Sci. Rep. Kanazawa Univ.
Vol. 29 No. 1 pp. 9-14
June 1984

Supplementary Notes on Galois Groups of Central Extensions of Algebraic Number Fields

Yoshiomi FURUTA

Department of Mathematics, Faculty of Science, Kanazawa University

(Received April 16, 1984)

Abstract. The relations between Galois groups of central extensions and those of everywhere locally abelian central extensions are studied.

Introduction

Let M/K/k be a tower of Galois extensions of algebraic number fields of finite degree. Denote by M_0 the maximal abelian extension over k contained in M. Let $K^*=KM_0$, which is called the genus field for M/K/k. We call L a central extension of K/k in M when L is Galois extension over k contained in M and Gal(L/K) is contained in the center of Gal(L/K). We call L an EL-abelian central extension of K/k in M when L is a central extension of K/k in M and each local completion of L is contained in the composite of the local completion of K and an abelian extension of the corresponding local completion of k. Denote by $\hat{K}$ resp. $\tilde{K}$ or by $\hat{K}_M$ resp. $\tilde{K}_M$ the maximal central extension resp. the maximal EL-abelian central extension of K/k in M. There are several papers[1)] concerning with Galois groups of $\hat{K}$ or $\tilde{K}$ over k and over K^*. However the relations between Galois groups of $\hat{K}$ and $\tilde{K}$ are not written explicitly[2)]. So we shall treat it in the present paper (Theorem 6), which is rather expository. Especially the first half part is well-known, but we repeat it with some changes of arrangement for the sake of convenience.

1. Throughout this paper, we use the following notation for an algebraic number field F and Galois extension L/F.

F^{ab}	Maximal abelian extension of F
F^{el}	Maximal Galois extension which is exerywhere locally abelian
F_p	Local completion of F at a prime p

1) See for instance the reference at the end of this paper.
2) Cf. Heider [5], which partly concerns with the relation.

J_F	Idele group of F
$F^{\times}$	Multiplicative group of non-zero elements of F, which coincides with the group of principal ideles of F.
$C_F = J_F/F^{\times}$	Idele class proup of F
$G(L/F)$	Galois group of L/F
D_F	Kernel of the norm residue map $J_F \rightarrow G(F^{ab}/F)$
$N_{L/F}$	Norm map of L to F

$H(L/F)=F^{\times}N_{L/F}J_L$ Kernel of the norm residue map $J_F \rightarrow G(L \cap F^{ab}/F)$, that is the idele group corresponding to L by class field theory

$C(L/F)=H(L/F)/F^{\times}$ Idele class group of F corresponding to L by class field theory

2. M/K/k, K^*, $\hat{K}$ and $\tilde{K}$ being as in Introduction, we add the following notation

$J^{\circ}_K = J^{\circ}_{K/k}$	Group of ideles a of J_K such that $N_{K/k}\, a = 1$
$J^{\Delta}_K = J^{\Delta}_{K/k}$	Group generated by all ideles $a^{1-\sigma}$, where $a \in J_K$ and $\sigma \in G(K/k)$
$J^*_K = J^*_{K/k}$	Group of ideles a of J_K such that $N_{K/k}\, a \in k^{\times}$

Let p be a prime of k and P a prime of K lying over p.

We start our discussion from the case $M=K^{ab}$.

PROPOSITION 1. *Let* $M=K^{ab}$. *Then*

(1) $H(K^*/K)=J^*_K D_K = N^{-1}_{K/k}(D_k)$

(2) $H(\hat{K}/K)=J^{\Delta}_K D_K$

(3) $H(\tilde{K}/K)=J^{\circ}_K D_K$

Proof. (1) and (2) follow immediately from class field theory. We have also $H(k^{e1}K \cap K^{ab}/K)=\prod_P N^{-1}_{K_P/k_p}(1) \cdot D_K$.

Put $G_P=G(K_P/k_p)$. Then since $H^{-1}(G, J_K) \leqq \sum_p H^{-1}(G_P, K_P)$, we have $J^{\circ}_K = J^{\Delta}_K \prod_P N^{-1}_{K_P/k_p}(1)$.

