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On Nilpotent Factors of Maximam Abelian Extensions of Algebraic Number Fields

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Abstract. For a finite Galois extension K of an algebraic number field k and an abelian extension L of K which is a Galois extension of k, let L_0 be the genus field of L with respect to K/k and L_n be the n-th central field of L with respect to K/k. Then the structure of $Gal(L_n/L_{n-1})$ is studied for $n \geqq 1$.

1. Let k be an algebraic number field and K be a finite Galois extension of k with the Galois group G. Let L be an abelian extension of K which is also a Galois extension of k. Let H be the Galois group of L/K. Let $\sigma \varepsilon G$ and S_σ be any extension of σ to L. Then $S_\sigma^{-1} x S_\sigma$ does not depend on the choice of S_σ for any $x \varepsilon H$. Hence G acts on H by $\sigma x = S_\sigma^{-1} x S_\sigma$. Let I_G be the augumentation ideal of the group ring $\mathbf{Z}[G]$ over the ring of integers $\mathbf{Z}$. Let I_G^n be the product of n-copies of I_G. Then we have a sequence $H \supset I_G H \supset I_G^2 H \supset \cdots$, and a sequence of the corresponding intermediate fields of L ; $K \subset L_1 \subset L_2 \subset \cdots$. We call L_n *the n-th central field* of L with respect to K/k. Let L_0 be the genus field of L with respect to K/k. Then it is known that $Gal(L_1/L_0)$ is a homomorpic image of the Schur multiplicator M(G) of G, and that there exists a finite extension L of K such that $Gal(L_1/L_0) \cong M(G)$. This field L_1 is called *an abundant central extension* of K/k in [3] . Let [G, G] be the commutator group of G and $G_0 = G/[G, G]$. We give an alternative proof of the existence of the abundant central extension, using the cohomology group of the maximam connected component of the idēle class group of K (Theorem 3). Secondly we prove that $Gal(L_{n+1}/L_n)$ is an infinite torsion group with the exponent dividing the order of G_0 when L is the maximam abelian extension of K and $G_0 \neq 1$ (Theorem 4).

2. Let C_K (resp. C_k) be the idēle class group of K (resp. k). Let $N_{K/k} : C_K \to C_k$ be the norm map and $H_{K/k}$ be its kernel. Let H_L be the closed subgroup of C_K corresponding to L.

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PROPOSITION 1. $\mathrm{Gal}(L/L_0)\cong H_{K/k}/H_L\cap H_{K/k}$.

PROOF. By class field theory, $\mathrm{Gal}(L/L_0)\cong N_{K/k}^{-1}(N_{K/k}(H_L))/H_L\cong H_L\cdot H_{K/k}/H_L\cong H_{K/k}/H_L\cap H_{K/k}$.

COROLLARY 2. $\mathrm{Gal}(L_1/L_0)\cong H_{K/k}/C_K^{I_G}\cdot(H_{K/k}\cap H_L)$, *where we denote by* $C_K^{I_G}$ *the group* $\{xa \mid a\in C_K, x\in I_G\}$.

Let $M(G)$ be the Schur multiplicator of G. Then $M(G)\cong H^{-3}(G,\mathbb{Z})\cong H^{-1}(G,C_K)$. We prove the following theorem in other way of [3] or [4].

THEOREM 3. *There exists an abundant central extension of* K/k.

PROOF. Let $\mathfrak{m}$ be a G-invariant divisor of K and $K_{\mathfrak{m}}$ be the ray class field mod $\mathfrak{m}$ over K. We put $L=K_{\mathfrak{m}}$ and $H_{\mathfrak{m}}=H_{K_{\mathfrak{m}}}$. By Corollary 2 we have the sequence

$$1\to C_K^{I_G}\cdot(H_{\mathfrak{m}}\cap H_{K/k})/C_K^{I_G}\to H_{K/k}/C_K^{I_G}\to \mathrm{Gal}(L_1/L_0)\to 1.$$

Let D_K be the maximam connected component of C_K. Then we have $D_K=\bigcap_{\mathfrak{m}} H_{\mathfrak{m}}$, where $\mathfrak{m}$ runs through all of the G-invariant divisors of K. We note that $C_K^{I_G}$ is a compact subgroup of C_K and $H^{-1}(G,D_K)\cong H_{K/k}\cap D_K/D_K^{I_G}=1$. $H_{K/k}/C_K^{I_G}\cong H^{-1}(G,C_K)$ is a discrete group, and $\bigcap(H_{K/k}\cap H_{\mathfrak{m}})=H_{K/k}\cap D_K\subset C_K^{I_G}$. Thus we have a G-invariant divisor $\mathfrak{m}$ of K such that $H_{K/k}\cap H_{\mathfrak{m}}\subset C_K^{I_G}$. Since $H_{K/k}/C_K^{I_G}\cong M(G)$, we have Theorem 3.

