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On the Decomposition Theorem for $(n$-2)-Connected $2n$-Dimensional π-Manifolds

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Abstract. Let M be an $(n$-2)-connected $2n$-dimensional closed π-manifold. H. Ishimoto showed in [1] that under some conditions the manifold M is decomposed as a connected sum of some copies of $S^n\times S^n$ and an $(n$-2)-connected $2n$-dimensional closed π-manifold which has the vanishing n-th homology group. The purpose of this paper is to prove the uniqueness of such decomposition.

1. Introduction

For every simply connected 6-dimensional closed manifold M, there exists a decomposition

$$M=M_0\,\#\,(S^3\times S^3)\,\#\,\cdots\,\#\,(S^3\times S^3)$$

with $\mathrm{rank}H_3(M_0)=0$ (See Theorem 1 in [3]). A.V. Zhubr showed in [4] that such decomposition is unique.

Generally, the following has been shown (See Theorem 3.1 in [1]).

SPLITTING THEOREM (H. Ishimoto). *Let M be an $(n$-2)-connected $2n$-dimensional closed π-manifold $(n\geqq 3)$ such that $H_{n-1}(M)$ has no torsion. We assume that the Arf invariant of M is zero if $n=2^k-1$. Then, there exsists a decomposition*

$$M=M_1\,\#\,(S^n\times S^n)\,\#\,\cdots\,\#\,(S^n\times S^n), \tag{1}$$

where M_1 is an $(n$-2)-connected $2n$-dimensional closed π-manifold such that

$$H_i(M_1)\cong\begin{cases} H_i(M) & \text{if}\quad i=n-1,\ n+1,\\ 0 & \text{if}\quad i=n.\end{cases}$$

In this paper, we prove the following theorem by using Zhubr's method ([4]).

UIQUENESS THEOREM. *In the above splitting theorem, the representation* (1) *is unique in* *the following sence : Let M have another decomposition*

$$M=M_2\,\#\,(S^n\times S^n)\,\#\,\cdots\,\#\,(S^n\times S^n)$$

with $H_n(M_2)=0$. *Then,* $M_1=M_2$ *if* $n\not\equiv 0,1$ (mod 8), *and* $M_1\#\Sigma_1=M_2\#\Sigma_2$ *for some homotopy spheres* Σ_1, $\Sigma_2\in\theta_{2n}$ *if* $n\equiv 0,1$ (mod 8).

Throughout this paper, we shall use "manifold" for compact, connected, and oriented smooth manifold. By $M=N$ we mean that M is diffeomorphic to N by an orientation preserving diffeomorphism. The sign $\#$ denotes the operation of the connected summation, and the connected sum of the manifold M and k-copies of the manifold N will be denoted by $M\ \# kN$.

We would like to express our gratitude to Professor H. Ishimoto for his kind help and advice.

2. Some propositions

The proof of the uniqueness theorem is roughly as follows (Compare[4]): First, we show that there exists a cobordism W between $M_1\#\Sigma_1$ and $M_2\#\Sigma_2$ for some Σ_1, $\Sigma_2\in\theta_{2n}$. The cobordism W will be $(n\text{-}2)$-connected and, if necessary, we can choose W to be parallelizable. Next, we show that W can be converted into an h-cobordism between $M_1\#\Sigma_1$ and $M_2\#\Sigma_2$ by surgeries of index $(n+1)$. This will complete the proof.

The following two propositions assert the above steps.

