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Conjugate Functions of Several Variables

Michitaka KOJIMA

Department of Mathematics, Faculty of Science, Kanazawa University

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Abstract. In this note, for $f \in L(\log^+ L)^\beta\ (T^n)$ $(\beta \geqq 0)$, the almost everywhere existence and integrability of the conjugate function $\tilde{f}_B$ of f with respect to the family of variables x_B with any set $B \subset \{1, 2, \dots, n\}$ is discussed. Let $k(B)$ be the number of elements of B. According to L.V. Zhizhiashvili, (i) the condition $f \in L(\log^+ L)^\beta\ (T^n)$ ensures the almost everywhere existence of $\tilde{f}_B$ for $k(B) \in [1, \beta+1]$ and does not for $k(B) \in (\beta+1, n]$, and (ii) the condition $f \in L(\log^+ L)^\beta\ (T^n)$ ensures the integrability of $\tilde{f}_B$ for $k(B) \in [1, \beta]$ and does not for $k(B) \in (\beta, n]$. Our main purpose is to give detail proofs to the negative parts of (i), (ii) for the general case $n \geqq 3$ along by the papers [6]– [9] where the case of $n=2$ is treated mainly.

§0. Introduction.

Let R^n $(n \geqq 1)$ be the n-dimensional Euclidean space with usual linear operations, Z^n the set of all lattice points in R^n and $T^n = R^n/2\pi Z^n$ the n-dimensional torus. When we take any non-empty subset B from the set $\{1, 2, \dots, n\}$, we denote the number of elements of the set B by $k(B)$, and also, for each point $x = (x_1, x_2, \dots, x_n) \in R^n$, we define the point $x_B = (x_{B1}, x_{B2}, \dots, x_{Bn}) \in R^n$ by

$$x_{Bi} = \begin{cases} x_i & \text{if} \quad i \in B \\ 0 & \text{if} \quad i \notin B. \end{cases}$$

The number of all such $B's$ is equal to $2^n - 1$.

Now, for $f \in L(T^n)$, we will define

$$\tilde{f}_B(x) = P.\ V.\ \left(\frac{1}{2\pi}\right)^{k(B)} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(x - t_B) \prod_{i \in B} \cot \frac{t_i}{2}\, dt_B$$

denoting $dt_B = dt_{i_1} \cdots dt_{i_m}$ for $B = \{i_1, \dots, i_m\}$, if the multiple integral exists in the sense of Pringsheim. Then we will call this integral the conjugate functions of n-variables with respect to the index set B.

The purpose of this paper is to give a detail proof to a few results for $n \geqq 3$ of L. V. Zhizhiashvili on the almost everywhere existence and integrability of $\tilde{f}_B$ for $f \in L(\log^+ L)^\beta (T^n)$ $(\beta \geqq 0)$ along by his papers [6] -[9], where the case of $n=2$ is treated mainly.

Here we shall recall that in the case of $n=1$, the conjugate function $\tilde{f}$ of $f \in L(T^1)$,

$$\tilde{f}(x) = P.V. \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) \cot \frac{t}{2}\, dt$$

exists almost everywhere in T^1 and satisfies the following inequalities;

$$|\{x \in T^1;\ |\tilde{f}(x)| > \alpha\}| \leqq \frac{A}{\alpha} \|f\|_1 \ (\alpha > 0)$$

$$\|\tilde{f}\|_1 \leqq A\{\int_{T^1} |f(x)| \log^+ |f(x)|\, dx + 1\}$$

$$\int_{T^1} |\tilde{f}(x)| (\log^+ |\tilde{f}(x)|)^\lambda dx \leqq A_\lambda \{\int_{T^1} |f(x)| (\log^+ |f(x)|)^{\lambda+1} dx + 1\} \ (\lambda \geqq 0)$$

where constants A, A_λ are independent of f.

§1. On the existence of conjugate functions.