3. Let $G_0^{(n)}$ be the tensor product $G_0\otimes\cdots\otimes G_0$ of n-copies of G_0. We have by [2] that $\mathrm{Gal}(L_{n+1}/L_n)$ is a torsion group with the exponent dividing $\#G_0$. Moreover $\mathrm{Gal}(L_{n+1}/L_n)=1$ for all $n\geqq 1$ if $G_0=1$.

THEOREM 4. *Let* L *be the maximam abelian extension of* K.

(1) *If* $G_0=1$, *we have* $\mathrm{Gal}(L_{n+1}/L_n)=1$ *for all* $n\geqq 1$.

(2) *If* $G_0\neq 1$, *we have* $\mathrm{Gal}(L_{n+1}/L_n)$ *is an infinitely generated torsion groups with the exponent dividing* $\#G_0$.

To prove Theorem 4, we need some lemmas. Let K_1 be an intermediate Galois extension of K/k. Let M be an abelian extension of K_1 which is also a Galois extension of k. Let L=MK. We denote by M_n (resp. L_n) the n-th central field of M (resp. L) with respect to K_1/k (resp. K/k).

LEMMA 5. $\mathrm{Gal}(L_{n+1}/L_n)$ *is finite iff* $\mathrm{Gal}(M_{n+1}/M_n)$ *is so.*

PROOF. Let $K^*=K\cap M$. Let $G=Gal(K/k)$, $G_1=Gal(K_1/k)$, $H=Gal(M/K_1)$, and $H'=Gal(M/K^*)\cong Gal(L/K)$. It is easily verified that $I_G^n\, Gal(L/K)\cong I_{G_2}^n H'$, where $G_2=Gal(K^*/k)$. Since H is abelian and'$Gal(K^*/K_1)$ is its factor group, $Gal(K^*/K_1)$ acts trivially on H'. We have $I_{G_2}^n H'=I_{G_1}^n H'$. We define the surjection $\varphi : I_{G_1}^n \otimes H/H' \rightarrow I_{G_1}^n H/I_{G_1}^n H'$ by φ $(x\,(a \bmod H'))=xa \bmod I_G^n H'$. Since H/H' is finite, so is $I_{G_1}^n H/I_{G_1}^n H'$. Hence $[I_{G_1}^n H : I_{G_1}^{n+1} H]$ $=[I_{G_1}^n H : I_{G_1}^n H']\ [I_{G_1}^n H' : I_{G_1}^{n+1} H']/\ [I_{G_1}^{n+1} H : I_{G_1}^{n+1} H']$. We have Lemma 5.

By Lemma 5, it is sufficient to prove (2) of Theorem 4 when K is a cyclic extension over k of order p which is a prime factor of $\#G_0$. Let G be a cyclic group of order p. In the following, we denote by I_G the augumentation ideal of the group ring $\mathbf{Z}_p[G]$, where $\mathbf{Z}_p$ is the ring of p-adic integers.

Let A be a finite p-abelian group such that $\mathbf{Z}_p[G]$ acts on it. We denote by I_G^n the product of n-copies of I_G. Let $I_G^n A=\{xa \mid x\in I_G^n,\ a\in A\}$. We define nil(A) to be I_G^n for the smallest natural number n such that $I_G^n A=0$. Let $X_1,\cdots, X_p$ be indeterminants. Let $S_n=\sum_{1\leqq i_1<<\cdots i_n\leqq k} X_{i_1}\cdot\cdots\cdot X_{i_n}$ for $1\leqq n\leqq p$ and $S_0=1$. Let $S_n^{(k)}=\sum_{1\leqq i_1<\cdots<i_n\leqq p} X_{i_1}\cdot\cdots\cdot X_{i_n}$, where we chose $i_1,\cdots, i_n$ so that any of them are not equal to k. Let $S_0^{(k)}=1$. Let $T_m=X_1^m+\cdots+X_p^m$. Then $S_n=X_k\, S_{n+1}^{(k)}+S_n^{(k)}$. Hence we have $S_1S_n=\sum_{k=1}^{p} X_kS_n=\sum X_k^2\, S_{n-1}^{(k)}+\sum X_k\, S_n^{(k)}=\sum X_k^2\,(S_{n-1}-X_k\, S_{n-2}^{(k)})+\sum X_k\, S_n^{(k)}=T_2\, S_{n-1}-\sum X_k^3\, S_{n-2}^{(k)}+\sum X_kS_n^{(k)}$ when $p-1\geqq n\geqq 2$. Moreover $\sum X_kS_n^{(k)}=(n+1)\, S_{n+1}$. Hence we have $S_1S_n=\sum_{i=2}^{n+1} (-1)^iT_iS_{n+1-i}+(n+1)\, S_{n+1}$. We substitute σ^i for X_i in the following, where σ is a generator of G.