PROPOSITION 1. *Let* M_1 *and* M_2 *be* $2n$*-dimensional closed* π*-manifolds* $(n\geqq 3)$. *We suppose that* $M_1\#\,k(S^n\times S^n)=M_2\#\,l(S^n\times S^n)$ *for some* k *and* l. *Then there exists a cobordism* W *between* M_1 *and* M_2 *such that the inclusions* i_s: $M_s\to W$ $(s=1,2)$ *induce the following isomorphisms :*

$$(i_s)_*:\ \pi_j(M_s)\cong\pi_j(W)\quad for\ j\leqq n\text{-}2,$$
$$(i_s)_*:\ H_{n-1}(M_s)\cong H_{n-1}(W).$$

If $n\equiv 2,4,5,6$ (mod 8), *then* W *will be parallelizable. If we require* W *to be parallelizable when* $n\equiv 0,1,3,7$ (mod 8), $n\neq 3,7$, *then it suffices to replace* M_1, M_2 *by* $M_1\#\Sigma_1$, $M_2\#\Sigma_2$ *for some* Σ_1, $\Sigma_2\in\theta_{2n}$.

PROPOSITION 2. *Let* M_1 *and* M_2 *be* $(n-2)$*-connected* $2n$*-dimensional closed* π*-manifolds* $(n\geqq 3)$ *with vanishing* n*-th homology groups. Let* W *be an* $(n-2)$*-connected cobordism between* M_1 *and* M_2 *such that the inclusions* i_s: $M_s\to W$ $(s=1,2)$ *induce isomorphisms* $(i_s)_*$: $H_{n-1}(M_s)\cong H_{n-1}(W)$ $(s=1,2)$. *We assume that* W *is parallelizable if* $n\not\equiv 3,5,6,7$ (mod 8). *Then,* W *can be converted into an* h*-cobordism between* M_1 *and* M_2 *by surgeries* *of index* $(n+1)$ (Compare Theorem 2 in [4]).

In order to perform surgeries of index $(n+1)$, we must assume that W is parallelizable if $n \not\equiv 3,5,6,7 \pmod 8$. (When $n \equiv 3,5,6,7 \pmod 8$, we can perform surgeries of index $(n+1)$ without the assumption that W is parallelizable, since $\pi_{n-1}(SO)$ is trivial). And then, we must choose the embeddings $S^n \times D^{n+1} \to W$ so that the modified manifold W_1 is also parallelizable. Therefore, the process is slightly different from that of Zhubr's. However, the assumption of torsion free will make the proof of Proposition 2 simpler than Zhubr's. By such reason, we depend mostly on the technique of Kervaire and Milnor [2, §§ 5–6].

3. Proof of Proposition 1

Since the product $S^n \times S^n$ can be modified into the standard $2n$-sphere S^{2n} by a surgery of index $(n+1)$ with respect to the standard embedding, it follows easily that if $M^{2n} = M_1^{2n} \# (S^n \times S^n)$, then there exists the embedding φ: $S^n \times D^n \to M$ such that $\chi(M, \varphi) = M_1 \# S^{2n} = M_1$. The standard cobordism W between M and $\chi(M, \varphi)$ (i.e. the manifold obtained from the disjoint union $M \times [0,1] \cup D^{n+1} \times D^n$ by identifying $\partial D^{n+1} \times D^n$ with $\varphi(S^n \times D^n) \times \{1\}$) has the homotopy type of M with an $(n+1)$-cell attached. W also has the homotopy type of the one-point union of M_1 and S^n, for it is seen that the dual embedding φ' : $D^{n+1} \times S^{n-1} \to M_1$ is homotopic to a constant map.

If $n \equiv 2,4,5,6 \pmod 8$ then $\pi_n(SO)$ is trivial, so W will be parallelizable. For, it is known that the standard cobordism between M and $M_1 = \chi(M, \varphi)$ has a trivialization F of the tangent bundle τ_W such that $F|_M = f$ if and only if a well defined element

$$\gamma(\varphi) \in \pi_n(SO_{2n+1}) \cong \pi_n(SO)$$

is zero, where f is a fixed trivialization of the stable tangent bundle $\tau_M \oplus \varepsilon_M^1$ (See §6 in [2]).