If $f \in L(\log^+ L)^\beta (T^n)$ $(\beta \geqq 0)$, then for each set B with $k(B) \in [1, \beta+1]$, $\tilde{f}_B$ exists almost everywhere in T^n and satisfies the following inequalities;

$$|\{x \in T^n;\ |\tilde{f}_B(x)| > \alpha\}|$$

$$\leqq \begin{cases} \dfrac{A_1}{\alpha} \int_{T^n} |f(x)|\, dx & \text{if } k(B)=1 \\[2ex] \dfrac{A_1}{\alpha} \{\int_{T^n} |f(x)| (\log^+ |f(x)|)^{k(B)-1} dx + 1\} & \text{if } k(B) \geqq 2 \end{cases}$$

$$\leqq \frac{A_2}{\alpha} \{\int_{T^n} |f(x)| (\log^+ |f(x)|)^\beta dx + 1\}$$

with some constants A_1, A_2 independent of f. These facts follows from the known results on the existence of conjugate functions of several variables and from the familiar theorems in the case of $n=1$ ([1], [10], [8], [9]). If $\beta \geqq n-1$, then the integers k with $k \in [1, \beta+1]$ are all of 1, 2, ..., n. On the other hand, when $0 \leqq \beta < n-1$, in general the condition $f \in L(\log^+ L)^\beta (T^n)$ does not ensure the existence of $\tilde{f}_B$ for B with $k(B) \in (\beta+1, n]$ in the following sense ([7, p. 85], [6, pp. 124-133 for $n=2$]).

THEOREM 1. *Let $\{\varepsilon_p\}_{p=1}^\infty$ be any given sequence with $\varepsilon_p > 0$, $\varepsilon_p \to 0$ $(p \to \infty)$. Then for each given $0 \leqq \beta < n-1$ and integer $k \in (\beta+1, n]$, there exists a function $f \in L(\log^+ L)^\beta (T^n)$ such that*

$$\overline{\lim_{p\to\infty}} \left| \int_{\varepsilon_p<|t_1-x_1|\leq\pi} \cdots\cdots\cdots \int_{\varepsilon_p<|t_k-x_k|\leq\pi} f(t_1,\ldots,t_k,x_{k+1},\ldots,x_n)\prod_{i=1}^{k}\cot\frac{x_i-t_i}{2}\,dt_1\cdots dt_k\right|$$

$= +\infty$ *for a. e.* $x=(x_1,\ldots,x_n)\in T^n$.

PROOF. We put $\Phi(t)=t[\log(t+a)]^{\beta}$ for $0\leq t<\infty$, where $a=\exp(\beta^2+\beta+1)$. Then $\Phi(0)=0$, $\Phi(t)$ is convex, $\Phi(t^{1/2})$ is concave and $\Phi(2t)\leq C\Phi(t)$ for some constant C. Let L_Φ be the set $\{f \text{ on } T^n;\ \Phi(|f|)\in L(T^n)\}$. For each $p=1,2,\ldots$, we set

$$T_p(x;f)$$
$$=\int_{\varepsilon_p<|t_1+x_1|\leq\pi}\cdots\int_{\varepsilon_p<|t_k+x_k|\leq\pi} f(t_1,\ldots,t_k,x_{k+1},\ldots,x_n)\prod_{i=1}^{k}\cot\frac{x_i+t_i}{2}\,dt_1\cdots dt_k$$

and

$$T^*(x;f)=\sup_p |T_p(x;f)|.$$

If the conclusion were not valid, we would have for any $f\in L_\Phi$

$$\overline{\lim_{p\to\infty}} |T_p(x;f)|<\infty$$

on some set of positive measure. By use of the theorem of E. M. Stein on limits of sequences of operators ([2]), we should get that there exists a positive constant A such that

$$|\{x\in T^n;\ T^*(x;f)>\alpha\}|\leq\int_{T^n}\Phi\Big(\frac{A}{\alpha}|f|\Big)\,dx$$

for all $f\in L_\Phi$ and for all $\alpha>0$, and moreover by the convexity of $\Phi(t)$ and the property $\Phi(0)=0$,

$$|\{x\in T^n;\ T^*(x;f)>\alpha\}|\leq\frac{1}{\alpha}\int_{T^n}\Phi(A|f|)\,dx$$

for all $f\in L_\Phi$ and for all $\alpha\geq1$. But if we can construct a function $f_0\in L_\Phi$ such that

$$|\{x\in T^n;\ T^*(x;f_0)>\alpha\}|\geq C\,\frac{(\log\log\alpha)^{n-1}}{\alpha}$$

for all large α, where C is some constant independent of α, we shall have a contradiction and so the theorem shall be proved.

