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A Note on an Algebraic Divisor in $\mathbf{C}^n$

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Abstract. We consider the condition that a divisor in $\mathbf{C}^n$ given by an entire function. It is well known that a divisor A is algebraic if and only if

$$\limsup_{r\to\infty} (N_f(r)/\log r) < \infty.$$

We shall give a simple proof of this theorem. Combining with the theorem of L. I. Ronkin, we shall give another characterization that A being algebraic.

§ 0. Introduction.

Let $f(z)$ be an entire function of n complex variables. It is an interesting problem that under what conditions the divisor $A = \{z \in \mathbf{C}^n ;\ f(z)=0\}$ becomes algebraic. In connection with this problem, R. A. Kramer has shown that A is algebraic if and only if there exists an $r \in \mathbf{R}^n_+$ such that $A \cap \Delta_r$ is compact. We shall give a simple proof of this theorem (Corollary of Theorem 1). In section 2, we shall give a characterization of different type that A being algebraic as an application of Theorem 1.

§ 1. Put

$$\mathbf{R}^n_+ = \{r = (r_1, r_2, \ldots, r_n) \in \mathbf{R}^n ;\ r_j > 0,\ 1 \leqq j \leqq n\} ,$$
$$|r|^2 = \sum_{j=1} r_j^2$$

and

$$\Delta_r = \{z = (z_1, z_2, \ldots, z_n) \in \mathbf{C}^n ;\ \frac{|z_1|}{r_1} = \frac{|z_2|}{r_2} = \ldots = \frac{|z_n|}{r_n}\}$$

here we mean that $\Delta_r = \mathbf{C}$ if n=1.

LEMMA 1. *For any homogeneous polynomial* $P_m(z)$ *of degree* m *which is not identically zero, there exists an* $r \in \mathbf{R}^n_+$ *such that* $P_m(z) \neq 0$ *if* $z \in \Delta_r - \{o\}$, *where* o *is the origin of* $\mathbf{C}^n$.

Proof. Put

$$\widetilde{\Delta}_r = \{z \in \mathbf{C}^n ; \frac{|z_1|}{r_1} = \frac{|z_2|}{r_2} = \dots = \frac{|z_n|}{r_n} = 1\} .$$

Then since P_m is homogeneous, we have only to show that $P_m(z) \neq 0$ if $z \in \widetilde{\Delta}_r$. We shall show the lemma by the induction on n and m. If $n=1$ or $m=0$, then it is trivial. Assume that the lemma holds for a homogeneous polynomial of at most n-1 complex variables or of degree at most m-1 of n complex variables. Put

$$P_m(z) = z_1 Q_1(z) + Q_2(z_2, z_3, \dots, z_n),$$

where Q_1 is identically zero or a homogeneous polynomial of degree m-1 and Q_2 is identically zero or a homogeneous polynomial of degree m of n-1 complex variables $z_2, z_3, \dots, z_n$. Let Q_2 be identically zero, then by the induction hypothesis there exists an $r \in \mathbf{R}^n_+$ such that $Q_1(z) \neq 0$ if $z \in \widetilde{\Delta}_r$. Then $z_1 Q_1(z) \neq 0$ if $z \in \widetilde{\Delta}_r$. In case that Q_2 is not identically zero, there exists an $r' = (r_2, r_3, \dots, r_n) \in \mathbf{R}^{n-1}_+$ such that $Q_2(z) \neq 0$ in

$$L_{r'} := \{z' \in \mathbf{C}^{n-1} ; |z_j| = r_j, 2 \leqq j \leqq n\} .$$

Since $L_{r'}$ is compact,

$$c = min \{ | Q_2(z') | ; z' \in L_{r'}\} > 0.$$

Then it holds that

$$| P_m(0, z') | = | Q_2(z') | > c$$

for all $z' \in L_{r'}$. Therefore there exists a positive number r_1 such that $P_m(z_1, z') \neq 0$ if $| z_1 | = r_1$ and $z' \in L_{r'}$. Put $r = (r_1, r')$, then we obtain the desired conclusion. q.e.d.

