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On the Hausdorff Dimension of Kleinian Groups with Parabolic Elements.

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Abstruct. In this paper we shall consider the limit set of the Kleinian group with parabolic elements and prove that the Hausdorff dimension of the limit set of this group is greater than $\frac{1}{2}$. A.F.Beardon showed in [3] the existance of a finitely generated Fuchsian group G of the second kind whose Hausdorff dimension d(G) is greater than $\frac{1}{2}$. He also proved in [4] that d(G) $>\frac{1}{2}$ for a Fuchsian group of the second kind with parabolic elements. Hence to extend this inequality for Kleinian groups with parabolic elements is significant.

1. Let Γ be a group of linear transformations

$$S: \quad z \longrightarrow \frac{az+b}{cz+d}\,(ad-bc=1),$$

on the complex sphere $\tilde{C}=C \cup \{\infty\}$. A point z_0 is called a limit point of Γ if there is a sequence $\{S_n\}$ consisting of distinct elements of Γ such that $\lim_{n\to\infty} S_n(z)=z_0$ for some $z \in \tilde{C}$. The set of limit points will be called the limit set of Γ denoted by $\Lambda(\Gamma)$. $\Lambda(\Gamma)$ is a nowhere dense and perfect set, and invariant under Γ. We shall denote its complementary set by $\Omega(\Gamma)$ and call it the set of discontinuity. If $\Omega(\Gamma)$ is not empty, it is said that Γ is discontinuous and further, if $\Lambda(\Gamma)$ contains more than two points, Γ is said a Kleinian group.

For a linear transformation of the form

$$S(z)=\frac{az+b}{cz+d}, \quad ad-bc=1, \quad c \neq 0,$$

the circle $I(S)=\{z \mid |cz+d|=1\}$ is called the isometric circle of the transformation S. The radius of $I(S)$ is equal to $\frac{1}{|c|}$.

2. Let us give some properties of a linear transformation.

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Lemma 1. *Let* $P=\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$, $\alpha\delta-\beta\gamma=1$, $(\alpha+\delta=2)$ *be a parabolic element. Then it holds for any integer* $n(\neq 0)$

$$(1)\qquad P^n=\begin{pmatrix} n\alpha-(n-1) & n\beta \\ n\gamma & n\delta-(n-1)\end{pmatrix}.$$

Proof. We can prove easily for $n\geqq 1$ by induction. Further, since $(P^n)^{-1}=\begin{pmatrix} n\delta-(n-1) & -n\beta \\ -n\gamma & n\alpha-(n-1)\end{pmatrix}$ and $\alpha+\delta=2$, we have $(P^n)^{-1}=P^{-n}$. q.e.d.

Lemma 2. *Let C be a circle of radius r with center at the point z_1 in $\mathbb{C}$. Let $S(z)=(az+b)/(cz+d)$, $(ad-bc=1)$ by any linear transformation. If $S^{-1}(\infty)=-\frac{d}{c}$ is not on C, the radius $r(C')$ of the image C' of C by S is equal to $\frac{1}{|c|^2}\frac{r}{|\rho^2-r^2|}$, where ρ denotes the distance $|S^{-1}(\infty)-z_1|$.*

Proof. If $S^{-1}(\infty)=-\frac{d}{c}$ lies in the outside of C, set $\theta=\arg\{(z-z_1)/(S^{-1}(\infty)-z_1)\}$. Then we have

$$(2)\qquad r(C')=\frac{1}{2\pi}\int_C\left|\frac{dS(z)}{dz}\right||dz| =\frac{1}{2\pi|c|^2}\int_0^{2\pi}\frac{rd\theta}{\rho^2-2\rho r\cos\theta+r^2}$$
$$=\frac{1}{|c|^2}\frac{r}{\rho^2-r^2}.$$

If $S^{-1}(\infty)=-\frac{d}{c}$ lies in the inside of C, we have the equality

$$r(C')=\frac{1}{|c|^2}\frac{r}{r^2-\rho^2}$$

by the same manner stated above. q.e.d.

As to the location of isometric circles, we know the following facts. If $I(U)$ and $I(V^{-1})$ are exterior to each other, then $I(UV)$ is contained in $I(V)$ $(U\neq V^{-1})$. If P is a parabolic element, $I(P)$ and $I(P^{-1})$ are tangent externally.

Now let us prove the following lemma.

Lemma 3. *Let Γ be a Kleinian group with parabolic elements. Then there exists a subgroup G of Γ which satisfies the following two conditions :* (i) *P and T are parabolic and loxodromic transformations, where any pair of four circles $I(P)$, $I(P^{-1})$, $I(T)$, $I(T^{-1})$ are external to one another except the case that $I(P)$ and $I(P^{-1})$ are tangent externally.* (ii) *G is a free group generated by P and T.*

Proof. Since Γ is a Kleinian group with parabolic elements, then we may assume no loss of generality that Γ has a parabolic element P where $P(\infty)\neq\infty$. For if $P_1(\infty)=\infty$, we may take $P=VP_1V^{-1}$ where V is a loxodromic element in Γ. Further, Γ has a loxodromic element W such that $W(\infty)\neq\infty$. Let ξ_1, ξ_2 be fixed points and $\varkappa$ be a multiplier of W, respectively, where $\varkappa\neq 1$. Then, it holds

$$W^n = \frac{1}{\varkappa^{\frac{n}{2}}(\xi_1-\xi_2)} \begin{pmatrix} \varkappa^n \xi_2-\xi_1 & (1-\varkappa^n)\xi_1\xi_2 \\ \varkappa^n-1 & \xi_2-\varkappa^n\xi_1 \end{pmatrix}.$$