Now we define the function $f_0(x)$ by

$$f_0(x)=\begin{cases}\dfrac{1}{x_1\cdots x_n}\,\dfrac{(\log\log 1/x_1\cdots x_n)^{n-1}}{(\log 1/x_1\cdots x_n)^{n-1+k}} & \text{if } x\in(0,\delta]^n\\[2ex] 0 & \text{if } x\in(-\pi,\pi]^n/(0,\delta]^n\end{cases}$$

where δ is positive small number such as $f_0(x)>1$ on $(0, \delta]^n$. Further we extend it to R^n periodically with period 2π with respect to each variable. Then since $\log^+ f_0(x)=\log f_0(x) \sim \log 1/x_1x_2\cdots x_n$ for $x\in(0, \delta]^n$, we have as $\varepsilon=k-(\beta+1)>0$,

$$\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f_0(x)\,(\log^+ f_0(x))^\beta dx$$

$$\leqq C_1\int_0^\delta\cdots\int_0^\delta \frac{1}{x_1\cdots x_n}\,\frac{(\log\log 1/x_1\cdots x_n)^{n-1}}{(\log 1/x_1\cdots x_n)^{n+\varepsilon}}\,dx_1\cdots dx_n$$

$$\leqq C_2\int_0^\delta\cdots\int_0^\delta \frac{1}{x_1\cdots x_n}\,\frac{1}{(\log 1/x_1\cdots x_n)^{n+(\varepsilon/2)}}\,dx_1\cdots dx_n$$

$$=C_3\int_0^{\delta^n}\frac{1}{t(\log 1/t)^{1+(\varepsilon/2)}}\,dt=C_3\int_{1/\delta^n}^{\infty}\frac{1}{t(\log t)^{1+(\varepsilon/2)}}\,dt$$

$$<\infty$$

and so $f_0\in L_\Phi$.

Next we shall estimate $T^*(x\,;\,f_0)$ from below. By the definition of $f_0(x)$, we have, denoting $x_{k+1}\cdots x_n$ by x^* for the sake of simplicity, for $x\in(0, \delta]^n$,

$$T^*(x\,;\,f_0)$$

$$\geqq\int_0^\delta\cdots\int_0^\delta\frac{1}{t_1\cdots t_kx^*}\,\frac{(\log\log 1/t\cdots t_kx^*)^{n-1}}{(\log 1/t_1\cdots t_kx^*)^{n-1+k}}\prod_{i=1}^{k}\cot\frac{x_i+t_i}{2}\,dt_1\cdots dt_k$$

$$\geqq C_1\,\frac{1}{x_1\cdots x_k}\int_0^{x_1}\cdots\int_0^{x_k}\frac{1}{t_1\cdots t_kx^*}\,\frac{(\log\log 1/t_1\cdots t_kx^*)^{n-1}}{(\log 1/t_1\cdots t_kx^*)^{n-1+k}}\,dt_1\cdots dt_k$$

$$\geqq C_1\,\frac{(\log\log 1/x_1\cdots x_n)^{n-1}}{x_1\cdots x_k}\int_0^{x_1}\cdots\int_0^{x_k}\frac{1}{t_1\cdots t_kx^*}\,\frac{1}{(\log 1/t_1\cdots t_kx^*)^{n-1+k}}\,dt_1\cdots dt_k$$

$$\geqq C_2\,\frac{1}{x_1\cdots x_n}\,(\log\log 1/x_1\cdots x_n)^{n-1}\int_0^{x_1\cdots x_n}\frac{1}{t(\log 1/t)^n}\,dt$$

$$=C_2\,\frac{1}{x_1\cdots x_n}\,(\log\log 1/x_1\cdots x_n)^{n-1}\int_{1/x_1\cdots x_n}^{\infty}\frac{1}{t(\log t)^n}\,dt$$

$$=C_3\,\frac{1}{x_1\cdots x_n}\,\frac{(\log\log 1/x_1\cdots x_n)^{n-1}}{(\log 1/x_1\cdots x_n)^{n-1}}$$

