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Maskit's Combination Theorems and the Residual Limit Sets of the First Kind

Katsumi INOUE

Mathematical Institute, Tôhoku University, Sendai, 980. *Japan*

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Abstract Let G be a finitely generated Kleinian group. A property of the residual limit set of G is found by using Maskit's Combination Theorems.

1. Introduction.

In this note we shall deal with the separators and the residual limit sets of the first kind of finitely generated Kleinian groups. We shall show that, if G is constructed from its subgroups $G_1, \cdots, G_s$ by a finite number of applications of Maskit's Combination Theorems, the set of separators for G is the union of translates under G of separators for these groups (Theorem 1). Next we shall prove that the residual limit point of the first kind of G is nested by a sequence of structure loops of G (Theorem 2).

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2. Preliminaries.

Let G be a Kleinian group and denote by $\Omega(G)$ and $\Lambda(G)$ the region of discontinuity and the limit set of G, respectively. If G has at most two limit points, G is called elementary. A non-elementary fintely generated Kleinian group is degenerate, if $\Omega(G)$ is connected and simply connected. A connected component of $\Omega(G)$ is also called a component of G. For each component Δ of G we denote by G_Δ the subgroup of G which keeps Δ invariant and call G_Δ the component subgroup of Δ. If a finitely generated Kleinian group G has two components Δ, Δ' and $G=G_\Delta=G_{\Delta'}$, G is called quasi-Fuchsian. A web group is a finitely generated Kleinian group for which each component subgroup is quasi-Fuchsian. Clearly quasi-Fuchsian groups are web groups.

Consider a sequence $\{C_n\}$ of Jordan curves on $\hat{C}$ and a point $z \in \hat{C}$. We say that $\{C_n\}$ nests about z, if C_{n+1} separates z from C_n for every number n and if the sequence of spherical diameters of $\{C_n\}$ forms a null sequence. From now on, we assume that G is finitely generated. A Jordan curve $C \subset \Lambda(G)$ is called a separator for G if there is a component Δ of G and a component Δ_1 of $\Omega(G_\Delta)-\Delta$ so that $C=\partial\Delta_1$, where we denote by $\partial\Delta_1$ the boundary of Δ_1. The set of all separators for G is denoted by $S(G)$. It is well known that any two separators for G do not cross each other. (See [1]). The residual limit set $\Lambda_0(G)$ of G is the set $\Lambda(G)-\cup_i\partial\Delta_i$, where $\{\Delta_i\}$ is the set of all components of G. A point $\lambda\in\Lambda_0(G)$ is said to be of the first kind ($\lambda\in L_1(G)$) if there exists a sequence $\{C_n\}$ of separators for G so that $\{C_n\}$ nests about λ. Otherwise, it is said to be of the second kind ($\lambda\in L_2(G)$).

3. Maskit's Combination Theorems.

Let G be a Kleinian group and let H be a subgroup of G. A set S on $\hat{C}$ is called precisely invariant under H in G, if $h(S)=S$ for every $h\in H$ and $g(S)\cap S=\phi$ for every $g\in G-H$. For a cyclic subgroup H of G, a precisely invariant disc B for H is the interior of a closed topological disc $\overline{B}$ on $\hat{C}$, where $\overline{B}-\Lambda(H)$ is precisely invariant under H in G and $\overline{B}-\Lambda(H)\subset\Omega(G)$. We use Maskit's Combination Theorems in the following forms.

Combination Theorem I. Let G_1 and G_2 be two Kleinian groups and let B_i $(i=1, 2)$ be a precisely invariant disc for H, a finite cyclic or a parabolic cyclic subgroup of both G_1 and G_2. Assume that B_1 and B_2 have the common boundary γ and $B_1\cap B_2=\phi$. Let G be the group generated by G_1 and G_2. Then the following hold:

(I−1) G is Kleinian.

(I−2) G is the free product of G_1 and G_2 with the amalgamated subgroup H.

(I−3) $\Omega(G)/G=(\Omega(G_1)/G_1-B_1/H)\cup(\Omega(G_2)/G_2-B_2/H)$,
where $(\Omega(G_1)/G_1-B_1/H)\cap(\Omega(G_2)/G_2-B_2/H)=\gamma\cap\Omega(H)/H$.