Let $r(W^n)$ be a radius of $I(W^n)$. Then we have

(3) $$r(W^n) = \frac{|\xi_1-\xi_2|}{|\varkappa^{\frac{n}{2}}-\varkappa^{-\frac{n}{2}}|}.$$

From (1) of lemma 1, we can easily find for the radius $r(P^n)$ of $I(P^n)$ that

(4) $$r(P^n) = \frac{1}{|n\gamma|}.$$

Since both radii of $r(W^n)$ and $r(P^n)$ tend to 0 for $n\to\infty$, we can select four circles with above condition (i). q.e.d.

3. From now we shall consider the subgroup G of Γ generated by P and T appeared in Lemma 3. Further let us consider the subgroup G′ of G generated by P^jTP^{-j} $(j=0, \pm1, \pm2, \ldots)$. Put $P^iTP^{-i}=T_i$ $(i\in Z)$, $P^0=E$ and $T_0=T$, where E is the identity. Now let us denote by $D(T_i^\varepsilon)$ the closed disc bounded by $P^i(I(T^\varepsilon))$ $(i\in Z)$, where $\varepsilon=\pm1$ and further denote by B and B' the exterior of four discs $\bigcup_{\varepsilon=\pm1}\{D(P^\varepsilon), D(T^\varepsilon)\}$ and $\bigcup_{j=-\infty}^{\infty}\bigcup_{\varepsilon=\pm1} D(T_j^\varepsilon)$, respectively. Then B and B' are the fundamental domains of G and G′, respectively.

Next we shall give the following important lemma.

Lemma 4. *Let* G *and* G′ *be the Kleinian groups defined in the above. Then there are two relations between* G *and* G′ *as follows*: (i) G′ *is a normal subgroup of* G *and the quotient group* G/G′ *is a cyclic group* $<P>$ *generated by P.* (ii) $\Lambda(G)=\Lambda(G')$.

Proof of (i). Let g be any element of G. Then we can choose adequately $n\in Z$ and $g'\in G'$ satisfying $g=P^ng'$. Considering the automorphisms $\overline{P}_n$: $g'\to P^ng'P^{-n}$ for each $g'\in G'$, we have $gG'g^{-1}\subset G'$ $(g\in G)$. Replacing g by g^{-1} we obtain $gG'g^{-1}\supset G'$ $(g\in G)$. q.e.d.

Proof of (ii). Denote by Y the set of generators of G and their inverses $\{P, P^{-1}, T, T^{-1}\}$. Any element $S(\neq E)$ of G has the form

$$S=P^{l_j}T^{m_j}\cdots\cdots P^{l_2}T^{m_2}P^{l_1}T^{m_1}\ (l_i, m_i\in Z).$$

Set $n=\sum_{i=1}^{j}(|l_i|+|m_i|)$. S is called an element of grade n of G and sometimes denote it by $S=S_{(n)}$. If $S_{(n)}V^{-1}=S_{(n-1)}$, $D(S_{(n)})=S_{(n)}(D(U))$ is called a closed disc of grade n of G where $U\in Y-\{V\}$. In other words, $D(S_{(n)})$ is a closed disc bounded by the inner boundary circle of $S_{(n)}(B)$. Since it is obvious that $\Lambda(G')\subset\Lambda(G)$, it is sufficient only to prove $\Lambda(G)\subset\Lambda(G')$. It is known $\bigcap_{n=0}^{\infty}\bigcup_{S(n)} D(S_{(n)})=\Lambda(G)$. Let z_0 be any point of $\Lambda(G)$. Then we can take a sequence of distinct elements $\{A_n\}$ in G for a sequence of neighbourhoods $U(z_0\,;\,\varepsilon_n)=\{z \mid |z-z_0|<\varepsilon_n\}$ $(\varepsilon_n\to 0)$ of z_0 such that $A_n(B)\subset U(z_0\,;\varepsilon_n)$. We have proved in (i) $A_nT^{\varepsilon}A_n^{-1}(\varepsilon=\pm 1)$ belongs to G′ where T is a generator of G′. There are fixed points z_n, z_n' of $A_nTA_n^{-1}$ in $U(z_0\,;\,\varepsilon_n)$. This concludes the existance of a sequence $\{z_n\}$ in $\Lambda(G')$ such that $\lim_{n\to\infty} z_n=z_0$. Therefore $z_0\in\Lambda(G')$. q.e.d.

4. Take a point z_1 in B and construct a closed disc $D_r(z_1)$ of radius r with the center z_1 such that $D_r(z_1)\subset B$. Then we have the following lemma.

