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A Note on the Functional Central Limit Theorem for Lacunary Walsh Series

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Abstract. Let X_n ($n=1, 2, \cdots$) be random functions defined by partial sums of lacunary Walsh series with gaps $n_{k+1}/n_k > 1+ck^{-\alpha}$ ($c>0$, $0 \leqq \alpha \leqq 1/2$). Then we shall show that in $(C, \mathscr{C})$, X_n converges in distribution to the Wiener process W.

1. Introduction.

Let $C=C[0, 1]$ and let $\mathscr{C}$ be the σ-field generated by all open sets of C. The underlying probability space on which we shall discuss is $(\Omega, \mathscr{B}, P)$ where $\Omega=[0, 1]$, $\mathscr{B}$ is the class of all Borel subsets of Ω and P is the Lebesgue measure on $\mathscr{B}$. The following system $\{r_n\}$ is called the Rademacher system:

$$r_0(\omega)=1 \quad \text{if } 0 \leqq \omega < 1/2, \qquad r_0(\omega)=-1 \quad \text{if } 1/2 \leqq \omega < 1,$$
$$r_0(\omega+1)=r_0(\omega) \quad \text{and} \quad r_n(\omega)=r_0(2^n\omega) \quad (n=1, 2, \cdots).$$

Then the Walsh system $\{w_n\}$ is defined by

$$w_0(\omega)=1 \qquad \text{and} \qquad w_n(\omega)=r_{n_1}(\omega)r_{n_2}(\omega)\cdots r_{n_k}(\omega),$$

where n_i's are uniquely determined integers by $n_{i+1} < n_i$ and $n=2^{n_1}+2^{n_2}+\cdots+2^{n_k}$.

In the following let $\{n_k\}$ be a sequence of positive integers such that

$$\text{(1. 1)} \qquad n_{k+1}/n_k > 1+ck^{-\alpha} \quad (k=1, 2, \cdots),$$

where $c>0$ and $0 \leqq \alpha \leqq 1/2$ and let $\{a_k\}$ be a sequence of real numbers satisfying

$$\text{(1. 2)} \qquad A_0=0,\ A_n=\Big(\sum_{k=1}^{n} a_k^2\Big)^{1/2} \to \infty \quad \text{and} \quad a_n=o(A_n/n^\alpha), \quad \text{as } n\to\infty.$$

Then we put for $\omega \in \Omega$ and $0 \leqq t \leqq 1$

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$$(1.3)\quad \begin{cases} S_0(\omega)=0,\ S_k(\omega)=\sum_{j=1}^{k} a_j w_{n_j}(\omega) \qquad (k=1, 2, \cdots), \\ X_k(t)=\frac{1}{A_k}\{S_j(\omega)+a_{j+1}w_{n_{j+1}}(\omega)\frac{tA_k^2-A_j^2}{A_{j+1}^2-A_j^2}\},\ \text{if}\ \frac{A_j^2}{A_k^2}\leqq t\leqq\frac{A_{j+1}^2}{A_k^2}\ (j=0, 1, \cdots, k-1). \end{cases}$$

THEOREM. *For random functions* $X_n(t)$ $(n=1, 2, \cdots)$ *defined in* (1. 3), X_n *converges in distribution to* W *in* $(C, \mathscr{C})$, *where* W *denotes the Wiener process.*

The conditions (1. 2) of our theorem are best possible, since we can show similarly in [5] that $X_n(1)$ does not converge in distribution to $W(1)$, as $n\to\infty$.

In [1] I. Berkes showed the a. e. invariance principle under the conditions (1. 1) for $0\leqq\alpha<1/2$ and $a_n=1$ $(n=1, 2, \cdots)$.

For the proof of the theorem we use the functional central limit theorem for martingale of B. M. Brown.

Theorem of B. M. Brown (cf. [2], Theorem 3). For a martingale difference sequence $\{Y_n, \mathscr{F}_n\}$, let $s_0=0$, $s_m^2=\sum_{j=1}^{m}EY_j^2<\infty$ $(m=1, 2, \cdots)$ and

$$\xi_n(t)=\frac{1}{s_n}\ \{\sum_{j=0}^{k}Y_j+Y_{k+1}\frac{ts_n^2-s_k^2}{s_{k+1}^2-s_k^2}\},\ \text{if}\ \frac{s_k^2}{s_n^2}\leqq t\leqq\frac{s_{k+1}^2}{s_n^2} \qquad (k=0, 1, \cdots, n-1).$$

If random functions $\xi_n(t)$ $(n=1, 2, \cdots)$ satisfy the following conditions:

$$(1.4)\qquad \frac{1}{s_n^2}\sum_{j=1}^{n}E(Y_j^2\mid\mathscr{F}_{j-1})\to 1, \quad \text{in prob.}\quad \text{as } n\to\infty,$$

and for any $\varepsilon>0$,

$$(1.5)\qquad \frac{1}{s_n^2}\sum_{j=1}^{n}E\,[\,Y_j^2\ I(\,|Y_j|\geqq\varepsilon s_n)\,]\to 0 \quad \text{as } n\to\infty,$$

then ξ_n converges in distribution to W in $(C, \mathscr{C})$ as $n\to\infty$.

