

Representation Modules and the Augmentation Ideal of a Finite group

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Abstract. For a finite group G and a G -module A , the structure of $I^n A / I^{n+1} A$ is studied, where I is the augmentation ideal of the group ring $Z[G]$.

1. Let G be a finite group and A be a G -module. Denote by I the augmentation ideal of the group ring $Z[G]$ of G with respect to the ring of integers. Put $G_o = G/[G, G]$, where $[G, G]$ stands for the commutator subgroup of G . Denote by $G_o^{\otimes n}$ the tensor product $G_o \otimes \cdots \otimes G_o$ of n -copies of G_o . Then $I^n A / I^{n+1} A$ is isomorphic to a factor group of $G_o^{\otimes n} \otimes (A/IA)$. This is shown in [1] together with other cohomological relationship by means of class field theory, when G is the Galois group of a Galois extension K of an algebraic number field k and A is a congruent ideal class group of K .

The purpose of the present paper is to get a simple proof of the purely group theoretical part of the above result.

2. Let I be as above the augmentation ideal of the group ring $Z[G]$

LEMMA 1. We have the following congruences in $Z[G]$ for any σ and τ of G :

- (i) $\sigma \tau - 1 \equiv (\sigma - 1) + (\tau - 1) \pmod{I^2}$
- (ii) $\tau (\sigma - 1) \equiv \sigma - 1 \pmod{I^2}$
- (iii) $\sigma \tau \sigma^{-1} \tau^{-1} - 1 \equiv 0 \pmod{I^2}$

PROOF. (i) and (ii) are implied from $(\sigma - 1)(\tau - 1) = (\sigma \tau - 1) - (\sigma - 1) - (\tau - 1)$ and $\tau(\sigma - 1) - (\sigma - 1) = (\tau - 1)(\sigma - 1)$. Moreover (i) implies $(\sigma - 1) + (\sigma^{-1} - 1) \equiv \sigma \sigma^{-1} - 1 \equiv 0 \pmod{I^2}$. Hence $\sigma^{-1} - 1 \equiv -(\sigma - 1) \pmod{I^2}$. This implies (iii) by using (i) again.

LEMMA 2. Let φ be a mapping of $G_o \otimes (A/IA)$ to $IA/I^2 A$ defined by

$$\varphi(\sigma \otimes \alpha) \equiv (\sigma - 1) a \pmod{I^2 A},$$

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where $\bar{\sigma}$ and α are elements of G_0 and A/IA represented by σ of G and a of A respectively. Then φ induces a surjective homomorphism.

PROOF. Let $\bar{\sigma}_1 = \bar{\sigma}_2$. Then there exists an element ρ of $[G, G]$ such that $\sigma_2 = \sigma_1\rho$, and we see that $\sigma_2 - 1 \equiv (\sigma_1 - 1) + (\rho - 1) \equiv \sigma_1 - 1 \pmod{I^2}$ by (i) and (iii) of Lemma 1. Moreover (i) of Lemma 1 implies that $(\sigma^n - 1)a \equiv n(\sigma - 1)a \pmod{I^2A}$ for any integer n . Hence φ is well defined, and it is clear that φ induces a surjective homomorphism.

3. We denote by G^n the direct product of n -copies of G , and denote by $C^n(G, A)$ the group of cochains of G of dimension n with A as its coefficient module. For $f \in C^n(G, A)$, let

$$\Delta^n f = \sum (\tau_1 - 1) \cdots (\tau_n - 1) f(\tau_1, \dots, \tau_n),$$

where the sum is taken over all (τ_1, \dots, τ_n) of G^n . Denote by $R^n(G, A)$ the subgroup of $C^n(G, A)$ consisted of f such that $\Delta^n f = 0$. Let μ be a homomorphism of $C^n(G, A)$ to $G_0^{\otimes n} \otimes (A/IA)$ defined by

$$\mu(f) = \sum \bar{\tau}_1 \otimes \cdots \otimes \bar{\tau}_n \otimes \bar{f}(\tau_1, \dots, \tau_n),$$

where $\bar{f}(\tau_1, \dots, \tau_n)$ stands for the element of A/IA represented by $f(\tau_1, \dots, \tau_n)$ and the sum is taken over all (τ_1, \dots, τ_n) of G^n .

