced from the well-known equality $\triangle(x_n|x|^{-1}) = 0$. Q. E. D.

LLEMMA 2. *If* $\phi^{1/2} f \in V^{-1}(\Omega)$, *then* $f \in H^{-1}(\Omega)$.

Proof. It is enough to show that

$$|(f, v)| \leqq C \|v\|_1 \qquad \text{for } v \in C_0^1(\Omega). \tag{3. 1}$$

By the assumption on f we have

$$|(f, v)| \leqq C |||\phi^{-1/2} v|||.$$

From Lemma 1 it follows that

$$|\phi^{-1}\nabla\phi|,\ |(-\nabla\phi)^{1/2}\phi^{-1/2}| \leqq C x_n^{-1} \quad \text{for } x_n > 0.$$

Further the inequality $\|x_n^{-1} v\| \leqq C\|\nabla v\|$ is well-known. Combining these inequalities, we obtain (3. 1). Q. E. D.

By (A. 3) and (A. 5) we have for $\xi \in R^n$

$$c_0|\xi|^2 \leqq g(\eta, \xi, \xi')\, c_{ij}\, \xi_i\, \xi_j.$$

And by (A. 1) there is a positive constant c_1 such that

$$g(\eta, \xi, \xi')\, c_{ij}\, \zeta_i\, \zeta_j \leqq c_1 |\zeta|^2.$$

Let $\{\lambda_i\}$ be the eigenvalues of the matrix (c_{ij}). Then $\lambda_i > 0$ and we get

$$c_0 |\zeta|^2 \leqq g(\eta, \xi, \xi') \sum_i \lambda_i\, \zeta_i^2 \leqq c_1 |\zeta|^2$$

uniformly for η, ξ, ξ' and ζ. This implies immediately that

(3. 2) $$c_2 \leqq g(\eta, \xi, \xi') \leqq c_3,$$

where $c_2 = c_0/\max_i (\lambda_i)$ and $c_3 = c_1/\min_i (\lambda_i)$. Hence, putting $\gamma = (c_2 + c_3)/2$, we have

(3. 3) $$|g(\eta, \xi, \xi') - \gamma| \leqq (c_3 - c_2)/2.$$

Now we take a non-singular matrix (b_{ij}) such that $(b_{ij})(c_{ij})^t(b_{ij}) = \delta_{ij}$, where δ_{ij} is the Kronecker's delta. Let us introduce two coordinates transformations $x' = (b_{ij})x$ and $x'' = (t_{ij})x'$, where (t_{ij}) is an orthogonal matrix satisfying that $x_n \geqq 0$ is equivalent to $x''_n \geqq 0$. We set

$$\psi(x) = (1 + x''_n |x''|^{-n})^\alpha \quad (= \phi(x'')).$$

Obviously $\psi(x)$ is equivalent to $\phi(x)$, that is,

(3. 4) $$c_4\, \phi(x) \leqq \psi(x) \leqq c_4^{-1}\, \phi(x) \quad (c_4 > 0)$$

in $x_n \geqq 0$.

We define

$$A_\psi(u, v) = -\psi \nabla \cdot \mathbf{A}(x, u, \nabla v) + \psi B(x, u, \nabla v)$$

and

$$A_\psi(u) = A_\psi(u, u).$$

We shall see that $A_\psi(u, v)$ is an operator from $V_\alpha(\Omega) \times V_\alpha(\Omega)$ to $V^{-1}(\Omega)$.

The following lemma holds:

LEMMA 3.
(i) $u \to A_\psi(u)$ *is bounded from* $V_\alpha(\Omega)$ to $V^{-1}(\Omega)$.
(ii) $\forall u \in V_\alpha(\Omega)$, $v \to A_\psi(u, v)$ *is hemicontinuous and bounded from* $V_\alpha(\Omega)$ *to* $V^{-1}(\Omega)$.
(iii) $\forall v \in V_\alpha(\Omega)$, $u \to A_\psi(u, v)$ *is hemicontinuous and bounded from* $V_\alpha(\Omega)$ *to* $V^{-1}(\Omega)$.

Proof. First we note the inequalities

$$|\nabla \psi| \leqq C |\nabla_{x''}\, \phi(x'')| \leqq C |\nabla \phi|,$$

where $\nabla_{x''} = (\partial_{x''_1}, \dots, \partial_{x''_n})$ and $x'' = (x''_1, \dots, x''_n)$. Combining these inequalities, Lemma 1, (A. 1) and (A. 4), We see that for $u, v, w \in V_\alpha(\Omega)$

$$|(A_\psi(u, v), w)| \leqq C |||w||| (1 + |||u||| + |||v|||).$$

Thus (i) and the boundedness in both (ii) and (iii) follow.