Hence[3] $H(\tilde{K}/K)=H(k^{e1}K \cap K^{ab}/K)$. $H(\hat{K}/K)=\prod_P N^{-1}_{K_P/k_p}(1) \cdot D_K \cdot J^{\Delta}_K \cdot D_k = J^{\circ}_K \cdot D_K$.

For a general case of M, replacing D_K by $H(M/K)=K^{\times} N_{M/K}J_M$, we have the following

PROPOSITION 2

(1′) $H(K^*_M/K)=J^*_K\, H(M/K)$

(2′) $H(\hat{K}_M/K)=J^{\Delta}_K\, H(M/K)$

3) Cf. Masuda [8] and Miyake [9]

(3′) $H(\tilde{K}_M/K)=J_K^{\circ}\ H(M/K)$

3. A commutative diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & K^\times & \xrightarrow{i_M} & H(M/K) & \xrightarrow{j_M} & C(M/K) & \to 0 \\
 & \downarrow & & \downarrow \varkappa & & \downarrow \lambda & \\
0\to & K^\times & \xrightarrow{i} & J_K & \xrightarrow{j} & C_K & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow \rho & \\
 & 0 & \to & J_K/H(M/K) & \to & J_K/H(M/K) & \to 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array}$$

implies the following commutative diagram (#) of long exact sequences of cohomology groups, where $H^i(A)$ stands for $H^i(G(K/k), A)$.

$$(\#)\quad \begin{array}{cccccccccc}
 & & & & \vdots & & & & & \\
 & & & & \downarrow & & & & & \\
\cdots\to H^{-1}(K^\times) & \xrightarrow{i_M^\#} & H^{-1}(H(M/K)) & \xrightarrow{j_M^\#} & H^{-1}(H(M/K)/K^\times) & \xrightarrow{\delta_M^\#} & H^0(K^\times) & \longrightarrow & H^0(H(M/K)) & \longrightarrow\cdots \\
\| & & \downarrow \varkappa^\# & & \downarrow \lambda^\# & & \| & & & \\
\cdots\to H^{-1}(K^\times) & \xrightarrow{i^\#} & H^{-1}(J_K) & \xrightarrow{j^\#} & H^{-1}(C_K) & \xrightarrow{\delta^\#} & H^0(K^\times) & \longrightarrow & H^0(J_K) & \longrightarrow\cdots \\
 & & & & \downarrow \rho^\# & & & & & \\
 & & & & H^{-1}(J_K/H(M/K)) & & & & & \\
 & & & & \downarrow \bar{\delta}^\# & & & & & \\
 & & & & H^0(H(M/K)/K^\times) & & & & & \\
 & & & & \downarrow & & & & & \\
 & & & & \vdots & & & & &
\end{array}$$

4. Let $\mathfrak{G}$ be a finite group and $\mathfrak{H}$ be a normal subgroup of $\mathfrak{G}$. Let A be a $\mathfrak{G}$-module and $N_{\mathfrak{H}}$ be an endomorphism of A defined by $N_{\mathfrak{H}}a=\sum_{\sigma\in\mathfrak{H}}\sigma a$. Put $O_{\mathfrak{H}}(A)=\mathrm{Ker}\ N_{\mathfrak{H}}$ and let $\tilde{N}_{\mathfrak{H}}$ be the homomorphism of $H^{-1}(\mathfrak{G}, A)$ to $H^{-1}(\mathfrak{G}/\mathfrak{H}, N_{\mathfrak{H}}A)$ induced from $N_{\mathfrak{H}}$. Denote by $I_{\mathfrak{G}}$ the augmentation ideal of the group ring $\mathbf{Z}\mathfrak{G}$.

Proposition 3 ([3, Proposition 6]). *Notation being as above, we have the exact sequence*

$$H^{-1}(\mathfrak{H}, A)\to H^{-1}(\mathfrak{G}, A)\xrightarrow{\tilde{N}_{\mathfrak{H}}} H^{-1}(\mathfrak{G}/\mathfrak{H}, N_{\mathfrak{H}}A)\longrightarrow 0$$

PROPOSITION[4] 4. *Let* M/K/k *be a tower of Galois extension with Galois group* $\mathfrak{G}=G(M/k)$, $\mathfrak{H}=G(M/K)$ *and* $G=G(K/k)$. *Then*

$$\lambda^{\#}\tilde{N}_{M/K}H^{-1}(\mathfrak{G}, C_M)=\mathrm{Def}_{\mathfrak{G}\to G}H^{-1}(\mathfrak{G}, C_M),$$

where $\lambda^{\#}$ *is defined in the exact sequence* (#) *in Section* 3.