Lemma 5. *Let r_n be the radius of the image of $D_r(z_1)$ by P^n $(n\neq 0)$. Then there exist constants c_1 and c_2 depending only on B and r such that it holds*

$$(5)\qquad 0<\frac{1}{n^2}c_1 \leqq r_n \leqq \frac{1}{n^2}c_2.$$

Proof. Because of $P^{-n}(\infty)=P^{-1}(\infty)+\frac{n-1}{n}\cdot\frac{1}{\gamma}$ from Lemma 1, we have

$$|P^{-1}(\infty)-z_1|-\frac{1}{|\gamma|}\leqq |P^{-n}(\infty)-z_1| \leqq |P^{-1}(\infty)-z_1|+\frac{1}{|\gamma|}$$

for $n>0$. Therefore we have from Lemma 2 the following inequality

$$(6)\qquad \frac{r}{|n\gamma|^2\{(|P^{-1}(\infty)-z_1|+|\gamma|^{-1})^2-r^2\}} \leqq r_n \leqq \frac{r}{|n\gamma|^2\{(|P^{-1}(\infty)-z_1|-|\gamma|^{-1})^2-r^2\}}.$$

In the similar manner, we can also prove for $n<0$ the following inequality

$$(6')\qquad \frac{r}{|n\gamma|^2\{(|P(\infty)-z_1|+|\gamma|^{-1})^2-r^2\}} \leqq r_n \leqq \frac{r}{|n\gamma|^2\{(|P(\infty)-z_1|-|\gamma|^{-1})^2-r^2\}}.$$

If we put $c_1=\min_{z_1\in B}\frac{r}{|\gamma|^2}\frac{1}{(|P^{-1}(\infty)-z_1|+|\gamma|^{-1})^2-r^2}$ and $c_2=\max_{z_1\in B}\frac{r}{|\gamma|^2}\frac{1}{(|P^{-1}(\infty)-z_1|-|\gamma|^{-1})^2-r^2}$ in (6), we have (5). In the case of $n<0$, we may put $c_1=\min_{z_1\in B}\frac{r}{|\gamma|^2}\frac{1}{(|P(\infty)-z_1)+|\gamma|^{-1})^2-r^2}$ and $c_2=\max_{z_1\in B}\frac{r}{|\gamma|^2}\frac{1}{(|P(\infty)-z_1|-|\gamma|^{-1})^2-r^2}$ in (6′). q.e.d.

By the same manner as Lemma 5, taking the closed disc $D(T^{\varepsilon})$ $(\varepsilon=\pm 1)$ in place of $D_r(z_1)$, we can find the constants c_1', c_2' depnding only on B for the radius $r(D(T_N^{\varepsilon}))$ of $D(T_N^{\varepsilon})$ such that the following inequality

$$(7)\qquad 0<\frac{1}{N^2}c_1' < r(D(T_N^{\varepsilon})) < \frac{1}{N^2}c_2',$$

holds for $N\in Z-\{0\}$ and $\varepsilon=\pm 1$.

5. Let $T=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, $(ad-bc=1)$ be a loxodromic transformation defined in Lemma 3. Denote by D_{ε} and R_{ε} be the closed disc $\{z \mid |z-T^{\varepsilon}(\infty)|\leqq r(D(T))+2\delta\}$ and the ring domain $\{z \mid r(D(T))\leqq |z-T^{\varepsilon}(\infty)|\leqq r(D(T))+2\delta\}$, respectively, where $\varepsilon=\pm 1$ and $\delta>0$ is taken such that $D_{-1}\cap D_1=\phi$ and $R_{\varepsilon}\subset \overline{B}$. Then we obtain from Lemma 2,

$$r(T^{-\varepsilon}(D_{\varepsilon}^{c})) = \frac{r(D(T))+2\delta}{|\,c\,|^{2}} \frac{1}{|\,T^{\varepsilon}(\infty)-z_{1}|^{2}-(r(D(T))+2\,\delta)^{2}}, \ (\varepsilon=\pm 1)$$

where D_{ε}^{c} is the complementary set of D_{ε} and z_1 is the center of $I(T^{-\varepsilon})$. Since the radius $r(D(T^{-\varepsilon}))$ of the isometric circle $I(T^{-\varepsilon})$ is $1/\,|\,c\,|$ and $T^{\varepsilon}(\infty)=z_1$, we have

$$(8) \qquad r(T^{-\varepsilon}(D_{\varepsilon}^{c})) = \frac{r^{2}(D(T))}{r(D(T))+2\delta}, \qquad (\varepsilon=\pm 1).$$

Let $D_{\delta}(z')$ denote the inscribed disc of radius δ with center z' in R_{ε}. Then we have also from Lemma 2

$$r(T^{\varepsilon}(D_{\delta}(z'))) = \frac{\delta}{|\,c\,|^{2}} \frac{1}{|\,T^{-\varepsilon}(\infty)-z'\,|^{2}-\delta^{2}} \qquad \text{(see Fig. 1)}$$

for the radius $r(T^{\varepsilon}(D_{\delta}(z')))$ of the image of $D_{\delta}(z')$ by T^{ε} ($\varepsilon=\pm 1$). Hence from the relation $|\,T^{-\varepsilon}(\infty)-z'| = r(D(T)) + \delta$ we have the following equality:

$$(9) \qquad r(T^{\varepsilon}(D_{\delta}(z'))) = \frac{\delta \cdot r(D(T))}{r(D(T))+2\delta}.$$

It is easily seen from the right hand side of (9) that $r(T^{\varepsilon}(D_{\delta}(z')))$ is independent of the location of the center of $D_{\delta}(z')$. Therefore we find from (8) and (9) that the two closed discs $T^{\varepsilon}(D_{-\varepsilon}^{c})$ and D_{ε} are concentric and hence it holds,

$$(10) \quad T^{\varepsilon}(D_{-\varepsilon}^{c}) \subset \{z \mid\, |\,z-T^{\varepsilon}(\infty)\,| \leqq \frac{r^{2}(D(T))}{r(D(T))+2\delta}\} \qquad (\varepsilon=\pm 1).$$

Now let us denote by R_{ε}' the ring domain bounded by two circles with radii $r^{2}(D(T))\,/(r(D(T))+2\delta)$ and $r(D(T))$, that is,

$$(11) \quad R_{\varepsilon}' = \{z \mid \frac{r^{2}(D(T))}{r(D(T))+2\delta} \leqq |\,z-T^{\varepsilon}(\infty)\,| \leqq r(D(T))\}$$

(see Fig. 1). Then we have the following lemma.