THEOREM. Notation being as above, we have

$$I^n A / I^{n+1} A \cong \frac{G_0^{\otimes n} \otimes (A/IA)}{\mu R^n(G, A)}.$$

PROOF. Applying Lemma 2 repeatedly, we have a surjective homomorphism $\varphi^{(n)}$ of $G_0^{\otimes n} \otimes (A/IA)$ to $I^n A / I^{n+1} A$, which is defined by

$$\varphi^{(n)}(\bar{\sigma}_1 \otimes \cdots \otimes \bar{\sigma}_n \otimes \alpha) \equiv (\sigma_1 - 1) \cdots (\sigma_n - 1) a \pmod{I^{n+1} A},$$

where $\bar{\sigma}_i$ and α are elements of G_0 and A/IA represented by $\sigma_i \in G$ and $a \in A$ respectively ($i = 1, \dots, n$). Let us determine $\text{Ker } \varphi^{(n)}$. Put $(\bar{\sigma}) = (\bar{\sigma}_1, \dots, \bar{\sigma}_n)$, which runs over all elements of G_0^n , and let representatives σ_i in G of each class of G_0 be fixed once for all. Now let $\alpha_{(\bar{\sigma})}$ be elements of A/IA such that $\varphi(\sum_{(\bar{\sigma})} \bar{\sigma}_1 \otimes \cdots \otimes \bar{\sigma}_n \otimes \alpha_{(\bar{\sigma})}) = 0$. Then $\sum (\sigma_1 - 1) \cdots (\sigma_n - 1) a_{(\bar{\sigma})} \equiv 0 \pmod{I^{n+1} A}$, where $a_{(\bar{\sigma})}$ are representatives of $\alpha_{(\bar{\sigma})}$ in A . There exist elements $b_{(\rho)}$ of A such that

$$\sum_{(\bar{\sigma})} (\sigma_1 - 1) \cdots (\sigma_n - 1) a_{(\bar{\sigma})} = \sum_{(\rho)} (\rho_1 - 1) \cdots (\rho_{n+1} - 1) b_{(\rho)},$$

where $(\rho) = (\rho_1, \dots, \rho_{n+1}) \in G^{n+1}$. Put $(\tau) = (\tau_1, \dots, \tau_n) \in G^n$, and let $b_{(\rho)}^{(\tau)} = \sum_{\rho_{n+1} \in G} (\rho_{n+1} - 1) b_{(\rho)}$,

where $b(\rho)$ runs over all (ρ) such that $(\rho_1, \dots, \rho_n) = (\tau_1, \dots, \tau_n)$. Put $f(\tau_1, \dots, \tau_n) = a_{(\bar{\sigma})} - b_{(\rho)}^{(\tau)}$ or $-b_{(\rho)}^{(\tau)}$ according as $(\tau_1, \dots, \tau_n) = (\sigma_1, \dots, \sigma_n)$ for some $(\bar{\sigma})$ or not. Then $f \in R^n(G, A)$. Moreover we have $\mu(f) = \sum \bar{\tau}_1 \otimes \dots \otimes \bar{\tau}_n \otimes \bar{f}(\tau_1, \dots, \tau_n) = \sum_{(\bar{\sigma})} \bar{\sigma}_1 \otimes \dots \otimes \bar{\sigma}_n \otimes \alpha_{(\bar{\sigma})}$, because $b_{(\rho)}^{(\tau)} \in IA$. Now we have $\text{Ker } \varphi^{(n)} \subset \mu R^{(n)}(G, A)$. The opposite inclusion is clear by definition.

References

- [1] Y. Furuta, On nilpotent factors of congruent ideal class groups of Galois extensions, Nagoya Math. J., **62** (1976), 13-28.