Sci. Rep. Kanazawa Univ.
Vol. 23, No. 2, pp. 69—75
December 1978

On the Continuability of Holomorphic Functions with Real Parameters

Mitsuru HARITA and Eiichi SAKAI

Department of Mathematics, Faculty of Science, Kanazawa University

(Received October 31, 1978)

To the memory of Professor Kiyoshi Oka

Abstract. The purpose of this note is to generalize the Hartogs-Osgood's theorem in C^n for the case of (complex and real) mixed several variables.

§1. Introduction.

The famous Hartogs-Osgood's theorem states that if K is a relatively compact open set in the space C^n $(n \geqq 2)$ of n complex variables $z_1, \cdots, z_n$ and the boundary ∂K is connected, then every function f holomorphic in a neighborhood of ∂K can be continued holomorphically to the whole K [6, 8]. F. Severi [9] and G. Fubini [4] gave the corresponding result for the case of (complex and real) mixed several variables*). But their proofs were imcomplete and in 1936 A. B. Brown completed the proof in the case where all variables are complex [3]. After that, various proofs were given by H. Fujimoto-K. Kasahara [5], H. B. Laufer [7] and etc.

In this paper, we shall prove a similar theorem as above for case of holomorphic functions in the space C^m $(m \geqq 1)$ with parameters in the space R^n of n real variables $u_1, \cdots, u_n$ $(n \geqq 1)$. Our proof is mainly due to H. B. Laufer [7] and J. Siciak [10, 11].

The authors are very grateful to Professor J. Kajiwara for his fruitful discussions and valuable suggestions.

§2. Holomorphic functions with real parameters.

We shall discuss the holomorphic continuation of functions as follows.

DEFINITION 1. Let D be an open set in $C^m \times R^n$ $(m, n \geqq 1)$ and $\mathfrak{F}(D)$ denote the family of all the functions f defined in D satisfying following conditions: for each point $(z^0, u^0) = (z_1^0, \cdots, z_m^0, u_1^0, \cdots, u_n^0) \in D$ there exists a complex neighborhood

*) A function f of mixed several variables is said to be *holomorphic* if f is locally expanded by the Taylor series.

$$C(z^0, u^0) = \{(z, w) \in C^{m+n}; |z_j - z_j^0| < r_j, |w_k - u_k^0| < s_k, \\ j=1, 2, \cdots, m, k=1, 2, \cdots, n\}$$

such that for each fixed point $(\zeta, \xi) \in C(z^0, u^0) \cap D$

(i) $f(\zeta_1, \cdots, \zeta_{j-1}, z_j, \zeta_{j+1}, \cdots, \zeta_m, \xi_1, \cdots, \xi_n)$ is a holomorphic function of z_j in the disc $\{|z_j - z_j^0| < r_j\}$ $(j=1, 2, \cdots, m)$,

(ii) $f(\zeta_1, \cdots, \zeta_m, \xi_1, \cdots, \xi_{k-1}, u_k, \xi_{k+1}, \cdots, \xi_n)$ is continuable to a holomorphic function of w_k in the disc $\{|w_k - u_k^0| < s_k\}$ $(k=1, 2, \cdots, n)$.

The complex neighborhood $C(z^0, u^0)$ may depend on (z^0, u^0) and on f. If K is a closed set in $C^m \times R^n$ and a function f satisfies the above conditions in some open neighborhood U of K, we shall write simply $f \in \mathfrak{F}(K)$. Every function $f \in \mathfrak{F}(D)$ is of course real analytic with respect to each variable u_k separately. The functions of $\mathfrak{F}(D)$ appeared usefully in the study of the following topics: analyticity of distribution kernels, the edge of the wedge theorem, bounded representation of the classical Lie groups on Hilbert space, Feynman integrals, analyticity of solutions for partial differential equations (see [2]).

