Sci. Rep. Kanazawa Univ.
Vol. 24 No. 1 pp. 9—12
June 1979

On the Almost Everywhere Convergence of Lacunary Spherical Partial Sums of Multiple Fourier Series

Michitaka KOJIMA

Department of Mathematics, Faculty of Science, Kanazawa University

(Received April 24, 1979)

Abstract. Let $S_R(f,x)$ be the spherical partial sum of multiple Fourier series of $f\in L^1(T^n)$. In this note for any Hadamard-lacunary sequence $\{R_k\}_{k=1}^{\infty}$ the problem for the almost everywhere convergence of $S_{R_k}(f,x)$ is considered. We shall prove that if $f\in L^p(T^n)$ then $S_{R_k}(f,x)$ converges almost everywhere as $k\to\infty$ for $p=2$ and that it is not valid for $1\leqq p<2$. This assertion is treated also in the more general situation of Bochner-Riesz means.

1. Introduction.

Let R^n be the $n(\geqq 2)$-dimensional Euclidean space, Z^n be the set of all lattice points in R^n, and T^n be the n-dimensional torus. For any $x=(x_1,\cdots,x_n)$, $y=(y_1,\cdots,y_n)$ in R^n, we denote $(x,y)=x_1y_1+\cdots+x_ny_n$ and $|x|=\sqrt{(x,x)}$.

For the Fourier series of a function $f\in L^1(T^n)$, we denote its Bochner-Riesz mean of order $\alpha\geqq 0$ by

$$S_R^\alpha(f,x)=\sum_{|m|<R}\left(1-\frac{|m|^2}{R^2}\right)^\alpha \hat{f}_m e^{i(m,x)},$$

where

$$\hat{f}_m=\frac{1}{(2\pi)^n}\int_{T^n} f(x)e^{-i(m,x)}dx \qquad (m\in Z^n).$$

Concerning to the problem for the almost everywhere convergence of $S_R^\alpha(f,x)$ as $R\to\infty$, the following are known by E.M. Stein and G. Weiss [5], C. Fefferman [3] etc. For each pair (p,α) in the region

$$A=\{1\leqq p<2,\ \alpha>(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)\}\cup\{2\leqq p<\infty, \alpha>0\},$$

$$\lim_{R\to\infty} S_R^\alpha(f,x)=f(x) \text{ a.e. for any } f\in L^p(T^n).$$

On the other hand for each pair (p,α) in the region

$$B=\{1\leqq p<\frac{2n}{n+1},\ 0\leqq\alpha<\frac{n}{p}-\frac{n+1}{2}\}\cup\{\frac{2n}{n+1}\leqq p<2,\ \alpha=0\}$$

besides $\{p=1,\ \alpha=\frac{n-1}{2}\}$, there exists a function $f\in L^p(T^n)$ such that

$$\overline{\lim_{R\to\infty}}\ |\ S_R^\alpha(f,x)\ | =+\infty \text{ a.e.}$$

I don't know whether or not it is true for any other pair (p,α).

In this note for any Hadamard-lacunary sequence $\{R_k\}$ in the place of $\{R\}$ we consider on the almost everywhere convergence of $S_{R_k}^\alpha(f,x)$ as $k\to\infty$. We shall prove the following.

THEOREM 1. *Let $\{R_k\}_{k=1}^\infty$ be any given lacunary sequence. Then for each pair (p,α) in the region $\{2\leqq p\leqq\infty,\ \alpha=0\}$ in addition to A, we have*

$$\lim_{k\to\infty} S_{R_k}^\alpha(f,x)=f(x) \text{ a.e. for any } f\in L^p(T^n).$$

THEOREM 2. *For each pair (p,α) in the region B, there exist a function $f\in L^p(T^n)$ and a lacunary sequence $\{R_k\}_{k=1}^\infty$ such that*

$$\overline{\lim_{k\to\infty}}\ |\ S_{R_k}^\alpha(f,x)\ | =+\infty \text{ a.e.}$$

2. Proof of Theorem 1.

It is enough to consider in the only case of $p=2$ and $\alpha=0$ and so we write simply $S_R^0(f,x)$ as $S_R(f,x)$. Let $\{R_k\}_{k=1}^\infty$ be any given lacunary sequence with $\frac{R_{k+1}}{R_k}\geqq q>1$ $(k=1,2,\cdots)$.