Combination Theorem II. Let G_1 be a Kleinian group. For $i=1, 2$, let B_i be a precisely invariant disc for a finite cyclic or a parabolic cyclic subgroup H_i, and let γ_i be the boundary of B_i. Assume that $g(\overline{B}_1)\cap\overline{B}_2=\phi$ for all $g\in G_1$. Let f be a Möbius transformation satisfying $f(\gamma_1)=\gamma_2$, $f(B_1)\cap B_2=\phi$ and $f^{-1}H_2f=H_1$ and let G be the group generated by G_1 and f. Then the following hold:

(II−1) G is Kleinian.

(II−2) Every relation in G is a consequence of the relation in G_1 and the relation $f^{-1}H_2f=H_1$.

(II−3) $\Omega(G)/G=\Omega(G_1)/G_1-(B_1/H_1\cup B_2/H_2)$, where $(\gamma_1\cap\Omega(G))/H_1$ is identified in $\Omega(G)/G$ with $(\gamma_2\cap\Omega(G))/H_2$.

Let G be a Kleinian group which is constructed from $G_1, \cdots, G_s$ and $f_1, \cdots, f_t$ by a finite number of applications of Maskit's Combination Theorems I and II. Put $\Lambda_N(G)=$

$\Lambda(G) - \cup_{g \in G} g(\cup_{i=1}^{s} \Lambda(G_i))$. For each point $z \in \Lambda_N(G)$ there is a Jordan curve γ which is invariant under a finite cyclic or a parabolic cyclic subgroup H of G, and wihich lies, except for the fixed point of H, in $\Omega(G)$ so that $\{g_n(\gamma)\}$ nests about z for a suitable sequence $\{g_n\}$ in G. (See [4] and [5]). The loop γ may be chosen so as to be the boundary of a precisely invariant disc which appears in some step of the use of Maskit's Combination Theorems in constructing the group G. We call the Jordan curve γ and the translates of γ under G the structure loops of G. It is known that any two structure loops of G do not cross each other. (See [5] and [6]).

4. The separators.

Lemma 1. Let G be a finitely generated Kleinian group which is constructed from G_1 and G_2 by application of Maskit's Combination Theorem I. Then $S(G) = \cup_{g \in G} g(S(G_1) \cup S(G_2))$.

Proof. Let $\{\Delta_{1,1}, \cdots, \Delta_{1,m}\}$ (resp. $\{\Delta_{2,1}, \cdots, \Delta_{2,n}\}$) be a complete list of non-conjugate components of G_1 (resp. G_2), and set $\Omega(G_1)/G_1 = S_{1,1} + \cdots + S_{1,m}$ (resp. $\Omega(G_2)/G_2 = S_{2,1} + \cdots + S_{2,n}$). We may assume $\gamma \subset (\Delta_{1,1} \cup \Delta_{2,1}) \cup \Lambda(H)$, where γ is the common boundary of precisely invariant discs B_1 and B_2 under $H = G_1 \cap G_2$. (The set $\Lambda(H)$ may be empty). From (I-3) we may set $\Omega(G)/G = S_1 + \cdots + S_p$, where $S_1 = (S_{1,1} - B_1/H) \cup (S_{2,1} - B_2/H)$ and $\{S_2, \cdots, S_p\} = \{S_{1,2}, \cdots, S_{1,m}, S_{2,2}, \cdots, S_{2,n}\}$. Let π: $\Omega(G) \to \Omega(G)/G$ be a natural projection and set $\pi^{-1}(S_i) = \cup_j \Delta'_{i,j}$, where $\Delta'_{i,j}$ is a connected component of G.