J. Siciak [10] proved the following theorem which is very useful for our problem.

THEOREM 1. *Let D_k be an open set in the complex z_k-plane and E_k be a compact line interval contained in D_k $(k=1, 2, \cdots, n)$. Assume that $f(z)=f(z_1, \cdots, z_n)$ is defined in*

$$X=(D_1 \times E_2 \times \cdots \times E_n) \cup (E_1 \times D_2 \times E_3 \times \cdots \times E_n) \cup \cdots \cup (E_1 \times \cdots \times E_{n-1} \times D_n)$$

and satisfies that for every fixed $(z_1^0, \cdots, z_{k-1}^0, z_{k+1}^0, \cdots, z_n^0) \in E_1 \times \cdots \times E_{k-1} \times E_{k+1} \times \cdots \times E_n$ the function f is holomorphic with respect to z_k in D_k $(k=1, 2, \cdots, n)$. Then there exist an open set Ω in C^n containing $E_1 \times \cdots \times E_n$ and a function $\tilde{f}$ holomorphic in Ω such that $\tilde{f}=f$ in $\Omega \cap X$.

§3. Cohomology vanishing lemma.

In this section we shall prove some lemmas in preparation for our main theorem. We identify $C^m \times R^n$ with R^{2m+n} and put $z_j = x_{2j-1} + ix_{2j}$, $j=1, 2, \cdots, m$.

LEMMA 1. *Let B_1 and B_2 be closed cubes in $C^m \times R^n$, that is,*

$$B_1 = \{(z, u)=(x_1, \cdots, x_{2m}, u_1, \cdots, u_n) \in C^m \times R^n; a_j \leqq x_j \leqq b_j, a_{2m+k} \leqq u_k \leqq b_{2m+k}, \\ j=1, 2, \cdots, 2m, k=1, 2, \cdots, n\}$$

and

$$B_2 = \{(z, u) \in C^m \times R^n; a_j' \leqq x_j \leqq b_j', a_{2m+k}' \leqq u_k \leqq b_{2m+k}', \\ j=1, 2, \cdots, 2m, k=1, 2, \cdots, n\}.$$

Assume that $a_{l_0} < a'_{l_0} < b_{l_0} < b'_{l_0}$ *for some integer* l_0 $(1 \leqq l_0 \leqq 2m+n)$ *and* $a_l = a'_l$, $b_l = b'_l$ *for all* $l \neq l_0$ $(1 \leqq l \leqq 2m+n)$. *Then for every* $f \in \mathfrak{F}(B_1 \cap B_2)$ *there exist* $f_i \in \mathfrak{F}(B_i)$, $i=1, 2$, *such that* $f = f_1 - f_2$ *in a neighborhood of* $B_1 \cap B_2$.

PROOF. By Theorem 1, $f \in \mathfrak{F}(B_1 \cap B_2)$ can be continued to a holomorphic function in an open neighborhood U of cube

$$\{(z, w) \in C^{m+n};\ a_j \leqq x_j \leqq b_j,\ a_{2m+k} \leqq u_k \leqq b_{2m+k},\ |v_k| \leqq \varepsilon,\ w_k = u_k + iv_k,\ j=1, 2, \cdots, 2m,\ k=1, 2, \cdots, n\}$$
$$\cap\ \{(z, w) \in C^{m+n};\ a'_j \leqq x_j \leqq b'_j,\ a'_{2m+k} \leqq u_k \leqq b'_{2m+k},\ |v_k| \leqq \varepsilon,\ w_k = u_k + iv_k,\ j=1, 2, \cdots, 2m,\ k=1, 2, \cdots, n\},$$

where ε is a sufficiently small positive number.

When $1 \leqq l_0 \leqq 2m$ and $j_0 = [\frac{l_0+1}{2}]$, there exists a closed curve Γ in z_{j_0}-plane such that

$$\{(z, w) \in C^{m+n};\ z_{j_0} \in \Gamma,\ a_j \leqq x_j \leqq b_j,\ a_{2m+k} \leqq u_k \leqq b_{2m+k},\ |v_k| \leqq \varepsilon,\ w_k = u_k + iv_k,\ j=1, 2, \cdots, 2j_0-2, 2j_0+1, \cdots, 2m,\ k=1, 2, \cdots, n\}$$

is contained in U.