If we require W to be parallelizable when $n \equiv 0,1,3,7 \pmod 8$, $n \neq 3,7$, we can obtain a parallelizable standard cobordism between M and $M_1 \# \Sigma_1$ for some $\Sigma_1 \in \theta_{2n}$ by the following alteration of the standard embedding φ: $S^n \times D^n \to M$. Let α: $S^n \to SO_n$ be a differentiable map, and define φ_α: $S^n \times D^n \to M$ by $\varphi_\alpha(x, y) = \varphi(x, \alpha(x) \cdot y)$. From Lemma 6.2 in [2], it follows that a map α can be chosen so that $\gamma(\varphi_\alpha) = 0$, provided $n \neq 1,3,7$. (Remark: It is verified that Lemma 6.2 in [2] is also valid for $p = q+1$, if $p \neq 1,3,7$). Therefore the corresponding standard cobordism will be parallelizable. Hence it remains now to show that $\chi(M, \varphi_\alpha) = M_1 \# \Sigma_1$ for some $\Sigma_1 \in \theta_{2n}$. But it suffices to show that the product $S^n \times S^n$ is modified into a homotopy sphere by a surgery with respect to the altered embedding φ_α of the standard embedding φ: $S^n \times D^n \to S^n \times S^n$.

LEMMA 1. *Let* $\varphi_\alpha: S^n\times D^n\to S^n\times S^n$ *be as the above. Then* $N=\chi(S^n\times S^n, \varphi_\alpha)$ *is a homotopy* $2n$*-sphere.*

PROOF. Let $N_0=S^n\times S^n-\text{Int }\varphi_\alpha(S^n\times D^n)$. The image of φ_α coincides with the image of φ. So we obtain

$$N_0 = S^n\times S^n-\text{Int }\varphi(S^n\times D^n) = S^n\times D^n_-\simeq S^n.$$

Therefore N_0 has the homology

$$H_j(N_0)\cong\begin{cases} Z & \text{for } j=0,\ n,\\ 0 & \text{otherwise.}\end{cases}$$

Consider the homology exact sequence of the pair (N, N_0)

$$H_{j+1}(N, N_0)\to H_j(N_0)\to H_j(N)\to H_j(N, N_0)\to H_{j-1}(N_0).$$

By excision,

$$H_j(N, N_0)\cong H_j(\varphi'_\alpha(D^{n+1}\times S^{n-1}), \varphi'_\alpha(S^n\times S^{n-1}))\cong\begin{cases} Z & \text{for } j=n+1,\ 2n,\\ 0 & \text{otherwise,}\end{cases}$$

where $\varphi'_\alpha: D^{n+1}\times S^{n-1}\to N$ is the dual embedding of φ_α. Thus the group $H_j(N)$ is trivial for $1\leqq j\leqq n-1$, $n+2\leqq j\leqq 2n-1$. If $j=n$, $n+1$, we have the sequence

$$0\to H_{n+1}(N)\to H_{n+1}(N, N_0)\xrightarrow{\partial_*} H_n(N_0)\to H_n(N)\to 0.$$

The group $H_{n+1}(N, N_0)$ is generated by the class corresponding to $\varphi'_\alpha(D^{n+1}\times y_0)$ where y_0 is a point of S^{n-1}, and ∂_* maps this generator into the class of $H_n(N_0)$ corresponding to $\varphi'_\alpha(S^n\times y_0)=\varphi_\alpha(S^n\times y_0)$. Since $\varphi_\alpha(S^n\times y_0)$ is homotopic to $S^n\times o$ (in $S^n\times D^n_-$) which represents the generator of $H_n(N_0)$, the map ∂_* is an isomorphism. Therefore, $H_{n+1}(N)$ and $H_n(N)$ are trivial. Thus, N has the homology of the $2n$-sphere. Clearly N is simply connected, so this completes the proof.

Note that the new standard cobordism also has the same homotopy type as the old. Thus we obtain the following lemma.