So, when we put

$$E(\alpha)=\{x\in(0, \delta]^n\,;\,\frac{1}{x_1\cdots x_n}\,\frac{(\log\log 1/x_1\cdots x_n)^{n-1}}{(\log 1/x_1\cdots x_n)^{n-1}}\geqq\alpha\}$$

we have only to prove

$$|\,E(\alpha)\,|\geqq C_4\,\frac{(\log\log\alpha)^{n-1}}{\alpha}$$

for all large α.

We put

$$\omega(t) = \frac{1}{t} \frac{(\log\log 1/t)^{n-1}}{(\log 1/t)^{n-1}}$$

for $0<t\leqq\delta^n$. Let t_0 be the point such that $\omega(t_0)=\alpha$. Then we get for all large α,

$$\frac{(\log\log\alpha)^{n-1}}{2\alpha\ (\log\alpha)^{n-1}} \leqq t_0 \leqq \frac{2(\log\log\alpha)^{n-1}}{\alpha\ (\log\alpha)^{n-1}}$$

This inequality can be shown by setting left hand side or right hand side by t_1 or t_2 respectively and by proving $\omega\ (t_1) \geqq \omega\ (t_0) \geqq \omega\ (t_2)$, which can be verified easily.
Therefore

$$|\ E(\alpha)\ | = |\ \{x\in(0,\ \delta]^n\ ;\ \omega(x_1\cdots x_n)\geqq\omega(t_0)\}\ |$$

$$= |\ \{x\in(0,\ \delta]^n\ ;\ x_1\cdots x_n\leqq t_0\}\ |$$

$$\geqq |\ \{x\in(0,\ \delta]^n\ ;\ x_1\cdots x_n\leqq \frac{(\log\log\alpha)^{n-1}}{2\alpha(\log\alpha)^{n-1}}\ \}\ |$$

We put

$$r = \frac{1}{\delta} \frac{(\log\log\alpha)^{n-1}}{2\alpha\ (\log\alpha)^{n-1}}$$

Then, since $r\leqq 1/\alpha$ for all large α, we have

$$|\ E(\alpha)\ |$$

$$\geqq \int_{\substack{(x_1,\ \ldots,\ x_{n-1})\in(0,\ \delta]^{n-1} \\ x_1\cdots x_{n-1}\geqq r}} \frac{1}{x_1\cdots x_{n-1}} \frac{(\log\log\alpha)^{n-1}}{2\alpha\ (\log\alpha)^{n-1}}\, dx_1\cdots dx_{n-1}$$

$$\geqq \frac{(\log\log\alpha)^{n-1}}{2\alpha\ (\log\alpha)^{n-1}} \int_{\substack{(x_1,\ \ldots,\ x_{n-1})\in(0,\ \delta]^{n-1} \\ x_1\cdots x_{n-1}\geqq 1/\alpha}} \frac{1}{x_1\cdots x_{n-1}}\, dx_1\cdots dx_{n-1}$$

$$= \frac{(\log\log\alpha)^{n-1}}{2\alpha\ (\log\alpha)^{n-1}} \frac{1}{(n-1)!} \left[\log\ (\alpha\ \delta^{n-1})\right]^{n-1}$$

$$\geqq C_4 \frac{(\log\log\alpha)^{n-1}}{\alpha(\log\alpha)^{n-1}} (\log\alpha)^{n-1} = C_4 \frac{(\log\log\alpha)^{n-1}}{\alpha}$$