Lemma 6. *Let D be the closed disc inscribed in R_{ε}'. Then there exist positive constants c_3 and c_4 depending only on B such that it holds*

$$(12) \quad 0 < c_{3} < \frac{r(P^{k}(D))}{r(P^{k}(D(T^{\varepsilon})))} < c_{4} < 1, \quad (\varepsilon=\pm 1)$$

for the radii $r(P^{k}(D))$ and $r(P^{k}(D(T^{\varepsilon})))$ of the image of D and $D(T^{\varepsilon})$ by P^{k} ($k \in Z$), where $P=\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$, ($\alpha\delta-\beta\gamma=1$) is the parabolic element defined in Lemma 3. (*see Fig.* 1)

Proof. Taking the large number d, we may assume that

$$D(P^{\varepsilon}) \cup D(T^{\varepsilon}) \subset \{z \mid |\,z\,| < d\} \qquad (\varepsilon=\pm 1).$$

Then we have also from Lemma 2

$$(13) \quad \frac{r(P^{k}(D))}{r(P^{k}(D(T^{\varepsilon})))} = \frac{r(D)}{r(D(T))} \frac{|\,P^{-k}(\infty)-\alpha(D(T^{\varepsilon}))\,|^{2}-r^{2}(D(T))}{|\,P^{-k}(\infty)-\alpha(D)\,|^{2}-r^{2}(D)} \geqq$$

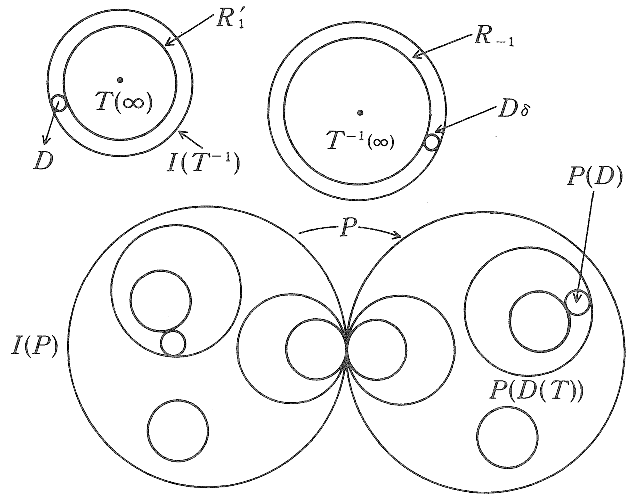

Fig. 1.

$$\frac{r(D)}{r(D(T))}\frac{(r(D(T))+2\delta)^2-r^2(D(T))}{(2d)^2-r^2(D)}=c_3>0,$$

where $\alpha(D(T^\varepsilon))$ and $\alpha(D)$ denote the centers of $D(T^\varepsilon)$ and D, respectively.

If we put $\ell=\min\{\,|z_1-z_2|\,|\,z_1\in I(P^\varepsilon),\ z_2\in I(T^\varepsilon),\ \varepsilon=\pm1\}$, we obtain from (13)

$$(14)\quad \frac{r(P^k(D))}{r(P^k(D(T^\varepsilon)))}\leqq\frac{r(D)}{r(D(T))}\frac{(2d)^2-r^2(D(T))}{(\ell+r(D))^2-r^2(D)}\ .$$

We obtain easily from (8)

$$(15)\quad r(D)=\frac{1}{2}\left\{r(D(T))-\frac{r^2(D(T))}{r(D(T))+2\delta}\right\}=\frac{\delta\cdot r(D(T))}{r(D(T))+2\delta}$$

If we substitute (15) into the right hand side of (14), we have

$$(16)\quad \frac{r(P^k(D))}{r(P^k(D(T^\varepsilon)))}\leqq\frac{\delta}{r(D(T))+2\delta}\frac{(2d)^2-r^2(D(T))}{(\ell+r(D))^2-r^2(D)}=c_4\ .$$

Since $\{(2d)^2-r^2(D(T))\}\ /\ \{(\ell+r(D))^2-r^2(D)\}$ is a constant, we can choose a small δ such that it holds $c_4<1$. q.e.d.

Let us denote by Y_N' the set $\bigcup_{|i|\leqq N}\{T_i,\ T_i^{-1}\}$ $(T_i=P^iTP^{-i})$. Then we shall get Y' from Y_N' as $N\to+\infty$. Let $S_{(n)}=T_{i_n}^{\varepsilon_n}\circ T_{i_{n-1}}^{\varepsilon_{n-1}}\circ\cdots\cdots\circ T_{i_1}^{\varepsilon_1}$ $(T_{i_j}^{\varepsilon_j}\in Y',\ i_j\in Z,\ \varepsilon_j=\pm1)$ be any element of G'. Then $S_{(n)}(D(T_j^{\varepsilon_j}))$ is contained in the inside of the closed disc $S_{(n)}(D(T_{i_1}^{\varepsilon_1}))$ and $S_{(n)}^{-1}(\infty)\in D(T_{i_1}^{\varepsilon_1})$ $(j\neq i_1)$.