The hemicontinuity in (ii) is clear from the inequality

$$|A_\psi(u, v_1 + \lambda v_2), w) - (A_\psi(u, v_1 + \mu v_2), w)|$$
$$\leqq C|\lambda - \mu| \cdot |||v_2||| \cdot |||w|||.$$

We prove the remained part. We easily see that

$$|(A_\psi(u_1 + \lambda u_2, v), w) - (A_\psi(u_1 + \mu u_2, v), w)|$$
$$\leqq \| \phi^{1/2}(\mathbf{A}(x, u_1 + \lambda u_2, \nabla v) - \mathbf{A}(x, u_1 + \mu u_2, \nabla v) \| \cdot |||w|||$$
$$+ \| \phi^{1/2}(B(x, u_1 + \lambda u_2, \nabla v) - B(x, u_1 + \mu u_2, \nabla v) \| \cdot |||w|||$$

and $\mathbf{A}(x, u_1 + \lambda u_2, \nabla v) \to \mathbf{A}(x, u_1 + \mu u_2, \nabla v)$, $B(x, u_1 + \lambda u_2, \nabla v) \to B(x, u_1 + \mu u_2, \nabla v)$ almost everywhere in Ω, if $\lambda \to \mu$. Hence by virtue of (A. 1) and (A. 4) we can apply the Lebesgue's convergence theorem. Hence the hemicontinuity in (iii) is valid. Q. E. D.

LEMMA 4. *If $\{u_\mu\}$ converges weakly to u in $V_\alpha(\Omega)$, there exists a subsequence $\{u_\nu\}$ such that for any $v \in V_\alpha(\Omega)$, $A_\psi(u_\nu, v)$ converges weakly to $A_\psi(u, v)$ in $V^{-1}(\Omega)$ and $(A_\psi(u_\nu, v), u_\nu)$ converges to $(A_\psi(u, v), u)$.*

Proof. By Lemma 1, $\phi^{1/2} u_\mu \in H_0^1(\Omega)$ and $\| \nabla(\phi^{1/2} u_\mu) \| \leqq C |||u_\mu||| \leqq C$. Hence $\{\phi^{1/2} u_\nu\}$ converges strongly to $\phi^{1/2} u$ in $L^2(\Omega)$, which implies $u_\nu(x) \to u(x)$ almost everywhere in Ω. From this $\{-\psi \nabla \cdot \mathbf{A}(x, u_\nu, \nabla v)\}$ converges weakly to $-\psi \nabla \cdot \mathbf{A}(x, u, \nabla v)$ in $V^{-1}(\Omega)$ similarly as in the proof of Lemma 3.. Further we see that $\{\phi^{1/2} B(x, u_\nu, \nabla v)\}$ is uniformly bounded in $L^2(\Omega)$ and $B(x, u_\nu, \nabla v) \to B(x, u, \nabla v)$ almost everywhere in Ω. Thus, as is well-known, $\{\phi^{1/2} B(x, u_\nu, \nabla v)\}$ converges weakly to $\phi^{1/2} B(x, u, \nabla v)$ (see e. g. page 12 in [5]). From the above it follows that $\{A_\psi(u_\nu, v)\}$ converges weakly to $A_\psi(u, v)$ in $V_\alpha(\Omega)$.

Obviously, $(A_\psi(u, v), u_\nu - u) \to 0$. Hence it is enough to show that $(A_\psi(u_\nu, v) - A_\psi(u, v), u_\nu - u) \to 0$. Indeed,

$$(A_\psi(u_\nu, v) - A_\psi(u, v), u_\nu - u)$$

$$=(\mathbf{A}(x, u_\nu, \nabla v)-\mathbf{A}(x, u, \nabla v), \nabla(\psi(u_\nu-u)))$$

$$+(B(x, u_\nu, \nabla v)-B(x, u, \nabla v), \psi(u_\nu-u)).$$

Similarly as in the proof of Lemma 3, we can apply the Lebesgue's convergence theorem for the first term on the right-hand side of this equality. The second term converges to zero, since $\{\phi^{1/2}u_\nu\}$ converges strongly to $\phi^{1/2}u$ in $L^2(\Omega)$. Thus we complete the proof. Q. E. D.