First we prove $S(G) \supset \cup_{g \in G} g(S(G_1) \cup S(G_2))$. Since $S(G)$ is invariant under G, it suffices to show $S(G) \supset S(G_i)$ for $i = 1, 2$. We may assume $i = 1$. The property (I-3) implies that for every $\Delta_{1,\nu}$ $(2 \leqq \nu \leqq m)$ there are $\Delta'_{i,j}$ $(2 \leqq i \leqq p)$ and $g \in G$ so that $\Delta_{1,\nu} = g(\Delta'_{i,j})$. It means $S(G) \supset S(G_1) \cap \cup_{g \in G_1} g(\cup_{\nu=2}^{m} \partial\Delta_{1,\nu})$. Let C be any separator for G_1 in $\partial\Delta_{1,1}$. Denote by G'_1 (resp. $G_{1,1}$) the component subgroup of $\Delta'_{1,1}$ (resp. $\Delta_{1,1}$) of G (resp. G_1). Since $G'_1 \supset G_{1,1}$, we see $\Lambda(G'_1) \supset \Lambda(G_{1,1})$, so $\Lambda(G'_1) \supset C$. Furthermore, if Δ_c is a component of $G_{1,1}$ which is bounded by C, we see $\Omega(G'_1) \supset \Delta_c$. It means $S(G) \ni C$, so $S(G) \supset S(G_1) \cap \cup_{g \in G} g(\partial\Delta_{1,1})$. Thus we have $S(G) \supset S(G_1)$. In the similar manner we have $S(G) \subset S(G_2)$.

Next we show $S(G) \subset \cup_{g \in G} g(S(G_1) \cup S(G_2))$. The property (I-3) shows that for every $\Delta'_{i,j}$ $(2 \leqq i \leqq p)$ there are a component $\Delta_{k,\nu}$ ($k = 1$ or 2 and $\nu \neq 1$) of G_k and $g \in G$ so that $\Delta'_{i,j} = g(\Delta_{k,\nu})$. Thus we see $S(G) \cap (\cup_{i=2}^{p} \cup_j \partial\Delta'_{i,j}) \subset \cup_{g \in G} g(S(G_1) \cup S(G_2))$. Let C' be any separator for G in $\partial\Delta'_{1,j}$ and x be any point in $\Delta'_{1,j}$. We may set $\Delta'_{1,j} = \Delta'_{1,1}$. For every point $z \in C'$ there is a path σ from x to z so that the number of the crossings of σ and the translates of γ under G is finite. It follows that $z \notin \Lambda_N(G)$ and we have $g(C') \subset \Lambda(G_k)$. Let Δ'_c be a component of G'_1 which is bounded by C'. Then $\Delta'_c \subset \Omega(G_{k,\nu})$. It shows $g(C') \subset S(G_k)$ and so $S(G) \cap \cup_j \partial\Delta'_{1,j} \subset \cup_{g \in G} g(S(G_1) \cup S(G_2))$. Thus we have $S(G) \subset \cup_{g \in G} g(S(G_1) \cup S(G_2))$ and our lemma is established.

Lemma 2. Let G be a finitely generated Kleinian group which is constructed from G_1 and f by application of Maskit's Combination Theorem II. Then $S(G)=\cup_{g\in G} g(S(G_1))$.

Proof. Let $\{\Delta_{1,1},\cdots,\Delta_{1,m}\}$ be a complete list of non-conjugate components of G_1 and set $\Omega(G_1)/G_1=S_{1,1}+\cdots+S_{1,m}$. We may assume $\gamma_1\subset\Delta_{1,1}\cup\Lambda(H_1)$. (The set $\Lambda(H_1)$ may be empty). In general the set $\gamma_2-\Lambda(H_2)$ is not necessarily contained in the component which contains $\gamma_1-\Lambda(H_1)$. But, whether $\gamma_1-\Lambda(H_1)$ and $\gamma_2-\Lambda(H_2)$ are contained in the same component or not gives no essential effect in our discussion. So we may assume $\gamma_2\subset\Delta_{1,1}\cup\Lambda(H_2)$. From (II$-$3) and our assumption, we have $\Omega(G)/G=S_1+\cdots+S_p$, where $S_1=S_{1,1}-(B_1/H_1\cup B_2/H_2)$ and $\{S_2,\cdots,S_p\}=\{S_{1,2},\cdots,S_{1,m}\}$. Let π: $\Omega(G)\to\Omega(G)/G$ be a natural projection and set $\pi^{-1}(S_i)=\cup_j\Delta'_{i,j}$, where $\Delta'_{i,j}$ is a connected component of G.