LEMMA 2. *Let* M *be a* $2n$*-dimensional closed* π*-manifold* $(n\geqq 3)$. *If* $M=M_1\#(S^n\times S^n)$, *then there exists such a cobordism* W *between* M *and* M_1 *that has the homotopy type of the one-point union of* M_1 *and* S^n, *or symmetrically, has the homotopy type of* M *with an* $(n+1)$*-cell attached. If* $n\equiv 2,4,5,6$ (mod 8), *then* W *is parallelizable. If we require* W *to be parallelizable when* $n\equiv 0,1,3,7$ (mod 8), $n\neq 3,7$, *then it suffices to replace* M, M_1 *by* M, $M_1\#\Sigma_1$ *for some* $\Sigma_1\in\theta_{2n}$.

Now, we prove the proposition. It suffices to consider the case that we require W to be parallelizable. Let $M=M_1 \# k(S^n\times S^n)=M_2 \# l(S^n\times S^n)$ and let f be a fixed trivialization of the stable tangent bundle $\tau_M \oplus \varepsilon^1_M$. By Lemma 2, we obtain a cobordism W_1 between M and M_1 (or $M_1 \# \Sigma_1$) which has the framing F_1 of the tangent bundle τ_{W_1} satisfying $F_1|_M=f$, and also obtain a cobordism W_2 between M and M_2 (or $M_2 \# \Sigma_2$) which has the framing F_2 of the tangent bundle τ_{W_2} satisfying $F_2|_M=f$. By pasting together $(-W_1)$ and W_2 along M, we have a cobordism W between M_1 and M_2 (or $M_1 \# \Sigma_1$ and $M_2 \# \Sigma_2$), and W acquires the framing F of the tangent bundle τ_W such that $F|_{W_1}=F_1$ and $F|_{W_2}=F_2$. It is clear that W has the homotopy type of

$$M_1 \vee \underbrace{S^n \vee \cdots \vee S^n}_{k} \cup \underbrace{e^{n+1} \cup \cdots \cup e^{n+1}}_{l},$$

or

$$M_2 \vee \underbrace{S^n \vee \cdots \vee S^n}_{l} \cup \underbrace{e^{n+1} \cup \cdots \cup e^{n+1}}_{k},$$

so, the isomorphisms of the proposition are obtained. This completes the proof of Proposition 1.

4. Proof of Proposition 2

Let M_1, M_2 and W be the manifolds which satisfy the conditions of Proposition 2. Using the homology exact sequence of the pair (W, M_s) $(s=1,2)$, we obtain the following:

$$(2) \qquad H_j(W, M_s) \cong \begin{cases} 0 & \text{if } j \leqq n-1 \text{ or } j \geqq n+2, \\ H_n(W) & \text{if } j=n, \\ H_n(W)/\mathrm{Tor}\, H_n(W) & \text{if } j=n+1. \end{cases}$$

The last isomorph is shown by the Poincaré duality theorem and the universal coefficient theorem. In fact,

$$\begin{aligned} H_{n+1}(W, M_1) &\cong H^n(W, W_2) \cong \mathrm{Hom}\,(H_n(W, M_2), Z) \\ &\cong \mathrm{Hom}\,(H_n(W), Z) \cong H_n(W)/\mathrm{Tor}\, H_n(W). \end{aligned}$$

By replacing M_1 with M_2, we also have $H_{n+1}(W, M_2) \cong H_n(W)/\mathrm{Tor}\, H_n(W)$.

Let φ: $S^n\times D^{n+1} \to W$ be an embedding. Since a surgery of index $(n+1)$ does not change the i-th homotopy group of W for $i\leqq n-1$, and therefore, the $(n-1)$-th homology group of W, the resulting cobordism W_φ satisfies the conditions of the proposition also. Hence, the new pair (W_φ, M_s) has such homology as (2). So, if we can kill the group $H_n(W)$ completely by surgeries of index $(n+1)$, then W can be converted into an h-cobordism.