LEMMA 5. *If α satisfies*

$$(c_3-c_2)\sqrt{\frac{\alpha}{1-\alpha}}<2\sqrt{c_2(c_2+c_3)},$$

then there is a constant C such that for $u, v\in V_\alpha(\Omega)$

$$(A_\psi(u, u)-A_\psi(u, v), u-v)\geqq -C\|\phi^{1/2}(u-v)\|^2.$$

Proof. We can write

$$(3.5)\qquad (A_\psi(u, u)-A_\psi(u, v), u-v)$$

$$=(\psi\partial_{x_j}(u-v)\cdot a_{ij}(x, u, \nabla v, \nabla(u-v)), \partial_{x_i}(u-v))$$

$$+(\partial_{x_j}(u-v)\cdot a_{ij}(x, u, \nabla v, \nabla(u-v))), (u-v)\partial_{x_i}\psi)$$

$$+(B(x, u, \nabla u)-B(x, u, \nabla v), \psi(u-v)).$$

$$\equiv I_1+I_2+I_3, \text{ say.}$$

From (A. 3), (A. 4) and (3. 4) we have

$$(3.6)\qquad I_1\geqq c_0\|\phi(x'')^{1/2}\nabla(u-v)\|^2$$

and

$$(3.7)\qquad I_3\geqq -C\|\phi^{1/2}\nabla(u-v)\|\cdot\|\phi^{1/2}(u-v)\|.$$

To estimate I_2, we decompose it into the two terms:

$$I_{2,1}=(\partial_{x_j}(u-v)\cdot a_{ij}(O, u, \nabla v, \nabla(u-v)), (u-v)\partial_{x_i}\psi)$$

and

$$I_{2,2}=(\partial_{x_j}(u-v)\cdot[a_{ij}(x, u, \nabla v, \nabla(u-v))$$

$$-a_{ij}(O, u, \nabla v, \nabla(u-v))], (u-v)\partial_{x_i}\psi).$$

By (A. 5) we see

$$I_{2,1}=(\partial_{x_j}(u-v)\cdot[g(u, \nabla v, \nabla(u-v))-\gamma], c_{ij}(u-v)\partial_{x_i}\psi)$$

$$+\gamma(\partial_{x_j}(u-v), c_{ij}(u-v)\partial_{x_i}\psi),$$

where γ is the number in (3. 3).

Let x'' be the new variable appearing in the definition of $\psi(x)$. For any two f, h, we easily see that

$$c_{ij}\partial_{x_i}\partial_{x_j}f=\Delta_{x''}f,\ c_{ij}\partial_{x_i}f\cdot\partial_{x_j}h=\nabla_{x''}f\cdot\nabla_{x''}h.$$

Hence rewriting the inner product (the norm) of $L^2(\Omega)$ by $(\ ,\ )_x$ ($\|\ \ \|_x$), we have from (3. 6)

$$I_1\geqq c_2\,\|\,\phi(x'')^{1/2}\nabla_{x''}(u-v)\,\|_{x}^{2},$$

where c_2 is the number in (3.2). Further integrating by parts, we obtain

$$I_{2,1}=\frac{1}{2}\gamma\,\|\,(-\nabla_{x''}\phi(x''))^{1/2}(u-v)\,\|_{x}^{2}$$

$$+(\ [g(u,\ \nabla v,\ \nabla(u-v))-\gamma](u-v)\nabla_{x''}\phi(x''),\ \nabla_{x''}(u-v))_x.$$

By (3.3) and Lemma 1, we see that

$$|\,(\ [g(u,\ \nabla v,\ \nabla(u-v))-\gamma]\ (u-v)\nabla_{x''}\phi(x''),\ \nabla_{x''}(u-v))_x\,|$$

$$\leqq\frac{1}{2}(c_3-c_2)\sqrt{\frac{\alpha}{1-\alpha}}\ \|\,(-\nabla_{x''}\phi(x''))^{1/2}(u-v)\,\|_x\ \|\,\phi(x'')^{1/2}\nabla_{x''}(u-v)\,\|_x.$$