LEMMA 6. $I_G^P=pI_G$.

PROOF. Let X be an indeterminant. Let σ be a generator of G. Put $\prod_{i=1}^{p} (X-\sigma^i)=X^p-S_1X^{p-1}+\cdots+(-1)^{p-1}S_{p-1}X+(-1)^{p-1}S_p$ for $S_i\in \mathbf{Z}_p[G]$. Obviously $(\sigma-1)S_1=0$. Let $T_i=\sigma^i+\sigma^{2i}+\cdots+\sigma^{pi}$. Then we have $0=(\sigma-1)\, S_1S_n=(\sigma-1)\sum_{i=2}^{n+1} (-1)^iT_iS_{n+1-i}+(\sigma-1)\,(n+1)\, S_{n+1}$ for $1\leqq n\leqq p-2$. Since $(\sigma-1)\, T_i=0$ for $2\leqq i\leqq p-1$ and $(n+1)\in \mathbf{Z}_p$, we have $(\sigma-1)\, S_{n+1}=0$. Put $\prod_{i=1}^{p-1} (x-\sigma^i)=\underline{X}^{p-1}-S_1'\, X^{p-1}+\cdots+(-1)^{p-2}S_{p-2}'\, X+(-1)^{p-1}S_{p-1}'$. Since $S_n'=S_n-S_{n-1}'$ for $2\leqq n\leqq p-1$, we have $(\sigma-1)S_n'=-(\sigma-1)\, S_{n-1}'$. Hence $(\sigma-1)\, S_n'=(-1)^{n-1}(\sigma-1)\, S_1'=(-1)^n(\sigma-1)$. We substitute 1 for X. Then $(\sigma-1)\prod_{i=1}^{p-1} (1-\sigma^i)=p\,(\sigma-1)$. Since $(\sigma-1)\prod (1-\sigma^i)$ generates I_G^P, we have $I_G^P=p\, I_G$.

The following Lemma 7 is obvious.

LEMMA 7. *Let* A *be a finite p-ablian group such that* $\mathbf{Z}_p[G]$ *acts on it. Then we have*
(1) $nil(A) \supset I_G^n \Leftrightarrow I_G^n A = I_G^{n+1} A$
(2) $A \neq 0 \Leftrightarrow A$ *has a G-invariant element* x *which is not* 0.

LEMMA 8. *Put* $A = (\mathbf{Z}/p^n\mathbf{Z})[G]$. *Let* m *be an integer such that* $p^n I_G = I_G^m$. *Then we have* $I_G^{m-1} A \neq 0$.

PROOF. Since $I_G A = I_G / p^n I_G$, we have $I_G^{m-1} A \cong I_G^{m-1}/I_G^m$. If $I_G^{m-1} A = 0$, then $I_G^{m-1} = I_G^m$ and moreover $I_G^m = I_G^{m+1} = \cdots$. Then $I_G^m = p^k I_G$ for any $k \geqq n$ by Lemma 6. Hence $I_G^m \subset \bigcap_{k=n}^{\infty} p^k I_G = 0$. This contradicts to the fact $I_G^m \neq 0$. Thus we have Lemma 8.

The following Lemma 9 is known in [5] .

LEMMA 9. *Let* k *be an algebraic number field. Let* $\mathfrak{c}$ *be an integral divisor of* k *and* n *be a natural number. Then there exists an integral divisor* $\mathfrak{d}$ *of* k *which is prime to* $\mathfrak{c}$, *such that*

$$E_k(\mathfrak{d}) \supset E_k^n$$

where $E_k(\mathfrak{d})$ *is the group of units* x *of* k *such that* $x = 1$ *mod* $\mathfrak{d}$.