Proof. By definition of $\mathrm{Def}_{\mathfrak{G}\to G}$, we have

$$\mathrm{Def}_{\mathfrak{G}\to G}H^{-1}(\mathfrak{G}, C_M)\cong \mathrm{Def}_{\mathfrak{G}\to G}\ (O_{\mathfrak{G}}(C_M)/I_{\mathfrak{G}}C_M)=$$

$$N_{\mathfrak{H}}O_{\mathfrak{G}}(C_M)/I_{\mathfrak{G}/\mathfrak{H}}C_M=O_{\mathfrak{G}/\mathfrak{H}}(N_{\mathfrak{H}}C_M)/I_{\mathfrak{H}/\mathfrak{G}}C_M=\lambda^{\#}\tilde{N}_{\mathfrak{H}}H^{-1}(\mathfrak{G}, C_M).$$

5. Now we can prove the following theorem by Proposition 2 and the exact sequence (#) of cohomology groups.

THEOREM 5. *Let* M/K/k *be a tower of Galois extensions of finite degree with Galois groups* $\mathfrak{G}=G(M/k)$ *and* $G=G(K/k)$. *Then we have*

$$(4)\quad G(\tilde{K}_M/K_M^*) \cong H^{-1}(G, C_K)/\lambda^{\#}H^{-1}(G, C(M/K))$$
$$\cong H^{-1}(G, C_K)/\mathrm{Def}_{\mathfrak{G}\to G}H^{-1}(\mathfrak{G}, C_M)$$
$$\cong H^{-3}(G, \mathbb{Z})/\mathrm{Def}_{\mathfrak{G}\to G}H^{-3}(\mathfrak{G}, \mathbb{Z})$$

$$(5)\quad G(\tilde{K}_M/K_M^*) \cong (k^{\times}\cap N_{K/k}\ J_K)/N_{K/k}\ K^{\times}\cdot(k^{\times}\cap N_{M/k}J_M)$$
$$\cong \delta^{\#}H^{-1}(G, C_K)/\delta^{\#}\lambda^{\#}H^{-1}(G, C(M/K))$$
$$\cong \delta^{\#}H^{-1}(G, C_K)/\delta^{\#}\ \mathrm{Def}_{\mathfrak{G}\to G}H^{-1}(\mathfrak{G}, C_M)$$
$$\cong \delta^{\#}H^{-3}(G, \mathbb{Z})/\delta^{\#}\ \mathrm{Def}_{\mathfrak{G}\to G}H^{-3}(\mathfrak{G}, \mathbb{Z})$$

$$(6)\quad G(\hat{K}_M/\tilde{K}_M) \cong J_K^{\circ}/J_K^{\Delta}(J_K^{\circ}\cap H(M/K))$$
$$\cong H^{-1}(G, J_K)/\kappa^{\#}H^{-1}(G, H(M/K))$$

Proof of (4). The exact sequence (#) implies $H^{-1}(G, C_K)/\lambda^{\#}H^{-1}(G, C(M/K))\cong \mathrm{Im}\ \rho^{\#}=$

$$\mathrm{Ker}\ \bar{\delta}^{\#}=\frac{H(K_M^*/K)\cdot H(M/K)/H(M/K)}{J_K^{\Delta}\cdot H(M/K)/H(M/K)}\cong H(K_M^*/K)/H(\hat{K}_M/K)\cong G(\hat{K}_M/K_M^*).$$

On the other hand, it follows from Proposition 3 and Proposition 4 that

$$H^{-1}(G, C_K)/\lambda^{\#}H^{-1}(G, C(M/K))=$$
$$H^{-1}(G, C_K)/\lambda^{\#}\tilde{N}_{M/K}H^{-1}(G, C_M)=H^{-1}(G, C_K)/\mathrm{Def}_{G\to G}H^{-1}(G, C_M)$$

---

4) Cf. Shirai [15] and Kuz'min [7, p.1152~]