Now let us give the following lemma.

Lemma 7. *For any $T_j^{\varepsilon_j}$ $(\neq T_{i_1}^{\varepsilon_1})\in Y'$, there exists a constant M depending only on B such that*

$$(17)\quad |\,S_{(n)}^{-1}(\infty)-\alpha(D(T_j^{\varepsilon_j}))\,|<M|\,P^{i_1}(T^{-\varepsilon_1}(\infty))-P^j(T^{-\varepsilon_j}(\infty))\,|\,,$$

where $S_{(n)}=T_{i_n}^{\varepsilon_n}\circ\cdots\cdots\circ T_{i_1}^{\varepsilon_1}$ *is an element of* G' *and* $\alpha(D(T_j^{\varepsilon_j}))$ *denotes the center of a closed disc* $D(T_j^{\varepsilon_j})$ $(\varepsilon_j=\pm 1,\ 1\leqq j\leqq k)$.

Proof. Put $\ell=\inf\{d(z_1, z_2)\mid z_1\in D(T_j^{\varepsilon_j}),\ z_2\in D(T_{i_1}^{\varepsilon_1})\}$. Since $S_{(n)}^{-1}(\infty)$ $(\in D(T_{i_1}^{\varepsilon_1}))$, $P^{i_1}(T^{-\varepsilon_1}(\infty))$ and $P^j(T^{-\varepsilon_j}(\infty))$ are not contained in the ring domains, that is, the images of R_ε' $(\varepsilon=\pm 1)$ by P^{i_1} and P^j, respectively, we have the following inequalities

(18) $\quad |S_{(n)}^{-1}(\infty)-\alpha(D(T_j^{\varepsilon_j}))|\leqq \ell+2\{r(D(T_j^{\varepsilon_j}))+r(D(T_{i_1}^{\varepsilon_1}))\}-\{r(P^j(D))+r(P^i(D))\}$

and

(19) $\quad |P^{i_1}(T^{-\varepsilon_1}(\infty))-P^j(T^{-\varepsilon_j}(\infty))|\geqq \ell+\{r(P^{i_1}(D))+r(P^j(D))\}$.

Using Lemma 6 we obtain from (18) and (19)

(20) $\quad |S_{(n)}^{-1}(\infty)-\alpha(D(T_j^{\varepsilon_j}))|\leqq \ell+(2-c_3)\{r(D(T_j^{\varepsilon_j}))+r(D(T_{i_1}^{\varepsilon_1}))\}$

and

(21) $\quad |P^{i_1}(T^{-\varepsilon_1}(\infty))-P^j(T^{-\varepsilon_j}(\infty))|\geqq \ell+c_3\{r(D(T_j^{\varepsilon_j}))+r(D(T_{i_1}^{\varepsilon_1}))\}$,

where c_3 (<1) is a positive constant in Lemma 6.
Hence we have from (20) and (21)

$$\frac{|S_{(n)}^{-1}(\infty)-\alpha(D(T_j^{\varepsilon_j}))|}{|P^{i_1}(T^{-\varepsilon_1}(\infty))-P^j(T^{-\varepsilon_j}(\infty))|}<\frac{2-c_3}{c_3}=M. \quad \text{(see Fig. 2)}$$

q.e.d.

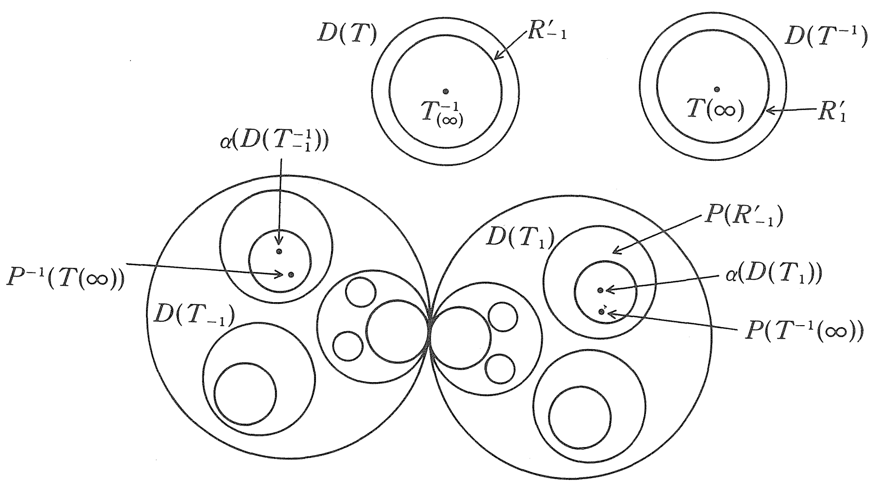

Fig. 2.

6. Considering the suitable conjugate group of G and using Lemma 1, we can take without loss of generality $P = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$ such that $\min\{\,|\gamma \circ T^{-1}(\infty)|,\ |\gamma \circ T(\infty)|\,\} > 2$.