We note that every class of $H_n(W)$ can be represented by an embedding $S^n \times D^{n+1} \to W$ by assumption, and if W is parallelizable then the embedding can be chosen within its homotopy class so that the modified manifold will also be parallelizable (See, Lemma 5.3 and Lemma 5.4 in [2]).

By the similar argument of Lemma 5.6 and Lemma 5.7 in [2], a generator λ in the infinite cyclic summands of $H_n(W)$ can be killed by a surgery of index $(n+1)$, where λ is primitive since $H_n(\partial W)=H_n(M_1 \cup M_2)$ is trivial and so $j_*: H_{n+1}(W) \to H_{n+1}(W, \partial W)$ is surjective. Therefore we can make the group $H_n(W)$ finite without changing the order of the torsion part by a sequence of surgeries of index $(n+1)$. Henceforth, we assume that $H_n(W)$ is finite.

Let us specialize to the case that n is even. In this case, we will apply the technique of Theorem 5.1, for k even, in [2]. If Lemma 5.8 in [2] is applicable to our manifold W, then the proof is completely analogous. Hence, it suffices to show the following.

LEMMA 3. *If n is even, then any surgery of index $(n+1)$ necessarily changes the n-th Betti number of W.*

PROOF. We use the technique of Zhubr (Lemma 2.5 in [4]). By X we denote the double of W and by X_φ we denote the closed manifold obtained by pasting together the manifolds W and W_φ along $M_1 \cup M_2$. Note that X_φ is obtained from X by a surgery of index $(n+1)$ with respect to the embedding $\varphi: S^n \times D^{n+1} \to W \subset X$. We use homology with rational coefficients. Consider the Mayer-Vietoris homology exact sequence of the triad (X, W, W)

$$0 = H_n(M_1 \cup M_2) \to H_n(W) \oplus H_n(W) \to H_n(X) \to H_{n-1}(M_1 \cup M_2) \to \cdots \to H_{n-2}(M_1 \cup M_2) = 0.$$

From its exactness follows the equality

$$2\dim H_n(W) = \dim H_n(X) - \dim H_{n-1}(M_1 \cup M_2) + 2\dim H_{n-1}(W) - \dim H_{n-1}(X).$$

Since $H_{n-1}(M_s) \cong H_{n-1}(W)$, we obtain

$$2\dim H_n(W) = \dim H_n(X) - \dim H_{n-1}(X). \tag{3}$$

Similarly, from the exact sequence of the triad $(X_\varphi, W, W_\varphi)$, we have

$$\begin{aligned}\dim H_n(W) + \dim H_n(W_\varphi) = \dim H_n(X_\varphi) &- \dim H_{n-1}(M_1 \cup M_2) + \dim H_{n-1}(W) \\ &+ \dim H_{n-1}(W_\varphi) - \dim H_{n-1}(X_\varphi).\end{aligned}$$

Since a surgery of index $(n+1)$ does not change the $(n-1)$-th homology groups of W and X, we obtain the equality

$$(4)\qquad \dim H_n(W)+\dim H_n(W_\varphi) = \dim H_n(X_\varphi) - \dim H_{n-1}(X).$$

Hence, by (3) and (4),

$$(5)\qquad \dim H_n(W)-\dim H_n(W_\varphi) = \dim H_n(X)-\dim H_n(X_\varphi).$$

Here, X is a closed manifold. Therefore the similar argument of Lemma 5.8 in [2] is applicable, where note that since $H_i(X) \cong H_i(X_\varphi)$ for $i \leqq n-1$,

$$e^*(X)-e^*(X_\varphi) \equiv \dim H_n(X)-\dim H_n(X_\varphi) \pmod 2$$

(e^* denotes the semi-characteristic). Thus, $\dim H_n(X) \neq \dim H_n(X_\varphi)$, and so $\dim H_n(W) \neq \dim H_n(W_\varphi)$. This completes the proof.