We put

$$\Gamma_1 = \Gamma \cap \{z_{j_0} \in C;\ x_{l_0} \geqq \frac{b_{l_0} + a'_{l_0}}{2}\},$$

$$\Gamma_2 = \Gamma \cap \{z_{j_0} \in C;\ x_{l_0} \leqq \frac{b_{l_0} + a'_{l_0}}{2}\},$$

$$f_1(z_1, z_2, \cdots, u_n) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(z_1, \cdots, z_{j_0-1}, \zeta, z_{j_0+1}, \cdots, u_n)}{\zeta - z_{j_0}}\, d\zeta$$

for $(z, u) \in B_1$ and

$$f_2(z_1, z_2, \cdots, u_n) = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(z_1, \cdots, z_{j_0-1}, \zeta, z_{j_0+1}, \cdots, u_n)}{\zeta - z_{j_0}}\, d\zeta$$

for $(z, u) \in B_2$. Then f_1 and f_2 answer the requirement of the lemma. In the same way, this lemma can be also proved when $2m+1 \leqq l_0 \leqq 2m+n$.

LEMMA 2. *Let* $l_1, l_2, \cdots, l_{2m+n}$ *be any natural numbers and* $a_1, \cdots, a_{2m+n}, b_1, \cdots, b_{2m+n}$ *be any real numbers satisfying* $a_1 < b_1, \cdots, a_{2m+n} < b_{2m+n}$. *We put*

$$\delta_j = \frac{b_j - a_j}{2l_j + 1} \qquad (j=1, 2, \cdots, 2m+n)$$

and

$$U_\nu = \{(z, u) \in C^m \times R^n;\ a_j + 2\delta_j\nu_j \leqq x_j \leqq a_j + \delta_j(2\nu_j + 3),\ a_{2m+k} + 2\delta_{2m+k}\nu_{2m+k} \leqq u_k \leqq a_{2m+k} + \delta_{2m+k}(2\nu_{2m+k} + 3),\ j=1, 2, \cdots, 2m,\ k=1, 2, \cdots, n\}$$

for $\nu=(\nu_1, \nu_2, \dots, \nu_{2m+n})$ *where* ν_j *is an integer satisfying* $0 \leqq \nu_j \leqq l_j - 1$, $j=1, 2, \cdots, 2m+n$. *Assume that* $g_{\nu\mu} \in \mathfrak{F}(U_\nu \cap U_\mu)$ *satisfies* $g_{\nu\mu}+g_{\mu\nu}=0$ *in* $U_\nu \cap U_\mu$ *and* $g_{\nu\mu}+g_{\mu\lambda}+g_{\lambda\nu}=0$ *in* $U_\nu \cap U_\mu \cap U_\lambda$. *Then there exists* $g_\nu \in \mathfrak{F}(U_\nu)$ *such that* $g_{\nu\mu}=g_\nu - g_\mu$ *in* $U_\nu \cap U_\mu$.

PROOF. We prove the lemma by induction for the number s of the cubes U_ν, that is, $s=l_1 l_2 \cdots l_{2m+n}$. When $s=2$, it is true by Lemma 1. We assume that it is proved for $l_1 l_2 \cdots l_{2m+n} \leqq s-1$. Now without loss of generality we may assume that $l_1 \geqq 2$. When $l_1 l_2 \cdots l_{2m+n}=s$, the numbers of $\{U_\nu;\ \nu_1=0\}$ and $\{U_\nu;\ \nu_1 \geqq 1\}$ are smaller than s respectively, and therefore there exist $\{g'_\nu\}$ and $\{g''_\nu\}$ such that

$$g_{\nu\mu}=g'_\nu-g'_\mu \quad \text{in } U_\nu \cap U_\mu \quad \text{where } \nu_1=\mu_1=0$$

and

$$g_{\nu\mu}=g''_\nu-g''_\mu \quad \text{in } U_\nu \cap U_\mu \quad \text{where } \nu_1 \geqq 1 \text{ and } \mu_1 \geqq 1.$$