At first we treat the case of $q\geqq\sqrt{n}$. For $m=(m_1,\cdots,m_n)\in Z^n$ we denote $|m|=[\sum_{j=1}^n |m_j|^2]^{\frac{1}{2}}$ and $\|m\| = \max_{1\leqq j\leqq n} |m_j|$. For each $k=1,2,\cdots$, setting

$$A_k= \{m\in Z^n;\ \ |m| \leqq R_k,\ \|m\| >\frac{1}{\sqrt{n}}R_k\} ,$$

$\{A_k\}_{k=1}^\infty$ is a sequence of pairwise disjoint sets since $R_{k+1}\geqq qR_k\geqq\sqrt{n}R_k$. Now for $f\in L^2(T^n)$ if we write

$$\sigma_{R_k}(f,x)=\sum_{\|m\|\leqq\frac{1}{\sqrt{n}}R_k} \hat{f}_m e^{i(m,x)},$$

then it is a square partial sum of the Fourier series of f and so

$$\lim_{k\to\infty}\sigma_{R_k}(f,x)=f(x) \text{ a.e.}$$

by the result due to C. Fefferman [4]. Furthermore since

$$\sum_{k=1}^\infty\int_{T^n} |\ S_{R_k}(f,x)-\sigma_{R_k}(f,x)\ |^2 dx$$
$$=\sum_{k=1}^\infty\int_{T^n} |\sum_{m\in A_k} \hat{f}_m e^{i(m,x)}\ |^2 dx=(2\pi)^n\sum_{k=1}^\infty\sum_{m\in A_k} |\hat{f}_m|^2$$

$$\leqq (2\pi)^n \sum_{m \in Z^n} |\hat{f}_m|^2 = \|f\|_2^2 < \infty$$

by Parseval's equality, so

$$\lim_{k\to\infty} [S_{R_k}(f,x) - \sigma_{R_k}(f,x)] = 0 \qquad \text{a.e.}$$

Therefore we get

$$\lim_{k\to\infty} S_{R_k}(f,x) = f(x) \qquad \text{a.e.}$$

in the case of $q \geqq \sqrt{n}$.

Next we consider our general case of $q>1$, but our assertion in this case is obtained easily by use of the above case. Let $M = M_q > 1$ be an integer such as $q^M \geqq \sqrt{n}$ and for each $\nu = 1, \cdots, M$ we put $R_k^{(\nu)} = R_{\nu+(k-1)M}$. Then for each $\nu = 1, \cdots, M$, $\{R_k^{(\nu)}\}_{k=1}^\infty$ is a lacunary sequence with $R_{k+1}^{(\nu)} \geqq q^M R_k^{(\nu)}$ for all k, and M-sequences $\{R_k^{(\nu)}\}_{k=1}^\infty$ $(\nu = 1, \cdots, M)$ are pairwise disjoint and $\{R_k\}_{k=1}^\infty = \bigcup_{\nu=1}^{M} \{R_k^{(\nu)}\}_{k=1}^\infty$. Therefore from the preceding result we have for each $\nu = 1, \cdots, M$

$$\lim_{k\to\infty} S_{R_k^{(\nu)}}(f,x) = f(x) \qquad \text{a.e.}$$

and hence

$$\lim_{k\to\infty} S_{R_k}(f,x) = f(x) \qquad \text{a.e.}$$

This proof is suggested from the paper [2] of A.I. Buadze.

3. Proof of Theorem 2.

If the conclusion were not valid, we wolud have for any $f \in L^p(T^n)$ and for any lacunary sequence $\{R_k\}$

$$\overline{\lim_{k\to\infty}} |S_{R_k}^\alpha(f,x)| < \infty$$

on some set of positive measure. Hence from the theorem on the limit of sequences of operators by E.M. Stein, the operator $S_{R_p}^\alpha$ from f to $S_{R_k}^\alpha(f)$ is of weak type (p,p) with a norm independent of k. Moreover since this operator is of strong type (2,2) with a norm independent of k also by Bessel's inequality, therefore we get that it is of strong type (r,r) for any r with $p < r \leqq 2$ by the interpolation theorem of J. Marcinkiewicz.

On the other hand from estimates for the operator norm due to K.I. Babenko [1] we know that if $1 \leqq r < \dfrac{2n}{n+1}$ and $0 \leqq \alpha < \dfrac{n}{r} - \dfrac{n+1}{2}$

$$\|S_R^\alpha\|_{L^r} \geqq CR^{(\frac{n}{r} - \frac{n+1}{2}) - \alpha},$$

and if $\dfrac{2n}{n+1} \leqq r < 2$ and $\alpha = 0$

$$\| S_R^\alpha \|^{L^r} \geqq C\,[\log R]^{(n-1)(\frac{1}{2}-\frac{1}{r})}$$

Hence for r enough close to p

$$\lim_{k\to\infty} \| S_{R_k}^\alpha \|^{L^r} = +\infty$$

and so we get a contradiction.

REFERENCES

〔1〕 K.I. BABENKO, On summability and convergence of eigenfunction expansions of a differential operator, Math. USSR Sbornik, **20**(1973), 157–211.

〔2〕 A.I. BUADZE, On the divergence of the spherical partial sums of multiple Fourier series, Soobšč. Akad. Nauk Gruzin. SSR., **84**(1976), 561–563.

〔3〕 C. FEFFERMAN, The multiplier problem for the ball, Ann. Math., **94**(1971), 330–336.

〔4〕 C. FEFFERMAN, On the convergence of multiple Fourier series, Bull. Amer. Math. Soc., **77**(1971), 744–745.

〔5〕 E.M. STEIN and G. WEISS, *Introduction to Fourier analysis on Euclidean spaces,* Princeton, 1971.