Now we shall give the following theorem by using the above lemmas.

Theorem 1. *Assume that* $0 \leqq \mu \leqq 1$. *Then there exists a sufficiently large integer* N (>0) *such that*

$$(22)\quad \sum_{|j| \leqq N,\ \varepsilon = \pm 1} \{r(S_{(n)}(D(T_j^{\varepsilon})))\}^{\mu/2} \geqq \{r(S_{(n)}(D(T_{i_1}^{\varepsilon_1})))\}^{\mu/2},$$

where $S_{(n)} = T_{i_n}^{\varepsilon_n} \circ \cdots\cdots \circ T_{i_1}^{\varepsilon_1}$ *is an element of* G'_N *and* $\varepsilon_j = \pm 1$ $(1 \leqq j \leqq n)$.

Proof. By using Lemma 2 and (17) of Lemma 7, it is easily seen for any μ (>0) and $S_{(n)} \in G'$ that

$$\sum_{|j| \leqq N,\ \varepsilon = \pm 1} \{r(S_{(n)}(D(T_j^{\varepsilon})))\}^{\mu/2}$$

$$(23) = \sum_{|j| \leqq N,\ \varepsilon = \pm 1} \left\{ \frac{r(D(T_j^{\varepsilon}))\, R_{S_{(n)}}^2}{|S_{(n)}^{-1}(\infty) - \alpha(D(T_j^{\varepsilon}))|^2 - r^2(D(T_j^{\varepsilon}))} \right\}^{\mu/2}$$

$$\geqq \sum_{|j| \leqq N} c_5 R_{S_{(n)}}^{\mu} \frac{r(D(T_j^{\varepsilon}))^{\mu/2}}{|P^{i_1}(T^{-\varepsilon_1}(\infty)) - P^j(T^{-\varepsilon_j}(\infty)|^{\mu}},$$

where $R_{S_{(n)}}$ is the radius of $I(S_{(n)})$ and $c_5 = M^{-\mu}$ is a constant depending only on B and μ.

We obtained from (7) for the estimate of $r(D(T_j^{\varepsilon}))$ in (23) that

$$(24)\quad 0 < \frac{1}{|j|^2} c_1' < r(D(T_j^{\varepsilon})) < \frac{1}{|j|^2} c_2' \quad (\epsilon = \pm 1).$$

Now let us give the estimation of the denominator of the right hand side in (23). Since P^k has the form $\begin{pmatrix} 1 & 0 \\ k\gamma & 1 \end{pmatrix}$, we have

$$|P^{i_1}(T^{-\varepsilon_1}(\infty)) - P^j(T^{-\varepsilon_j}(\infty))| = \left| \frac{T^{-\varepsilon_1}(\infty)}{\gamma i_1(T^{-\varepsilon_1}(\infty))+1} - \frac{T^{-\varepsilon_j}(\infty)}{\gamma j(T^{-\varepsilon_j}(\infty))+1} \right|$$

$$(25)\quad = \frac{|\gamma T^{-\varepsilon_1}(\infty) T^{-\varepsilon_j}(\infty)\,(j - i_1) + T^{-\varepsilon_1}(\infty) - T^{-\varepsilon_j}(\infty)|}{|\gamma i_1 T^{-\varepsilon_1}(\infty)+1|\ |\gamma j T^{-\varepsilon_j}(\infty)+1|}$$

$$\leqq \frac{|\gamma T^{-\varepsilon_1}(\infty) T^{-\varepsilon_j}(\infty)\,(j - i_1)| + |T^{-\varepsilon_1}(\infty) - T^{-\varepsilon_j}(\infty)|}{|i_1|\,|j|\,(|\gamma T^{-\varepsilon_1}(\infty)| - |i_1|^{-1})\,(|\gamma T^{-\varepsilon_j}(\infty)| - |j|^{-1})}.$$

If we consider only the terms of $\varepsilon_1 = +1$ and $\varepsilon_j = +1$ in (25) and note the assumption $|\gamma T^{\varepsilon}(\infty)| > 2$, $(\varepsilon = \pm 1)$ we obtain from (25)

$$(26)\quad |P^{i_1}(T^{-\varepsilon_1}(\infty)) - P^j(T^{-\varepsilon_j})|^{\mu} \leqq \frac{c_6\,|j - i_1|^{\mu}}{|i_1|^{\mu}\,|j|^{\mu}}$$

where c_6 is a constant depending only on B and μ. If we substitute the estimates (24) and (26) into the right hand side of (23), we have

$$(27)\quad \sum_{|j|\leqq N,\ \varepsilon=\pm1} \{r(S_{(n)}(D(T_j^{\varepsilon})))\}^{\mu/2} \geqq c_7 R^{\mu}_{S_{(n)}} \sum_{|j|\leqq N}{}' \left|\frac{i_1}{i_1-j}\right|^{\mu} (T_{i_1}^{\varepsilon_1} \neq T^{\varepsilon}),$$

In particular, if $T_{i_1}^{\varepsilon_1}= T^{\varepsilon}$, we obtain by the analogous method

$$(27)'\quad \sum_{|j|\leqq N,\ \varepsilon=\pm1} \{r(S_{(n)}(D(T_j^{\varepsilon})))\}^{\mu/2} \geqq c_7' R^{\mu}_{S_{(n)}} \sum_{|j|\leqq N}{}' \left|\frac{1}{j}\right|^{\mu}, (T_{i_1}^{\varepsilon_1}= T^{\varepsilon}).$$