Proof of (5). We have $H(K^*_M/K)=\{a\in J_K;\ N_{K/k}a\in H(M/k)=k^\times N_{M/k}J_M\}$ by translation theorem of class field theory. Hence Proposition 2 implies

$$
\begin{aligned}
G(\tilde{K}_M/K^*_M) &\cong H(K^*_M/K)/J^\circ_K H(M/K)\\
&\cong N_{K/k}H(K^*_M/K)/N_{K/k}H(M/K)\\
&= (k^\times N_{M/k}J_M\cap N_{K/k}J_K)/N_{K/k}K^\times\cdot N_{M/k}J_M\\
&\cong (k^\times\cap N_{K/k}J_K)/((k^\times\cap N_{K/k}J_K)\cap N_{K/k}K^\times\cdot N_{M/k}J_M)\\
&= (k^\times\cap N_{K/k}J_K)/N_{K/k}K^\times\cdot(k^\times\cap N_{M/k}J_M).
\end{aligned}
$$

It is well known that $k^\times\cap N_{K/k}J_K/N_{K/k}K^\times=\mathrm{Ker}\ i^\#=\mathrm{Im}\ \delta^\#$, and also $(k^\times\cap N_{M/k}J_M)\cdot N_{K/k}K^\times/N_{K/k}K^\times=(k^\times\cap N_{K/k}(K^\times\cdot N_{M/K}J_M))/N_{K/k}K^\times=\mathrm{Ker}\ i^\#_M=\mathrm{Im}\ \delta^\#_M=\mathrm{Im}\ (\delta^\#\circ\lambda^\#)$.

Proof of (6). By Proposition 2, (2′) and (3′), we have

$$
\begin{aligned}
G(\hat{K}_M/\tilde{K}_M) &\cong J^\circ_K\cdot K^\times\cdot N_{M/K}J_M/J^\Delta_K\cdot K^\times\cdot N_{M/K}J_M\\
&\cong J^\circ_K/J^\circ_K\cap(J^\Delta_K\cdot K^\times\cdot N_{M/K}J_M)\\
&= J^\circ_K/J^\Delta_K(J^\circ_K\cap K^\times\cdot N_{M/K}J_M)\\
&\cong (J^\circ_K/J^\Delta_K)/(J^\Delta_K\cdot(J^\circ_K\cap H(M/K))/J^\Delta_K)\\
&\cong H^{-1}(G,J_K)/\varkappa^\# H^{-1}(G,H(M/K)).
\end{aligned}
$$

Remark. (i) The formula (4) of Theorem 5 is obtained[5] immediately, if we use Hochschild-Serre exact sequence.

(ii) If M is sufficiently large, we have[6] $D_{\mathfrak{G}\to G}H^{-3}(\mathfrak{G},Z)=0$. Hence Theorem 5 implies $G(\hat{K}_M/K^*_M)\cong H^{-3}(G,\mathbb{Z})$ and[7] $G(\tilde{K}_M/K^*_M)\cong\delta^\# H^{-3}(G,\mathbb{Z})\cong(k^\times\cap N_{K/k}J_K)/N_{K/k}K^\times$, which is called number knot.

THEOREM 6. *We have the following exact sequence.*

$$
\begin{array}{ccccccccc}
0\longrightarrow & G(\hat{K}_M/\tilde{K}_M) & \longrightarrow & G(\hat{K}_M/K^*_M) & \longrightarrow & G(\tilde{K}_M/K^*_M) & \longrightarrow 0\\
& \wr\Vert & & \wr\Vert & & \wr\Vert & \\
0\longrightarrow & \dfrac{H^{-1}(G,J_K)}{\varkappa^\# H^{-1}(G,H(M/K))} & \xrightarrow{j^*} & \dfrac{H^{-1}(G,C_K)}{\lambda^\# H^{-1}(G,C(M/K))} & \xrightarrow{\delta^*} & \dfrac{\delta^\# H^{-1}(G,C_K)}{\delta^\#\lambda^\# H^{-1}(G,C(M/K))} & \longrightarrow 0\\
& & & \wr\Vert & & \wr\Vert &
\end{array}
$$

5) See Heider [4, §2]

6) See Heider [4, §4], Miyake [9] and Yamashita [16].