Here we use (3.4) and Lemma 1. Noting that $-\Delta_{x''}\phi(x'')$ is equivalent to $-\Delta_x\phi(x)$, we have by our hypothesis

$$I_1+I_{2,1}\geqq c_5\,|||\,u-v\,|||^{2}. \tag{3.8}$$

Let ε be any given positive number. From (A. 2) there is a number $\delta>0$ such that

$$|\,a_{ij}(x,\ \eta,\ \xi,\ \xi')-a_{ij}(O,\ \eta,\ \xi,\ \xi')\,|<\varepsilon$$

for $|\,x\,|<\delta$. We take a function $\chi(x)\in C_0^\infty(\{\,|\,x\,|<\delta\})$ such that $\chi(x)=1$ in $|\,x\,|<\delta/2$. We write

$$\begin{aligned}I_{2,2}=&\ (\partial_{x_j}(u-v)\cdot\chi[a_{ij}(x,\ u,\ \nabla v,\ \nabla(u-v))\\&\quad-a_{ij}(O,\ u,\ \nabla v,\ \nabla(u-v))]\ ,\ (u-v)\partial_{x_i}\psi)\\&+(\partial_{x_j}(u-v)\cdot(1-\chi)[a_{ij}(x,\ u,\ \nabla v,\ \nabla(u-v))\\&\quad-a_{ij}(O,\ u,\ \nabla v,\ \nabla(u-v))]\ ,\ (u-v)\partial_{x_i}\psi).\end{aligned}$$

Then using again Lemma 1, we obtain

$$I_{2,2}\geqq-C\varepsilon\,|||\,u-v\,|||^{2}-C(\varepsilon)\,\|\,u-v\,\|^{2}.$$

Taking previously ε as sufficiently small and combining (3.7), (3.8) and this inequality, we complete the proof. Q. E. D.

4. Propositions.

If the stronger inequality

$$(A_\psi(u, u)-A_\psi(u, v), u-v)\geqq 0$$

holds in place of the conclusion in Lemma 5, $A_\psi(u, v)$ is just pseudo-monotone in the sense of [5]. However, from Lemmas 3, 4 and 5 $A_\psi(u, v)$ becomes pseudo-monotone. Its proof is quite parallel to that of Proposition 2.6, page 181 in [5]. Thus we have

PROPOSITION 1. *If α satisfies*

$$(c_3-c_2)\sqrt{\frac{\alpha}{1-\alpha}} < 2\sqrt{c_2(c_2+c_3)},$$

the operator $A_\psi(u, v)$ is pseudo-monotone from $V_\alpha(\Omega)\times V_\alpha(\Omega)$ to $V^{-1}(\Omega)$.

We set

$$M_1=\sup_{x,\eta,\xi'} |a_{ij}(x, \eta, O, \xi')-a_{ij}(O, \eta, O, \xi')|.$$

We shall prove

PROPOSITION 2. *If α satisfies*

$$2\sqrt{\frac{\alpha}{1-\alpha}}\left[\frac{1}{2}(c_3-c_2)+nM_1(\min_i(\lambda_i))^{-1}\right] < \sqrt{c_2(c_2+c_3)}, \tag{4.1}$$

the operator $A_\psi(u)$ is coercive from $V_\alpha(\Omega)$ to $V^{-1}(\Omega)$.

Proof. From the proof of Lemma 5 we have for $u\in V_\alpha(\Omega)$

$$(A_\psi(u), u)=(A_\psi(u, u)-A_\psi(u, 0), u-0)+(A_\psi(u, 0), u)$$

$$\geqq c_2\|\phi(x'')^{1/2}\nabla_{x''}u\|_x^2+\frac{1}{2}\gamma\|(-\Delta_{x''}\phi(x''))^{1/2}u\|_x^2$$

$$-\frac{1}{2}(c_3-c_2)\sqrt{\frac{\alpha}{1-\alpha}}\|(-\Delta_{x''}\phi(x''))^{1/2}u\|_x\|\phi(x'')^{1/2}\nabla_{x''}u\|_x$$

$$-|(\partial_{x_j}u\cdot[a_{ij}(x. u, O, \nabla u)-a_{ij}(O. u, O, \nabla u)], u\partial_{x_i}\psi)|$$

$$-|(\mathbf{A}(x, u, O), \nabla(\psi u))|-|(B(x, u, \nabla u), \psi u)|.$$

Since $c_{ij}\partial_{x_j}f\cdot\partial_{x_i}f=|\nabla_{x''}f|^2$ for any function f, we see

$$(\min_i(\lambda_i))|\nabla f|^2\leqq|\nabla_{x''}f|^2.$$

Hence

$$\sum_{i,j}|\partial_{x_i}\psi|\cdot|\partial_{x_j}u|\leqq n(\min_i(\lambda_i))^{-1}|\nabla_{x''}\phi(x'')|\cdot|\nabla_{x''}u|.$$