First we show $S(G)\supset\cup_{g\in G}g(S(G_1))$. It suffices to show $S(G)\supset S(G_1)$. In the similar manner to the proof of Lemma 1, we see $S(G)\supset S(G_1)\cap\cup_{g\in G_1}g(\cup_{\nu=2}^{m}\partial\Delta_{1,\nu})$. Let C be any separator for G_1 in $\partial\Delta_{1,1}$. Denote by G'_1 (resp. $G_{1,1}$) the component subgroup of $\Delta'_{1,1}$ (resp. $\Delta_{1,1}$) of G (resp. G_1). Since $G'_1\supset G_{1,1}$, we have $\Lambda(G'_1)\supset\Lambda(G_{1,1})$, so $S(G')\ni C$. It means $S(G)\supset S(G_1)\cap\cup_{g\in G_1}g(\partial\Delta_{1,1})$. Thus we have $S(G)\supset S(G_1)$. By the similar argument to that of the proof of Lemma 1 we see $S(G)\subset\cup_{g\in G}g(S(G_1))$ and Lemma 2 is proved.

By using Lemma 1 or Lemma 2 in each step of the use of Maskit's Combination Theorem I or II, we have the following theorem.

Theorem 1. Let G be a finitely generated Kleinian group which is constructed from $G_1,\cdots,G_s$ and $f_1,\cdots,f_t$ by a finite number of applications of Maskit's Combination Theorems I and II. Then $S(G)=\cup_{g\in G}g(\cup_{i=1}^{s}S(G_i))$.

5. The residual limit sets of the first kind.

In [2], Abikoff and Maskit proved that every finitely generated Kleinian group can be constructed from $G_1,\cdots,G_s$ and $f_1,\cdots,f_t$ by a finite number of applications of Maskit's Combination Theorems I and II, where each G_i is an elementary group, a degenerate group or a web group. Here each elementary group has at most one limit point. From now on, we assume that G is finitely generated and is constructed in the way mentioned above.

Theorem 2. $L_1(G)\subset\Lambda_N(G)$.

Proof. Assume the contrary. Since $\Lambda(G)-\Lambda_N(G)=\cup_{g\in G}g(\cup_{i=1}^{s}\Lambda(G_1))$, there exist a point $z_0 \epsilon L_1(G)$ and an element $g \epsilon G$ so that $g(z_0) \epsilon \Lambda(G_i)$ for some i $(1 \leqq i \leqq s)$. If G_i is elementary, then $g(z_0)$ is a parabolic fixed point and we see $\Lambda(G_i) = \{g(z_0)\}$. By conjugation we may set $g(z_0) = \infty$, and may assume that the parabolic generator g_0 of G_i which fixes ∞ is a translation in the form $g_0 : z \to z+1$. The point ∞ is contained in $L_1(G)$ and there exists a sequence $\{C_n\}$ of separators for G so that $\{C_n\}$ nests about ∞. For sufficiently large numbers m_0 and n_0, $g_0^{m_0}(C_{n_0})$ and C_{n_0} cross each other. This is a contradicition. Thus G_i is not elementary. Next we assume that G_i is a degenerate group. From Theorem 1, there exist a web group $G_j (1 \leqq j \leqq s)$, a separator C_0 for G_j and a sequence $\{g_n\}$ in G so that $\{g_n(C_0)\}$ nests about $g(z_0)$. Since the limit set $\Lambda(G_i)$ is connected, g_n^{-1} $(\Lambda(G_1))$ cuts the separator C_0. This can not occur and G_i must be a web group. If $g(z_0)$ is contained is some separator C' for G_i, then $g_n(C_0)$ and C' cross each other for a sufficiently large number n. This is absurd and we have $g(z_0) \epsilon \Lambda_0(G_i) = L_2(G_i)$. From Theorem 1, any separator for G is contained in the closure of a component of G and the set of separators for G can not about $g(z_0)$. Thus G_i is not a web group and we complete the proof of Theorem 2.

In the proof Theorem 2 we have seen the following result.

Corollary 1. Any parabolic fixed point of G is not contained in $L_1(G)$.

We say that a limit point $z \epsilon \Lambda(G)$ is a point of approximation if there is a sequence $\{g_n\}$ of G and a point $x \epsilon \hat{C} - \{z\}$ so that the spherical distance $d(g_n(z), g_n(x))$ does not converge to zero. In [6] Maskit proved that every point of $\Lambda_N(G)$ is a point of approximation. Thus we have

Corollary 2. Every point of $L_1(G)$ is a point of approximation.

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