Now for an entire function f which is not identically zero, by removing the origin if necessary, we may assume that $f(o) \neq 0$. Let $\varkappa$ be a complex line which contains the origin of $\mathbf{C}^n$. Let P^{n-1} be the n-1 dimensional complex projective space with volume element $d\omega_{n-1}$ and with voume ω_{n-1}. Then $\varkappa$ can be considered as a point of p^{n-1}. For a point $a = (a_1, a_2, \dots, a_n) \in \mathbf{C}^n - \{o\}$, the slice function $f_a(w)$ of one complex variables w is defined by

$$f_a(w) = f(a_1 w, a_2 w, \dots, a_n w).$$

Since $f(o) \neq 0$, $f_a(w)$ is not identically zero. Let $S(1) = \{z \in \mathbf{C}^n ; | z | = 1\}$ and let $n_f(t : \varkappa)$ be the number of the zeros of $f_a(w)$ in $\{w \in \mathbf{C} ; | w | < t\}$, where $a \in S(1)$ and $\varkappa = \varkappa(a)$ is the complex line passing through the origin and the point a. Put

$$n_f(t) = \frac{1}{\omega_{n-1}} \int_{P^{n-1}} n_f(t : \varkappa)\, d\omega$$

and

$$N_f(t) = \int_0^t \frac{n_f(s)}{s}\, ds.$$

Since f(o) $\neq 0$ there exists a positive number t_0 such that

$$N_f(t) = \int_{t_0}^t \frac{n_f(s)}{s}\, ds.$$

Since Δ_r is a real submanifold of p^{n-1}, as was shown in [5] pp. 123–124, we can define the volume element $d\omega_\Delta$ on Δ_r which is induced from $d\,\omega_{n-1}$. Let ω_Δ be the volume of Δ_r. Put

$$n_{f,\Delta}(t) = \int_0^t \frac{n_{f\Delta}(s)}{s}\, ds.$$

Let $A = \{z \in \mathbf{C}^n;\ f(z)=0\}$ and let $\mathrm{Vol}(A \cap B(r))$ be the volume of A in the ball B(r) of radius r with center at the origin. Then it is well known that

$$A \text{ is algebraic if and only if } \frac{Vol\,(A \cap B(r))}{r^{2n-2}} = O(1). \tag{1}$$

Let $\nu_f(r:\varkappa(a))$ be the number of the zeros of $f_a(w)$ in the disc $\{w \in \mathbf{C};\ |w| < r\}$ without counting the multiplicities. It is shown in [7] that

$$Vol(A \cap B(r)) = \frac{r^{2n-2}}{2\pi} \int_{a \in S(1)} \nu_f(r:\varkappa(a))\, d\sigma \tag{2}$$

where $d\sigma$ is the surface element of S(1). By the definition,

$$\nu_f(r:\varkappa(a)) \leqq n_f(r:\varkappa(a)).$$

If P is a polynomial of degree m and if $P_a(w)$ is not identically zero, it is easily seen that

$$n_p(r:\varkappa(a)) \leqq m\,\nu_p(r:\varkappa(a)).$$

Now if A is algebraic, then there exists a polynomial P with P(o) $\neq 0$ such that $A = \{z \in \mathbf{C}^n;\ P(z)=0\}$. Since

$$n_f(r:\varkappa(a)) = n_p(r:\varkappa(a)),$$

it holds that

$$\int_{a \in S(1)} n_f(r:\varkappa(a))\, d\sigma = \int_{a \in S(1)} n_p(r:\varkappa(a))\, d\sigma$$
$$\leqq m \int_{a \in S(1)} \nu_f(r:\varkappa(a))\, d\sigma = O(1).$$

Conversely, if

$$n_f(r:\varkappa(a))\, d\sigma = O(1),$$

then by (2)

$$\frac{Vol\ (A\cap B(r))}{r^{2n-2}}=\frac{1}{2\pi}\int_{a\in S(1)}\nu_f(r:\varkappa(a))d\sigma$$

$$\leqq\frac{1}{2\pi}\int_{a\in S(1)}n_f(r:\varkappa(a))a\sigma=O(1).$$

Thus A is algebraic by (1). Therefore we have

LEMMA 2. A *is algebraic if and only if*

$$\int_{a\in S(1)}n_f(r:\varkappa(a))d\sigma=O(1).$$

Now since $n_f(t:\varkappa(\lambda a))=n_f(t:\varkappa(a))$ for any non zero complex number λ, as was shown in Stoll [6] pp. 142–143, it holds that

$$\int_{a\in S(1)}n_f(r:\varkappa(a))d\sigma=2\pi\int_{P^{n-1}}n_f(r:\varkappa)d\omega_{n-1}.$$