Q. E. D.

REMARK. Theorem 1 in the case of $n=2$ is reduced as follows. For any given $0 \leqq \beta < 1$, there exists a function $f \in L(\log^+ L)^\beta\ (T^2)$ such that $\tilde{f}_{\{1,2\}}$ does not exist almost everywhere in T^2. This result was first announced by E. M. Stein in the paper [2], but L. V. Zhizhiashvili indicated in [4] that his proof could not be proceeded without correction, stated in [5] that still this result is true, and gave the proof with some modification in [6]. Here we have only to consider the function

$$f_0(x) = \frac{1}{x_1 x_2} \frac{\log \log 1/x_1 x_2}{(\log 1/x_1 x_2)^3}$$

on $(0, \delta]^2$.

§2. On the integrability of conjugate functions.

As we can see from the facts on the existence of conjugate functions of several variables and from the wellknown results in the case of $n=1$, if $f \in L\ (\log^+ L)^\beta (T^n)$ $(\beta \geqq 1)$, then for any set B with $k(B) \in [1, \beta]$, $\tilde{f}_B$ exists almost everywhere in T^n, belongs to $L(T^n)$ and satisfies the following inequality ;

$$\| \tilde{f}_B \|_1 \leqq A\ \{\textstyle\int_{T_n} | f(x) | (\log^+ | f(x) |)^{k(B)} dx + 1\}$$

$$\leqq A\ \{\textstyle\int_{T_n} | f(x) | (\log^+ | f(x) |)^{\beta}\ dx + 1\}.$$

If $\beta \geqq n$, then the integers k with $k \in [1, \beta]$ are all of $1, \ldots\ldots, n$. On the other hand, when $0 \leqq \beta < n$, in general the condition $f \in L\ (\log^+ L)^\beta\ (T^n)$ does not ensure the integrability of $\tilde{f}_B$ for B with $k(B) \in (\beta, n]$ in the following sense ([7, p. 86], ([6, pp. 138-140 for $n=2$]).

THEOREM 2. *For each given $0 \leqq \beta < n$, there exists a function $f \in L\ (\log^+ L)^\beta\ (T^n)$ such that for any set B with $k(B) \in (\beta, n]$, $\tilde{f}_B$ is not integrable even if it exists.*

PROOF. We take ε so small that $0 < \varepsilon \leqq \min\ \{k - \beta\ ;\ k \in (\beta,\ n] \cap Z\}$, and define the function $f_1(x)$ by

$$f_1(x) = \begin{cases} \dfrac{1}{x_1 \cdots x_n} \dfrac{1}{(\log 1/x_1 \cdots x_n)^{n+\beta+\varepsilon}} & \text{if } x \in (0,\ \delta]^n \\ 0 & \text{if } x \in [-\pi,\ \pi)^n / (0,\ \delta]^n \end{cases}$$

where δ is positive small number such as $f_1(x) > 1$ on $(0,\ \delta]^n$. Further we extend it to R^n periodically with period 2π with respect to each variable. Then since $\log^+ f_1(x) = \log f_1(x)$ $\sim \log 1/x_1 \cdots x_n$ for $x \in (0,\ \delta]^n$, we have

$$\textstyle\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f_1(x)\ (\log^+ f_1(x))^\beta\ dx$$

$$\leqq C_1 \int_0^\delta \cdots \int_0^\delta \frac{1}{x_1 \cdots x_n} \frac{1}{(\log 1/x_1 \cdots x_n)^{n+\varepsilon}} dx_1 \cdots dx_n$$

$$= C_2 \int_0^{\delta^n} \frac{1}{t(\log 1/t)^{1+\varepsilon}} dt = C_2 \int_{1/\delta^n}^{\infty} \frac{1}{t(\log t)^{1+\varepsilon}} dt < \infty$$

and so $f_1 \in L (\log^+ L)^\beta (T^n)$.