From this and Lemma 1 it follows that

$$|(\partial_{xj}\, u\cdot[a_{ij}(x,\ u,\ O,\ \nabla u)-a_{ij}(O,\ u,\ O,\ \nabla u)]\ ,\ u\partial_{x_i}\psi)|$$

$$\leqq nM_1(\min_i(\lambda_i))^{-1}\sqrt{\frac{\alpha}{1-\alpha}}\ \|(-\Delta_{x''}\phi(x'')^{1/2}u\|_x\cdot\|\phi(x'')^{1/2}\nabla_{x''}u\|_x.$$

Further from (A. 1) and (A. 2) we see that

$$|(\mathbf{A}(x,\ u,\ O),\ \nabla(\psi u))|\leqq C\,\|k(x)\,|x|^{(1-n)/2}\|\cdot$$

$$(\|(-\Delta_{x''}\phi(x''))^{1/2}u\|_x+\|\phi(x'')^{1/2}\nabla_{x''}u\|_x)$$

$$\leqq C\,\|u\|$$

and

$$|(B(x,\ u,\ \nabla u),\ \psi u)|\leqq M_0\,\|\phi(x'')^{1/2}u\|_x\ (\|\phi(x'')^{1/2}u\|_x+\|\phi(x'')^{1/2}\nabla u\|_x)$$

$$+C\,\|k(x)\,|x|^{(1-n)/2}\|\cdot\|\phi(x'')^{1/2}u\|_x.$$

Using (3. 4) and Lemma 1, we have

$$\|\phi(x'')^{1/2}u\|_x\leqq C\,\|\nabla(\phi^{1/2}u)\|$$

$$\leqq C\,|||u|||.$$

Hence we get

$$|(B(x,\ u,\ \nabla u),\ \psi u)|\leqq C\,|||u|||(M_0\,|||u|||+1).$$

By taking the constant M_0 suitably, we obtain the following estimate from the above -mentioned:

$$(A\psi(u),\ u)\geqq\frac{1}{2}\,c_2\,\|\phi(x'')^{1/2}\nabla_{x''}u\|_x^2$$

$$+\frac{1}{4}\gamma\,\|(-\Delta_{x''}\phi(x''))^{1/2}u\|_x^2$$

$$-\sqrt{\frac{\alpha}{1-\alpha}}\ [\frac{1}{2}(c_3-c_2)+nM_1(\min_i\,(\lambda_i))^{-1}]\cdot$$

$$\|(-\nabla_{x''}\phi(x''))^{1/2}u\|_x\cdot\|\phi(x'')^{1/2}\nabla_{x''}u\|_x$$

$$-C\,|||u|||.$$

From our assumption on α, this becomes

$$(A_\psi(u),\ u)\geqq c_6\,|||u|||^2-C\,|||u|||,$$

where $c_6>0$. Therefore, if $u\in V_\alpha\ (\Omega)$ and $\|u\|\to\infty$,

then

$$(A\psi(u),\ u)/\,|||u|||\to\infty,$$

which implies that $A_\psi(u)$ is coercive. Q.E.D.

Combining Propositions 1, 2 and the result of Leray-Lions [4], we conclude

PROPOSITION 3. *Under the assumption of Proposition 2, there exists* $u \in V_\alpha(\Omega)$ *of the equation* $A_\psi(u)=f$ *for any given* $f \in V^{-1}(\Omega)$.

5. Proof of Theorem.

We substitute ψf for f in Proposition 3. Then there exists $\tilde{u} \in V_\alpha(\Omega)$ of $A_\psi(\tilde{u})=\psi f$. Thus for any $w \in V_\alpha(\Omega)$

$$(\mathbf{A}(x, \tilde{u}, \nabla \tilde{u}), \nabla(\psi w))+(B(x, \tilde{u}, \nabla \tilde{u}), \psi w)=(f, \psi w).$$

Setting $w=\psi^{-1}v$ for any given $v \in C_0^1(\Omega)$, we see that $w \in C_0^1(\Omega)$. Therefore $\tilde{u}$ is the solution of the equation (2. 3). By the uniqueness of (2. 3) we conclude that $u=\tilde{u} \in V_\alpha(\Omega)$. This completes the proof. Q.E.D.

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