Next we shall give the estimate of the right hand side of (22). If we see also Lemma 2, we obtain

$$(28)\quad r(S_{(n)}(D(T_{i_1}^{\varepsilon_1}))) = \frac{r(D(T_{i_1}^{\varepsilon_1}))R^2_{S_{(n)}}}{r^2(D(T_{i_1}^{\varepsilon_1}))- |S^{-1}_{(n)}(\infty)-\alpha(D(T_{i_1}^{\varepsilon_1}))|^2}\ .$$

Since $S^{-1}_{(n)}(\infty)$ ($\epsilon\ D(T_{i_1}^{\varepsilon_1})$) is not contained in the ring domain, that is, the image of R'_{ε} by P^{i_1}, we have from Lemma 6

$$(29)\quad \begin{aligned} \frac{R^2_{S_{(n)}}}{r(D(T_{i_1}^{\varepsilon_1}))} &\leqq r(S_{(n)}(D(T_{i_1}^{\varepsilon_1}))) \\ &\leqq \frac{r(D(T_{i_1}^{\varepsilon_1}))\, R^2_{S_{(n)}}}{r^2(D(T_{i_1}^{\varepsilon_1}))- \{r(D(T_{i_1}^{\varepsilon_1}))-c_3 r(D(T_{i_1}^{\varepsilon}))\}^2} \\ &= \frac{R^2_{S(n)}}{r(D(T_{i_1}^{\varepsilon_1}))c_3(2-c_3)},\quad (0 < c_3 < 1). \end{aligned}$$

Hence we obtain from (7)

$$(30)\quad c_8 |i_1|^{\mu} R^{\mu}_{S_{(n)}} \leqq r(S_{(n)}(D(T_{i_1}^{\varepsilon_1})))^{\mu/2} \leqq c_9 |i_1|^{\mu} R^{\mu}_{S_{(n)}} (T_{i_1}^{\varepsilon_1} \neq T^{\varepsilon}),$$

Where c_8, c_9 are the constants depending only on B and μ. In paticular, if $T_{i_1}^{\varepsilon_1}= T^{\varepsilon}$, we obtain by the analogous method

$$(30)'\quad c_8' R^{\mu}_{S_{(n)}} \leqq r(S_{(n)}(D(T_{i_1}^{\varepsilon_1})))^{\mu/2} \leqq c_9' R^{\mu}_{S_{(n)}},\ (T_{i_1}^{\varepsilon_1}= T^{\varepsilon}),$$

where c_8', c_9' are the constants depending only on B and μ. Therefore we have from (27) and (30) (or (27)′ and (30)′) the following estimation :

$$(31)\quad \frac{\sum_{|j|\leqq N,\varepsilon=\pm1} \{r(S_{(n)}(D(T_j^{\varepsilon})))\}^{\mu/2}}{r(S_{(n)}(D(T_{i_1}^{\varepsilon_1})))^{\mu/2}} \geqq c_{10} \sum_{|j|\leqq N}{}' \frac{1}{|i_1-j|^{\mu}},\ (T_{i_1}^{\varepsilon_1} \neq T^{\varepsilon})$$

$$\left(or \geqq c_{10}' \sum_{|j|\leqq N}{}' \frac{1}{|j|^{\mu}} (T_{i_1}^{\varepsilon_1}= T^{\varepsilon})\right).$$

where $c_{10}(c_{10}')$ is a constant depending only on B and μ. If we take a large number N, the right hand side of (31) is greater than 1 from the the divergence of the Dirichlet series for $0 \leqq \mu \leqq 1$. Thus we can prove the theorem. q.e.d.

7. Now denote by G_N' a Schottky group appeared in Theorem 1 and by B_N' the exterior of closed discs $\bigcup_{|j|\leqq N} \bigcup_{\varepsilon=\pm1} D(T_j^{\varepsilon})$. Then B_N' is a fundametal domain of G_N'. Let $S_{(m+1)}= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $(ad-bc = 1)$ be a transformation of grade $m+1$ in G_N'. Then the radius $r(S_{(m+1)}(D))$ of a closed disc D by $S_{(m+1)}(z)$ is given by

$$r(S_{(m+1)}(D)) = \frac{1}{2\pi \mid c \mid^2} \int_{\partial D} \frac{\mid dz \mid}{\mid z+d/c \mid^2},$$

where D is a suitable one in $\{D(T_i^\varepsilon) \mid \ \mid i \mid \leqq N$ and $\varepsilon = \pm 1\}$. Here we note that the point $S_{(m+1)}^{-1}(\infty) = -d/c$ is in the outside of B_N'.

If we put $t_1 = \max_{z \in \partial D} \mid z+d/c \mid$ and $t_2 = \min_{z \in \partial D} \mid z+d/c \mid$, then we have,

$$(32) \quad \frac{r(D)}{t_1^2} \cdot \frac{1}{\mid c \mid^2} \leqq r(S_{(m+1)}(D)) \leqq \frac{r(D)}{t_2^2} \cdot \frac{1}{\mid c \mid^2} ,$$

where $r(D)$ is the radius of D. Hence we have the following lemma.