Next we take a set B with $k(B) \in (\beta, n]$ arbitarily. For the sake of simplicity we may suppose $B = \{1, 2, \ldots, k\}$, where $\beta < k \leqq n$. Similarly in the proof of Theorem 1, we take any sequence of positive number $\{\varepsilon_p\}_{p=1}^{\infty}$ with $\varepsilon_p \to 0$ $(p \to \infty)$, and define $T_p(x\,;f_1)$ as before. Then we have for $x \in (0, \delta]^n$, denoting again $x_{k+1} \cdots x_n$ by x^*,

$$\lim_{p\to\infty} T_p(x\,;f_1)$$

$$= \int_0^\delta \cdots \int_0^\delta \frac{1}{t_1 \cdots t_k x^*} \frac{1}{(\log 1/t_1 \cdots t_k x^*)^{n+\beta+\varepsilon}} \prod_{i=1}^{k} \cot \frac{x_i + t_i}{2} dt_1 \cdots dt_k$$

$$\geqq C_1 \frac{1}{x_1 \cdots x_k} \int_0^{x_1} \cdots \int_0^{x_k} \frac{1}{t_1 \cdots t_k x^*} \frac{1}{(\log 1/t_1 \cdots t_k x^*)^{n+\beta+\varepsilon}} dt_1 \cdots dt_k$$

$$\geqq C_2 \frac{1}{x_1 \cdots x_n} \int_0^{x_1 \cdots x_n} \frac{1}{t(\log 1/t)^{n+\beta+\varepsilon-(k-1)}} dt$$

$$= C_2 \frac{1}{x_1 \cdots x_n} \int_{1/x_1 \cdots x_n}^{\infty} \frac{1}{t(\log t)^{n+\beta+\varepsilon-(k-1)}} dt$$

$$= C_3 \frac{1}{x_1 \cdots x_n} \frac{1}{(\log 1/x_1 \cdots x_n)^{n+\beta+\varepsilon-k}}$$

Since ε is choosen as $\varepsilon \leqq k-\beta$, so $n+\beta+\varepsilon-k \leqq n$. Therefore $\lim_{p\to\infty} T_p(x\,;f_1) \notin L(0, \delta]^n$ and it is concluded that $(\tilde{f}_1)_B$ can not belong to $L(T^n)$, even if it exists. Q. E. D.

REMARK. Theorem 2 in the case of $n=2$ implies the following result. For any given $1 \leqq \beta < 2$, there exists a function $f \in L(\log^+ L)^\beta (T^2)$ such that $\tilde{f}_{\{1,2\}}$ exists almost everywhere in T^2 and does not belong to $L(T^2)$. This fact is stated in [3] and is proved in [6] .

REFERENCES

[1] K. SOKOL-SOKOLOWSKI, On trignometric series conjugate to Fourier series of two variables, Fund. Math., **34** (1947), 166-182.

[2] E. M. STEIN, On limits of sequences of operators, Ann. Math., **74** (1961), 140-170.

[3] L. V. ZHIZHIASHVILI, Conjugate functions of two variables and double Fourier series, Soviet Math. Doklady, 4 (1963), 458-461.

[4] L. V. ZHIZHIASHVILI, Fourier series and maximal theorem, Soviet Math. Doklady, 8(1967), 617-619.

[5] L. V. ZHIZHIASHVILI, On conjugate functions and trigonometric series, Doctoral Dissertation, Math. Notes, **2** (1967), 910-914.

[6] L. V. ZHIZHIASHVILI, *Conjugate functions and trigonometric series,* Izdat. Tbilisi Univ., Tbilisi, 1969.

[7] L. V. ZHIZHIASHVILI, Some problems in the theory of simple and multiple trigonometric and orthogonal series, Russian Math. Surveys, **28** (1973), 65-127.

[8] L. V. ZHIZHIASHVILI, On conjugate functions of n-variables, Soviet Math. Doklady, **15** (1974), 1372-1373.

[9] L. V. ZHIZHIASHVILI, Validity of A. N. Kolmogoroff's theorem for conjugate functions of several variables, Math, Notes, **32** (1982), 485-490.

[10] A. ZYGMUND, On the boundary values of functions of several complex variables, I, Fund. Math., **36** (1949), 207-235.