Now consider the case that n is odd. Let $j_s\colon H_n(W)\to H_n(W, M_s)$ be a standard homomorphism ($s=1,2$). From the exact sequence

$$0 = H_n(M_s)\to H_n(W)\xrightarrow{j_s} H_n(W, M_s)\to H_{n-1}(M_s)\xrightarrow[\cong]{(i_s)_*} H_{n-1}(W),$$

it follows that j_s is a isomorphism ($s=1,2$). Let

$$L : \mathrm{Tor}\, H_n(W, M_1)\otimes \mathrm{Tor}\, H_n(W, M_2)\to Q/Z$$

be the linking pairing. Define the pairing

$$L' : \mathrm{Tor}\, H_n(W)\otimes \mathrm{Tor}\, H_n(W)\to Q/Z$$

by the formula $L'(\eta, \xi)=L(j_1(\eta), j_2(\xi))$, where $\eta, \xi \in \mathrm{Tor}\, H_n(W)$. Since n is odd and $j_s(s=1, 2)$ are isomorphisms, L' is symmetric and non-singular. We assumed that $H_n(W)$ is finite. Note that Lemma 6.3 and Lemma 6.4 in [2] are also applicable for W. So, if $L'(\lambda, \lambda)\neq 0$ for some $\lambda \in H_n(W)$ the group $H_n(W)$ can be simplified by a surgery of index $(n+1)$, that is, we can obtain the manifold W_1 such that $\mathrm{Ord}\, H_n(W_1)<\mathrm{Ord}\, H_n(W)$. Therefore we may assume that $L'(\lambda, \lambda)=0$ for every $\lambda \in H_n(W)$. According to Lemma 6.5 in [2] (or Lemma 3.6 in [4]), all the elements of such group are of order 2 and hence $H_n(W)$ must be a direct sum of cyclic groups of order 2.

LEMMA 4. *If the embedding φ represents the non-zero class $\lambda \in H_n W$ then* $\dim H_n(W; Z_2)\neq \dim H_n(W_\varphi; Z_2)$.

PROOF. Define the manifolds $X=W\cup W$ and $X_\varphi = W\cup W_\varphi$ similarly as before. Since X is closed, it follows from Lemma 2.1 in [4] that $\dim H_n(X; Z_2)\neq \dim H_n(X_\varphi; Z_2)$. So, by applying the equality(5) in Z_2-coefficient, it follows that $\dim H_n(W; Z_2)\neq \dim H_n(W_\varphi; Z_2)$.

We note that $H_{n-1}(W)$ remains always torsion free since $H_{n-1}(W) \cong H_{n-1}(M_s)$ by assumption. Thus the proof of the proposition when n is odd is completely analogous to that of Theorem 5.1, for k odd, in [2].

This completes the proof of Proposition 2.

5. Proof of Uniqueness Theorem

We suppose that M has two representations $M_1 \# k(S^n \times S^n)$ and $M_2 \# l(S^n \times S^n)$. By Proposition 1, there exists a parallelizable cobordism W between M_1 and M_2 if $n \equiv 2,4,5,6 \pmod 8$ and between $M_1 \# \Sigma_1$ and $M_2 \# \Sigma_2$ for some $\Sigma_1, \Sigma_2 \in \theta_{2n}$ if $n \equiv 0,1,3,7$ (mod 8), $n \neq 3,7$. W is $(n-2)$-connected and satisfies the homology condition of Proposition 2. Hence, by Proposition 2, W is converted into an h-cobordism. But, when $n \equiv 3,7 \pmod 8$ we need not require W to be parallelizable, therefore Σ_1 and Σ_2 may be the standard $2n$-sphere. So, if $n \not\equiv 0,1 \pmod 8$, then $M_1 = M_2$, and if $n \equiv 0,1 \pmod 8$, $M_1 \# \Sigma_1 = M_2 \# \Sigma_2$ for some $\Sigma_1, \Sigma_2 \in \theta_{2n}$. This completes the proof.

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