Lemma 8. *There exists a positive constant* c_{11} *depending only on* B_N' *such that*

$$(33) \quad r(S_{(m+1)}(D(T_j^{\varepsilon'}))) \geqq c_{11} r(S_{(m+1)}(D(T_k^\varepsilon))),$$

where $S_{(m+1)} = S_{(m)} \cdot T_k^\varepsilon$ $(T_j^{\varepsilon'} \neq T_k^\varepsilon, \varepsilon' = \pm 1, \varepsilon = \pm 1)$.

8. Denote by F_{n_o} the family of all closed discs of grade n $(\geqq n_o)$. It is easy to see F_{n_o} is a covering of the limit set of Γ and that the diameter of any disc of F_{n_o} is less than a given $\delta\,(>0)$ for a sufficiently large n_o. Let $I(\delta, \Lambda(\Gamma))$ be a family of a countable number of closed discs U of the diameter $\ell_U \leqq \delta$ such that every point of $\Lambda(\Gamma)$ is a point of at least one U. We call the quantity :

$$(34) \quad M_\eta(\Lambda(\Gamma)) = \lim_{\delta \to 0} \Big[\inf_{\{I(\delta, \Lambda(\Gamma))\}} \sum_{U \in I(\delta, \Lambda(\Gamma))} \ell_U^\eta \Big]$$

the η-dimensional Hausdorff measure of $\Lambda(\Gamma)$.

9. From now we consider the limit set of G_N' stated in Theorem 1. Since $\Lambda(G_N')$ is compact, $\Lambda(G_N')$ is covered by a finite number of discs $\boldsymbol{D}_1, \boldsymbol{D}_2, \cdots, \boldsymbol{D}_k$ of any covering system of $\{I(\eta, \Lambda(G_N'))\}$. Take an arbitrary $\boldsymbol{D}_i$ among these k discs and let ℓ_i $(\leqq \delta/2)$ be the radius of $\boldsymbol{D}_i$. For a fixed $\boldsymbol{D}_i$, we can find closed discs $S_{(m_1)}(D_1^i), \cdots, S_{(m_{q(i)})}(D_{q(i)}^i)$ in $\bigcup_{n=0}^{\infty} F_n$ satisfying the following three conditions.

(i) The radius $r(S_{(m_j)}(D_j^i))$ $(i \leqq j \leqq q(\mathrm{i}))$ is larger than ℓ_i .

(ii) There exists at least one disc of grade m_j+1 among the discs contained in the disc $S_{(mj)}(D_j^i)$, meeting $\boldsymbol{D}_i$ and of radius $r(S_{(m_j+1)}(D_k^i))$ not larger than r_i.

$$(\mathrm{iii}) \quad \bigcup_{j=1}^{q(i)} S_{(m_j)}(D_j^i) \supset \boldsymbol{D}_i \cap \Lambda(G_N'),$$

where D_j^i is a suitable one in $\{D(T_j^\varepsilon) \mid \ \mid j \mid \leqq N, \varepsilon = \pm 1\}$.

Hence we obtain from the preceding Lemma 8

$$(35) \quad c_{11} r(S_{(m_j)}(D_j^i)) \leqq r(S_{(m_j+1)}(D_k^i)) \leqq \ell_i < r(S_{(m_j)}(D_j^i)) .$$

Construct such discs $\{S_{(m_j)}(D_j^i)\}$ for each $\boldsymbol{D}_i$ $(1 \leqq i \leqq k)$. Then it is obvious that $\bigcup_{i=1}^{k} \bigcup_{j=1}^{q(i)} S_{(m_j)}(D_j^i) \supset \Lambda(G_N')$ and

$$(36)\qquad \sum_{i=1}^{k}\sum_{j=1}^{q(i)} \} \ r(S_{(m_j)}(D_j^i))\}^{\mu/2} \leqq c_{12} \sum_{i=1}^{k} \ell_i^{\ \mu/2},$$

where c_{12} is a constant depending only on G_N'. Hence we obtain the following inequality from Theorem 1 :

$$(37)\quad 0 < \sum_{|j| \leqq N} (r(D(T_j)))^{\mu/2} \leqq c_{12} \sum_{i=1}^{k} \ell_i^{\mu/2} \ (0 \leqq \mu \leqq 1).$$

Thus we have the following lemma.

Lemma 9. *Let G_N' be the Schottky group defined in Theorem 1. Then it holds $M_{\frac{\mu}{2}}(\Lambda(G_N'))>0$ for $0\leqq \mu \leqq 1$.*

10. The Hausdorff dimension of a point set F in the z-plane is defined as the unique non-negative number $d(F)$ satisfying

$$M_{\frac{\mu}{2}}(F)=0, \quad \text{if } \mu/2 > d(F)$$

and

$$M_{\frac{\mu}{2}}(F)=\infty, \quad \text{if } 0\leqq \mu/2 < d(F),$$

where $M_{\frac{\mu}{2}}(F)$ denotes the $\mu/2$-dimensional Hausdorff measure of F. It is known that the Hausdorff dimension increases according to the increment of the number of defining circles [2] ; that is, $d(G_{N+1}') > d(G_N')$.
Thus we obtain the following main theorem.

Theorem 2. *Let Γ be a kleinian group with parabolic elements. Then the Hausdorff dimension $d(\Gamma) = d(\Lambda(\Gamma))$ is larger than 1/2.*

Remark. It is known that $\frac{1}{2}$ is the sharp estimate from [5] .

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