We put $\nu'=(\nu_2, \dots, \nu_{2m+n})$ and $\mu'=(\mu_2, \dots, \mu_{2m+n})$ briefly and

$$g=g'_{(0,\nu')}+g_{(1,\nu')(0,\nu')}-g''_{(1,\nu')}$$

in $U_{(0,\nu')} \cap U_{(1,\nu')}$ for every ν', then

$$\begin{aligned}&(g'_{(0,\nu')}+g_{(1,\nu')(0,\nu')}-g''_{(1,\nu')})-(g'_{(0,\mu')}+g_{(1,\mu')(0,\mu')}-g''_{(1,\mu')})\\&=g_{(0,\nu')(0,\mu')}+g_{(1,\nu')(0,\nu')}+g_{(1,\mu')(1,\nu')}+g_{(0,\mu')(1,\mu')}\\&=0\end{aligned}$$

in $U_{(0,\nu')} \cap U_{(1,\nu')} \cap U_{(0,\mu')} \cap U_{(1,\mu')}$. Therefore g is a well-defined function in

$$\begin{aligned}\Delta&=\bigcup_{\nu'}(U_{(0,\nu')} \cap U_{(1,\nu')})\\&=\{(z,u) \in C^m \times R^n;\ a_1+2\delta_1 \leqq x_1 \leqq a_1+3\delta_1,\ a_2 \leqq x_2 \leqq b_2, \cdots,\\&\qquad a_{2m+n} \leqq u_n \leqq b_{2m+n}\}.\end{aligned}$$

By Lemma 1, there exist $g_i \in \mathfrak{F}(\Delta_i)$, $i=1, 2$, such that $g=g_1-g_2$ in Δ where

$$\Delta_1=\{(z,u) \in C^m \times R^n;\ a_1 \leqq x_1 \leqq a_1+3\delta_1,\ a_2 \leqq x_2 \leqq b_2, \cdots, a_{2m+n} \leqq u_n \leqq b_{2m+n}\}$$

and

$$\Delta_2=\{(z,u) \in C^m \times R^n;\ a_1+2\delta_1 \leqq x_1 \leqq b_1,\ a_2 \leqq x_2 \leqq b_2, \cdots, a_{2m+n} \leqq u_n \leqq b_{2m+n}\}.$$

We put

$$g_{(0,\nu_2,\dots,\nu_{2m+n})}=g'_{(0,\nu_2,\dots,\nu_{2m+n})}-g_1$$

and

$$g_{(\nu_1,\nu_2,\dots,\nu_{2m+n})}=g''_{(\nu_1,\nu_2,\dots,\nu_{2m+n})}-g_2 \quad \text{for } \nu_1\geqq 1,$$

then these functions satisfy the requirement of the lemma.

LEMMA 3. *Let $\mathcal{F}$ be the sheaf of all the germs of functions of $\mathfrak{F}(D)$ defined in Definition 1 where D is an open cube in $C^m\times R^n$, that* is,

$$D=\{(z,u)\in C^m\times R^n;\ a_j<x_j<b_j,\ a_{2m+k}<u_k<b_{2m+k},\ j=1,2,\cdots,2m,\ k=1,2,\cdots,n\}$$

where a_j and b_j are any real numbers ($j=1,2,\cdots,2m+n$). Then the first cohomology group of D with coefficients in the sheaf $\mathcal{F}$ vanishes, that is,

$$H^1(D,\mathcal{F})=0.$$

PROOF. Let $\{V_\alpha;\ \alpha\in A\}$ be any open covering of D and let

$$K(\lambda)=\{(z,u)\in C^m\times R^n;\ a_j+\frac{1}{\lambda}\leqq x_j\leqq b_j-\frac{1}{\lambda},\ a_{2m+k}+\frac{1}{\lambda}\leqq u_k\leqq b_{2m+k}-\frac{1}{\lambda},\ j=1,2,\cdots,2m,\ k=1,2,\cdots,n\}.$$

We put $V_\alpha^\lambda=V_\alpha\cap K(\lambda)$,

$$\delta_j=\frac{(b_j-\frac{1}{\lambda})-(a_j+\frac{1}{\lambda})}{2l_j+1}$$

for natural numbers l_j ($j=1,2,\cdots,2m+n$) and

$$U_\nu=\{(z,u)\in C^m\times R^n;\ a_j+\frac{1}{\lambda}+2\delta_j\nu_j\leqq x_j\leqq a_j+\frac{1}{\lambda}+\delta_j(2\nu_j+3),$$
$$a_{2m+k}+\frac{1}{\lambda}+2\delta_{2m+k}\nu_{2m+k}\leqq u_k\leqq a_{2m+k}+\frac{1}{\lambda}+\delta_{2m+k}(2\nu_{2m+k}+3),$$
$$j=1,2,\cdots,2m,\ k=1,2,\cdots,n\}$$