2. Proof of the Theorem.

Let us put

$$(2.1)\qquad p(0)=0, \qquad p(k)=\max\ \{j: n_j<2^k\} \quad (k=1, 2, \cdots).$$

Then if $p(k)+1<p(k+1)$, then by (1. 1),

$$2>\frac{n_{p(k+1)}}{n_{p(k)+1}}>\prod_{j=p(k)+1}^{p(k+1)-1}(1+cj^{-\alpha}) > 1+cp(k+1)^{-\alpha}(p(k+1)-p(k)-1).$$

Hence we have

$$(2.2)\qquad p(k+1)-p(k)=O(p(k)^{\alpha}) \qquad \text{as } k\to\infty.$$

Next putting

$$(2.3)\qquad \Delta_k(\omega) = \sum_{j=p(k)+1}^{p(k+1)} a_j w_{n_j}(\omega) \qquad (k=0, 1, 2, \cdots),$$

(1. 2) and (2. 2) imply

$$\sup_{\omega\in\Omega}|\Delta_k(\omega)| \leqq \sum_{j=p(k)+1}^{p(k+1)} |a_j| \leqq \max_{p(k)<j\leqq p(k+1)} |a_j| \{p(k+1)-p(k)\} = o(A_{p(k+1)}) \quad \text{as } k\to\infty. \tag{2. 4}$$

and

$$A_{p(k+1)}/A_{p(k)} = 1+o(1) \quad \text{as } k\to\infty. \tag{2. 5}$$

Here we shall describe a lemma of S. Takahashi which plays an essential role in the proof.

LEMMA 1. ([4], Lemma 3). *Under the conditions of the theorem, we have*

$$E(|A^{-2}_{p(n+1)}\sum_{j=0}^{n}\Delta_j^2 - 1|^2)\to 0 \quad \text{as } n\to\infty. \tag{2. 6}$$

LEMMA 2. *Let $\tilde{W}_n(t)$ $(n=1, 2, \cdots)$ be random functions defined as follows: let $\Delta_{-1}=0$ and*

$$\tilde{W}_n(t)=\frac{1}{A_{p(n)}}\{\sum_{j=-1}^{k-1}\Delta_j+\Delta_k\frac{tA^2_{p(n)}-A^2_{p(k)}}{A^2_{p(k+1)}-A^2_{p(k)}}\}, \quad \text{if } \frac{A^2_{p(k)}}{A^2_{p(n)}}\leqq t\leqq \frac{A^2_{p(k+1)}}{A^2_{p(n)}} \tag{2. 7}$$

$(k=0, 1, \cdots, n-1)$. *Then $\tilde{W}_n$ converges in distribution to W in $(C, \mathscr{C})$, as $n\to\infty$.*

PROOF. Put $\mathscr{F}_0=\{\phi, \Omega\}$, $\mathscr{F}_{k+1}=\sigma(r_0, r_1, \cdots, r_k)$ $(k=0, 1, \cdots)$ and $Y_j=\Delta_{j-1}$ $(j=1, 2, \cdots)$. Then $\{Y_j, \mathscr{F}_j\}$ is a martingale difference sequence and $E(\Delta_k^2 \mid \mathscr{F}_k)=\Delta_k^2$ $(k=0, 1, \cdots)$. Clearly (2. 6) implies (1. 4), and (2. 6) and (2. 4) imply (1. 5). Thus by the Theorem of B. M. Brown, the proof is complete.

Now for any given N, we can find an integer $M=M(N)$ such that $p(M)<N\leqq p(M+1)$ and define the random function $W_N(t)$ by

$$W_N(t)=\frac{1}{A_N}\{S_{p(k)}+\Delta_k\frac{tA_N^2-A^2_{p(k)}}{A^2_{p(k+1)}-A^2_{p(k)}}\}, \quad \text{if } \frac{A^2_{p(k)}}{A_N^2}\leqq t\leqq \frac{A^2_{p(k+1)}}{A_N^2} \tag{2. 8}$$