7) See Masuda [8, Theorem 8] and Heider [5, Satz 7].

$$\frac{H^{-1}(G, C_K)}{\mathrm{Def}_{\mathfrak{G}\to G} H^{-1}(\mathfrak{G}, C_M)} \quad\quad \frac{\delta^{\#}H^{-1}(G, C_K)}{\mathrm{Def}_{\mathfrak{G}\to G}\delta^{\#} H^{-1}(\mathfrak{G}, C_M)}$$

$$\wr\| \quad\quad\quad\quad \wr\|$$

$$\frac{H^{-3}(G, \mathbb{Z})}{\mathrm{Def}_{\mathfrak{G}\to G} H^{-3}(\mathfrak{G}, Z)} \quad\quad \frac{\delta^{\#}H^{-3}(G, Z)}{\mathrm{Def}_{\mathfrak{G}\to G}\delta^{\#} H^{-3}(\mathfrak{G}, \mathbb{Z})}$$

where j^* *and* δ^* *are induced from* $j^{\#}$ *and* $\delta^{\#}$ *is as in diagram* (#) *in* **3**.

Proof. The theorem follows immediately from Theorem 5 and the following equalities.

$j^{\#}H^{-1}(G, J_K) \cap \lambda^{\#}H^{-1}(G, C(M/K)) = \mathrm{Ker}\ \delta^{\#} \cap \mathrm{Im}\ \lambda^{\#} = \lambda^{\#}(\mathrm{Ker}\ \delta^{\#}_M) = \lambda^{\#} j^{\#}_M H^{-1}(G, H(M/K))$
$= j^{\#}\varkappa^{\#}H^{-1}(G, H(M/K))$ and $\mathrm{Ker}\ j^{\#} = \mathrm{Im}\ i^{\#} = \mathrm{Im}(\varkappa^{\#}\circ i^{\#}_M) \in \varkappa^{\#}H^{-1}(G, H(M/K))$.

References

[1] A. Fröhlich, On fields of class two, Proc. London Math. Soc., 4(1954), 235-256.

[2] Y. Furuta, On class field towers and the rank of ideal class groups, Nagoya Math. J., **48**(1972), 147-157.

[3] ———, On nilpotent factors of congruent ideal class groups of Galois extensions, Nagoya Math. J., **62**(1976), 13-28.

[4] F.-P. Heider, Strahlknoten und Geschlechterkörper mod. m, J. reine angew. Math., **320**(1980), 52-67.

[5] ———, Zahlentheoretische Knoten unendlicher Erweiterungen, Arch. Math., **37**(1981), 341-352.

[6] W. Jehne, On knots in algebraic number theory, J. reine angew. Math., **311/312**(1979), 215-254.

[7] L. V. Kuz'min, Homology of profinite groups, Schur multipliers, and class field theory, Math. USSR Izvestija, **3**(1969), 1149-1181.

[8] K. Masuda, An application of the generalized norm residue symbol, Proc. Amer. Math. Soc., **10**(1969), 245-252.

[9] K. Miyake, Central extensions and Schur multiplicators of Galois groups, Nagoya Math. J., **90**(1983), 137-144.

[10] ———, On central extensions of a Galois extension of algebraic number fields, Nagoya Math. J., **93**(1984), 133-148.

[11] H. Opolka, Geschlechter von zentralen Erweiterungen, Arch. Math., **37**(1981), 418-424.

[12] M. Razar, Central and genus class fields and the Hasse norm theorem, Compositio Math., **35**(1977), 281-298.

[13] A. Scholz, Totale Normenreste, die keine Normen sind, als Erzeuger nicht abelscher Körpererweiterungen, I, II, J. reine angew. Math., **172**(1936), 100-107; **182**(1940), 217-234.

[14] S. Shirai, On the central class field mod m of Galois extensions of an algebraic number field, Nagoya Math. J., **71**(1978), 61-85.

[15] ———, A remark on the Corestriction-Deflation sequence, Math. Rep. Toyama Univ., **2**(1979), 23-32.

[16] H. Yamashita, On nilpotent factors of maximum abelian extensions of algebraic number fields, Sci. Rep. Kanazawa Univ., **28**(1983), 1-5.