for $\nu=(\nu_1,\nu_2,\dots,\nu_{2m+n})$ where $0\leqq\nu_j\leqq l_j-1$ ($j=1,2,\cdots,2m+n$). We can choose the natural numbers $l_1,l_2,\cdots,l_{2m+n}$ such that $\{U_\nu\}$ is a refinement of $\{V_\alpha^\lambda;\ \alpha\in A\}$, that is, for any $\nu=(\nu_1,\nu_2,\dots,\nu_{2m+n})$ there exists $\alpha=\alpha(\nu)\in A$ such that $U_\nu\subset V_{\alpha(\nu)}^\lambda$.

Let us consider a pair $\{g_{\alpha\beta},\ V_\alpha\cap V_\beta\}$ such that

$$g_{\alpha\beta}+g_{\beta\alpha}=0 \qquad \text{in } V_\alpha\cap V_\beta$$

and

$$g_{\alpha\beta}+g_{\beta\gamma}+g_{\gamma\alpha}=0 \qquad \text{in } V_\alpha\cap V_\beta\cap V_\gamma.$$

We put $g^\lambda_{\alpha\beta}=g_{\alpha\beta} \mid V^\lambda_\alpha \cap V^\lambda_\beta$ and $h_{\nu\mu}=g^\lambda_{\alpha(\nu)\alpha(\mu)}$. Then, by Lemma 2, there exists $\{h_\nu \in \mathfrak{F}(U_\nu)\}$ such that $h_{\nu\mu}=h_\nu-h_\mu$ in $U_\nu\cap U_\mu$.

Let

$$g^\lambda_\alpha(z, u)=h_\nu(z, u)-g^\lambda_{\alpha(\nu)\alpha}(z, u)$$

for a point $(z, u)\in U_\nu\cap V^\lambda_\alpha$, then g^λ_α is a well-defined function in V^λ_α. Because, for a point $(z, u)\in U_\nu\cap U_\mu\cap V^\lambda_\alpha$,

$$\begin{aligned}&(h_\nu-g^\lambda_{\alpha(\nu)\alpha})-(h_\mu-g^\lambda_{\alpha(\mu)\alpha})\\&=h_{\nu\mu}-g^\lambda_{\alpha(\nu)\alpha(\mu)}=0.\end{aligned}$$

Moreover,

$$g^\lambda_\alpha-g^\lambda_\beta=-g^\lambda_{\alpha(\nu)\alpha}+g^\lambda_{\alpha(\nu)\beta}=g^\lambda_{\alpha\beta} \quad \text{in } U_\nu\cap V^\lambda_\alpha\cap V^\lambda_\beta \text{ for every } \nu$$

and

$$(g^{\lambda+1}_\alpha-g^\lambda_\alpha)-(g^{\lambda+1}_\beta-g^\lambda_\beta)=0 \quad \text{in } V^\lambda_\alpha\cap V^\lambda_\beta.$$

We put $f^\lambda=g^{\lambda+1}_\alpha-g^\lambda_\alpha$, then f^λ is a well-defined function in a neighborhood of $K(\lambda)$ and by Theorem 1 f^λ can be continued to a holomorphic function in a neighborhood U^λ of $K(\lambda)$ in C^{m+n}. Hence there exists a polynomial P^λ such that $|f^\lambda(z, w)-P^\lambda(z, w)|<\frac{1}{\lambda^2}$ for every point $(z, w)\in U'$ where U' is some open neighborhood of $K(\lambda)$ in C^{m+n} so that $U'\Subset U^\lambda$. If we put

$$g_\alpha(z, u)=\lim_{\lambda\to\infty}(g^\lambda_\alpha(z, u)-\sum_{k=1}^{\lambda-1}P^k(z, u)),$$

then $\{g_\alpha; \alpha\in A\}$ satisfies the relation $g_{\alpha\beta}=g_\alpha-g_\beta$ in $V_\alpha\cap V_\beta$ and $g_\alpha\in\mathfrak{F}(V_\alpha)$.