$(k=0, 1, \cdots, M)$. Then to prove the theorem it is sufficient to show that for any $\varepsilon>0$,

$$P\{\max_{0\leqq t\leqq 1}|X_N(t)-\tilde{W}_{M+1}(t)|\geqq\varepsilon\}\to 0 \quad \text{as } N\to\infty. \tag{2. 9}$$

We have

$$\begin{aligned}&\max_{0\leqq t\leqq 1}|X_N(t)-\tilde{W}_{M+1}(t)|\\ &\leqq \max_{0\leqq t\leqq 1}|X_N(t)-W_N(t)| + \max_{0\leqq t\leqq 1}|W_N(t)-\frac{A_{p(M+1)}}{A_N}\tilde{W}_{M+1}(t)|\\ &+(\frac{A_{p(M+1)}}{A_N}-1)\max_{0\leqq t\leqq 1}|\tilde{W}_{M+1}(t)| = I_{1,N}+I_{2,N}+I_{3,N}, \quad \text{say.}\end{aligned}$$

Since $X_N(A^2_{p(k)}/A^2_N)=S_{p(k)}/A_N = W_N(A^2_{p(k)}/A^2_N)$ $(k=0, 1, \cdots, M)$, we have by (2.4)

$$I_{1,N} \leqq \max_{0 \leqq k \leqq M} \{ \max_{\frac{A^2_{p(k)}}{A^2_N} \leqq t \leqq \frac{A^2_{p(k+1)}}{A^2_N}} | X_N(t) - X_N(\frac{A^2_{p(k)}}{A^2_N}) | + \frac{|\Delta_k|}{A_N} \}$$

$$\leqq \max_{0 \leqq k \leqq M} (2 \sum_{j=p(k)+1}^{p(k+1)} |a_j| / A_N) \to 0 \quad \text{as } N \to \infty.$$

By Lemma 2, $\mathscr{P}_n = P\tilde{W}_n^{-1}(n=1, 2, \cdots)$ are relatively compact, and by the well-known Prohorov's theorem, $\{\mathscr{P}_n\}$ is tight. Therefore by the equivalent proposition (cf. [3], p. 214), for every $\eta>0$ and every $\varepsilon>0$ there exist $\delta>0$ and an integer n_0 such that

$$(2.10) \qquad P \{ \max_{\substack{|t-s|<\delta \\ t,s \in [0,1]}} | \tilde{W}_n(t) - \tilde{W}_n(s) | \geqq \frac{\varepsilon}{2} \} < \eta \qquad \text{if } n \geqq n_0.$$

On the other hand, (2. 5) implies that $| A^2_N / A^2_{p(M+1)} - 1 | < \delta$, if $N \geqq N_0$. Hence we have

$$\max_{0 \leqq t \leqq 1} | W_N(t) - \frac{A_{p(M+1)}}{A_N} \tilde{W}_{M+1}(t) | = \frac{A_{p(M+1)}}{A_N} \max_{0 \leqq t \leqq 1} | \tilde{W}_{M+1}(\frac{A^2_N}{A^2_{p(M+1)}} t) - \tilde{W}_{M+1}(t) |$$

$$\leqq 2 \max_{\substack{|s-t|<\delta \\ s,t \in [0,1]}} | \tilde{W}_{M+1}(s) - \tilde{W}_{M+1}(t) | \qquad \text{if } N \geqq N_0$$

and, by (2. 10)

$$P(| I_{2,N} | \geqq \varepsilon) \leqq P \{ \max_{\substack{|t-s|<\delta \\ t,s \in [0,1]}} | \tilde{W}_{M+1}(t) - \tilde{W}_{M+1}(s) | \geqq \frac{\varepsilon}{2} \} < \eta \qquad \text{if } N \geqq \max(p(n_0), N_0).$$

Since $\max_{0 \leqq t \leqq 1} | \tilde{W}_{M+1}(t) | \leqq \max_{0 \leqq k \leqq M+1} (| S_{p(k)} | + | \Delta_k |) / A_{p(M+1)}$, we have, by (2. 4) and the submartingale inequality,

$$P(| I_{3,N} | \geqq \varepsilon) \leqq P \{ \max_{0 \leqq k \leqq M+1} | S_{p(k)} | \geqq \frac{\varepsilon}{2} A_{p(M+1)} (\frac{A_{p(M+1)}}{A_N} - 1)^{-1} \}$$

$$\leqq (\frac{\varepsilon}{2} A_{p(M+1)})^{-2} (\frac{A_{p(M+1)}}{A_N} - 1)^2 E S^2_{p(M+1)} = (\frac{2}{\varepsilon})^2 (\frac{A_{p(M+1)}}{A_N} - 1)^2 \qquad \text{if } N \geqq N_1,$$

which tends to zero as $N \to \infty$, by (2. 5). Hence we obtain (2. 9).

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