4. Let ζ_n be a primitive p^n-th root of 1. Let n_1 be the natural number such that $\mathbf{Q}(\zeta_{n_1}) \cap k \subsetneqq \mathbf{Q}(\zeta_{n_1})$. Let $k_n = k(\zeta_n)$. Let $n \geqq n_1$ and σ be a generator of $\mathrm{Gal}(k_{n+1}/k_n)$. Let $\mathfrak{q}$ be a prime ideal of k which is completely decomposed in K/k, which is prime to p, and whose Artin synbol $\left(\frac{k_n/k}{\mathfrak{q}}\right)$ equals σ. Let $q^f = N_{k/\mathbf{Q}}(\mathfrak{q})$. Since the restriction of σ onto $\mathbf{Q}(\zeta_{n+1})$ is a generator of $\mathrm{Gal}(\mathbf{Q}(\zeta_{n+1})/\mathbf{Q}(\zeta_n))$, we have $q^f \equiv 1$ mod p^n but $q^f \not\equiv 1$ mod p^{n+1}. Let $\mathfrak{P}$ be the prime ideal of K such that $\mathfrak{P} \mid \mathfrak{q}$. Then the residue field of $\mathfrak{P}$ contains the primitive p^n-th root of 1, but does not contain the primitive p^{n+1}-th root of 1. Let $\mathfrak{L}$ be the set of all of such prime ideals $\mathfrak{P}$ of K. This set is a infinite set. Let $\mathfrak{P}_1 \in \mathfrak{L}$ and $\mathfrak{p}_1 = N_{K/k}\mathfrak{P}_1$. We apply Lemma 9 for $p^n$and $\mathfrak{p}_1$. Then we have an ideal $\mathfrak{m}_1$ such that $(\mathfrak{m}_1, \mathfrak{p}_1) = 1$ and $E_K(\mathfrak{m}_1) \subset E_K^{p^n}$. We can take $\mathfrak{m}_1$ to be G-invariant. Let K_1 be the ray class field mod $\mathfrak{m}_1\mathfrak{p}_1$ of K, which is a Galois extension over k. Then we have $\mathrm{Gal}(K_1/K) \cong J_K / K^\times \cdot U_K(\mathfrak{m}_1\mathfrak{p}_1)$, where J_K is the idéle group of K and $U_K(\mathfrak{m}_1\mathfrak{p}_1)$ is the group of unit idéles u such that $u \equiv 1$ mod $\mathfrak{m}_1\mathfrak{p}_1$. Let ζ be the primitive p^n-th root of 1 contained in the completion of K at $\mathfrak{P}_1$. Let $j_{\mathfrak{P}_1} : \mathfrak{P}_1 \to J_K$ be the canonical injection. We put $\mathfrak{a} = j_{\mathfrak{P}_1}(\zeta)$. We denote by $\mathfrak{P}_1^\sigma$ the conjugate ideal of $\mathfrak{P}_1$ with respect to $\sigma \in G$ and by $\mathfrak{a}^\sigma$ the conjugate of $\mathfrak{a}$ with respect to $\sigma \in G$. Let T be the subgroup of J_K generated by $\mathfrak{a}^\sigma$ for $\sigma \in G$. Let $au \in K^\times \cdot U_K(\mathfrak{m}_1\mathfrak{p}_1) \cap T$, where $a \in K^\times$ and $u \in U_K(\mathfrak{m}_1\mathfrak{p}_1)$. Then $au = \prod_{\sigma \in G} \mathfrak{a}^{\sigma b_\sigma}$ for certein $b_\sigma \in \mathbf{Z}$. Since $a = u^{-1} \prod \mathfrak{a}^{\sigma b_\sigma} \in U_K(\mathfrak{m}_1) \cap K^\times = E_K(\mathfrak{m}_1)$, we have an element $a_1 \in E_K$ such that $a_1^{p^n}=a$. Since $a=a_1^{p^n}=\mathfrak{a}^{\sigma b \sigma}$ mod $\mathfrak{P}_1^{\sigma}$, we have $b_\sigma=0$ for any $\sigma \in G$. Hence $K^\times \cdot U_K(\mathfrak{m}_1 \mathfrak{p}_1) \cap T=\{1\}$ and $T \cong T \cdot K^\times \cdot U_K(\mathfrak{m}_1 \mathfrak{p}_1)/K^\times \cdot U_K(\mathfrak{m}_1 \mathfrak{p}_1)$. Let M_1 be the subfield of K_1 corresponding to $T \cdot K^\times \cdot U_K(\mathfrak{m}_1 \mathfrak{p}_1)/K^\times \cdot U_K(\mathfrak{m}_1 \mathfrak{p}_1)$. Then $\mathrm{Gal}(K_1/M_1) \cong T \cong (\mathbf{Z}/p^n \mathbf{Z})[G]$. Let $\mathfrak{P}_2$ be an element of L which is prime to $\mathfrak{m}_1 \mathfrak{p}_1$. We apply Lemma 9 for p^n and $\mathfrak{m}_1 \mathfrak{p}_1 \mathfrak{p}_2$, where $\mathfrak{p}_2=N_{K/k} \mathfrak{P}_2$. Then we obtain a G-invariant divisor $\mathfrak{m}_2$ such that $(\mathfrak{m}_2, \mathfrak{m}_1 \mathfrak{p}_1 \mathfrak{p}_2)=1$ and $E_K(\mathfrak{m}_2) \in E^{p^n}_K$. Let K_2 be the ray class field mod $\mathfrak{m}_2 \mathfrak{p}_2$ over K and M_2 be its subfield such that $\mathrm{Gal}(K_2/M_2)=(\mathbf{Z}/p^n\mathbf{Z})[G]$. In this way, we have ray class fields K_i over K and its subfield M_i for $i=1, 2, 3, \cdots$. Put $L_t=K_1 \cdots K_t$. Then L_t contains the Hilbert class field H over K. Let A be the p-Sylow subgroup of $\mathrm{Gal}(L_t/K)$, A_1 be that of $\mathrm{Gal}(L_t/H)$, and A_2 be that of $\mathrm{Gal}(H/K)$. Let $\mathrm{nil}(A_2)=I^j_G$. Then we have $I^j_G A \subset A_1$, and further $I^l_G A \supset I^l_G A_1 \supset I_G^{l+j+1} A_1 \supset I_G^{l+2j+1} A$. Hence we have $[I^l_G A : I_G^{l+2j+1} A] \geqq [I^l_G A_1 : I_G^{l+j+1} A_1]$.