§4. Hartogs-Osgood's Theorem.

In this section we shall prove the Hartogs-Osgood's Theorem for holomorphic functions with real parameters. To prove our theorem, we need one more lemma which is well-known in the case of C^n $(n\geqq 2)$.

LEMMA 4. *Let* $m, n\geqq 1$,

$$B(R)=\{(z, u)\in C^m\times R^n; |z_1|^2+\cdots+|z_m|^2+u_1^2+\cdots+u_n^2<R^2\}$$

and

$$B(R, r)=\{(z, u)\in C^m\times R^n; r^2<|z_1|^2+\cdots+|z_m|^2+u_1^2+\cdots+u_n^2<R^2\}.$$

Then every function $f\in\mathfrak{F}(B(R, r))$ *is continued to* $B(R)$, *that is, there exists* $\tilde{f}\in\mathfrak{F}(B(R))$

such that $\tilde{f} \mid B(R, r) = f$.

This lemma can be proved by using Cauchy's integral formula and the theorem of identity in the same way as in F. Severi [9] or S. Bochner-T. W. Martin [1].

THEOREM 2. *Let* K *be a compact set in* $C^m \times R^n$ $(m, n \geqq 1)$, *the boundary* ∂K *be connected and* U *be an open neighborhood of* ∂K. *Then, for every* $f \in \mathfrak{F}(U)$ *there exists a function* $\tilde{f} \in \mathfrak{F}(U \cup K)$ *such that* $\tilde{f} \mid U = f$.

PROOF. For the compact set K there exist a ball B and a cube D in $C^m \times R^n$ such that $K \subset B \Subset D$. We put $V_1 = (D \cap K^c) \cup U$ and $V_2 = K \cup U$, then $\{V_1, V_2\}$ is an open covering of D and by Lemma 3 there exist $f_1 \in \mathfrak{F}(V_1)$ and $f_2 \in \mathfrak{F}(V_2)$ such that $f = f_1 - f_2$ in U. By Lemma 4, f_1 can be continued to B and therefore f is continuable to $K \cup U$.

REMARK. Needless to say, every function $f \in \mathfrak{F}(D)$ is holomorphic*) in D, where D is an open set in $C^m \times R^n$ (see [10], Theorem 4).

References

[1] S. Bochner and W. T. Martin, *Several complex variables,* Princeton, 1948.

[2] F. E. Browder, On the "edge of the wedge" theorem, Canad. J. Math., 15(1963), 125–131.

[3] A. B. Brown, On certain analytic continuations and analytic homeomorphisms, Duke Math. J., 2(1936), 20–28.

[4] G. Fubini, Un teorema sulle equazioni alle derivate parziali di tipo ellittico che generalizza un teorema dell'Hartogs ed uno del Severi, Atti Accad. Naz. Lincei, Rend., Ser. 6, 15(1932), 499–501.

[5] H. Fujimoto and K. Kasahara, On the continuability of holomorphic functions on complex manifolds, J. Math. Soc. Japan, 16(1964), 183–213.

[6] F. Hartogs, Einige Folgerungen aus der Cauchyschen Integralformel bei Funktionen mehrerer Veränderlichen, Münch. Sitzungsber., 36(1906), 223–242.

[7] H. B. Laufer, Some remarks about a theorem of Hartogs, Proc. Amer. Math. Soc., 17(1966), 1244–1249.

[8] W. F. Osgood, *Lehrbuch der Funktionentheorie*, II. 1, 2nd edition, Teubner, Leibzig, 1929.

[9] F. Severi, Una proprietà fondamentale dei campi di olomorfismo di una funzione analitica di una variabile reale e di una variabile complessa, Atti Accad. Naz. Lincei, Rend., Ser. 6, 15(1932), 487–490.

[10] J. Siciak, Analyticity and separate analyticity of functions defined on lower dimensional subsets of C^n, Zeszyty Nauk. UJ, 13(1969), 53–70.

[11] J. Siciak, Separately analytic functions and envelopes of holomorphy of some lower dimensional subsets of C^n, Ann. Polon. Math., 22(1969), 145–171.

*) See footnote at p. 69.