Let B_i be the p-Sylow subgroup of $\mathrm{Gal}(K_i/H)$. Then $A_1 \cong \prod_{i=1}^{t} B_i$. Let $p^n I_G=I_G^m$ by Lemma 6. Since $I_G^{m-1}\mathrm{Gal}(K_i/M_i) \neq 0$ by Lemma 8, we have $I_G^{m-1} B_i \neq 0$. Since $I_G^{m-1} A_1/I_G^m A_1 \cong \prod_{i=1}^{t} I_G^{m-1} B_i/I_G^m B_i$ and $I_G^{m-1} B_i/I_G^m B_i \neq 0$ by Lemma 7, we have $[I_G^{m-1} A_1 : I_G^m A_1] \geqq p^t$. Let $m>j+1$. Let $l=m-1$. Then $[I_G^{m-1} A : I_G^{m+2j} A] \geqq [I_G^{m-1} A_1 : I_G^m A_1] \geqq p^t$ because $I_G^{m-1} A_1 \supset I_G^m A_1 \supset I_G^{m+2j} A_1$. Let σ be a generator of G. Then we have the surgection $\varphi : I_G^k A/I_G^{k+1} A \to I_G^{k+1} A/I_G^{k+2} A$ by $\varphi(a \bmod I_G^{k+1} A)=(\sigma-1)a \bmod I_G^{k+2} A$ for $a \in A$. Hence $[I_G^k A : I_G^{k+1} A] \geqq [I_G^{k+1} A : I_G^{k+2} A]$. Thus $[I_G^{m-1} A : I_G^{m+2j} A]=\prod_{i=0}^{2j} [I_G^{m-1+i} A : I_G^{m+i} A] \leqq [I_G^{m-1} A : I_G^m A]^{2j+1}$. We have $p^t \leqq [I_G^{m-1} A : I_G^m A]^{2j+1}$. Moreover we have $[I_G^k A : I^{k+1} A] \geqq [I_G^{m-1} A : I_G^m A] \geqq \sqrt[2j+1]{p^t}$ for $1 \leqq k \leqq m-1$.

Let L be the maximal abelian extension of K and $H=\mathrm{Gal}(L/K)$. We denote by J_G the augumentation ideal of $\mathbf{Z}[G]$. Let L_t' be the maximam p-extension over K which is contained in L_t. Then $\mathrm{Gal}(L_t'/K)=A$. Let π be the restriction map of the Galois group from L onto L_t'. Then we have $\pi(J_G^n H)=J_G^n A=I_G^n A$. Hence $[I_G^n A : I_G^{n+1} A]=[J_G^n H \cdot \mathrm{Ker}\ \pi : J_G^{n+1} H \cdot \mathrm{Ker}\ \pi]=[J_G^n H : J_G^{n+1} H(J_G^n H \cdot \mathrm{Ker}\ \pi)] \leqq [J_G^n H : J_G^{n+1} H]$. We conclude the proof of Theorem 4.

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