LEMMA 3. A *is algebraic if and only if*

$$\limsup_{r\to\infty}\frac{N_f(r)}{\log r}<\infty$$

Proof. By (1) and Lemma 2, the necessity is trivial. Suppose that

$$\limsup_{r\to\infty}\frac{N_f(r)}{\log r}=K.$$

If A is not algebraic, then there exists a sequence $\{r_\mu\}$ of positive numbers with $r_\mu\to\infty$ $(\mu\to\infty)$ which satisfies

$$\int_{a\in S(1)}n_f(r_\mu:\varkappa(a))\,d\sigma\geqq\mu.$$

Take a μ_0 with $\mu_0/(2\pi\omega_{n-1})\geqq 2K$, then since $n_f(r)$ is monotone increasing with respect to r,

$$N_f(t)=\int_{t_0}^{t}\frac{n_f(s)}{s}\,ds\geqq\int_{r_{\mu_0}}^{t}\frac{n_f(s)}{s}\,ds$$

$$\geqq n_f(r_{\mu_0})(\log t-\log r_{\mu_0})\geqq 2K(\log t-\log r_{\mu_0}).$$

This is a contradiction. q.e.d.

Now between $N_f(t)$ and $N_{f,\Delta}(t)$, the following relation is known.

LEMMA 4 (Ronkin [5], Theorem 2). *Let* $|r|=1$ *and let* $r_0=\min\{r_j;\ 1\leqq j\leqq n\}$. *For any positive number* $\delta>1$, *there exists a positive number* C_δ *such that*

$$N_f(r_0t)\leqq N_{f,\Delta}(t)\leqq C_\delta N_f(\delta t).$$

From Lemma 3 and 4, we have

THEOREM 1. A *is algebraic if and only if*

$$\limsup_{r\to\infty} \frac{N_{f,\Delta}(r)}{\log r} = O(1).$$

COROLLARY of THEOREM 1 (Kramer [4], Theorem 2.1). A *is algebraic if and only if there exists an* $r \in \mathbf{R}^n_+$ *such that* $A \cap \Delta_r$ *is compact.*

Proof. Suppose that A is algebraic. Let P(z) be a polynomial such that $A = \{z \in \mathbf{C}^n ; P(z)=0\}$. Let

$$P(z) = P_0 + P_1(z) + \ldots + P_m(z)$$

where P_j is a homogeneous polynomial of degree j and P_m is not identically zero. By Lemma 1 there exists an $r \in \mathbf{R}^n_+$ such that $P_m(z) \neq 0$ if $z \in \Delta_r - \{o\}$. Let $\varkappa \subset \Delta_r$ be a complex line which is given by $z_j = a_j w$ $(1 \leqq j \leqq n)$, where $a = (a_1, a_2, \ldots, a_n) \in S(1)$. Then

$$P_a(w) = \sum_{j=0}^{j=m} P_j(a) w^j.$$

Put $c = \min\{|P_m(z)| ; z \in S(1) \cap \Delta_r\}$ and $c' = \max\{|P_{m-j}(z)| ; z \in S(1) \cap \Delta_r\ 1 \leqq j \leqq m\}$. Then $c > 0$ and

$$|P_a(w)| \geqq c|w|^m - c' \sum_{j=1}^{j=m} w^{m-j}.$$

Since c and c' do not depend on the choice of the complex line $\varkappa \subset \Delta_r$, we can choose a positive number R_0 such that $P(z) \neq 0$ if $z \in \Delta_r \cap \{z \in \mathbf{C}^n ; |z| > R_0\}$. Therefore $A \cap \Delta_r$ is compact.

Conversely, if $A \cap \Delta_r$ is compact for some $r \in \mathbf{R}^n_+$ then there exists a positive number R_0 such that $n_f(t : \varkappa) \leqq n_f(R_0 : \varkappa)$ for any complex line $\varkappa \subset \Delta_r$ which through the origin. Then

$$N_{f,\Delta}(t) = \frac{1}{\omega_\Delta} \int_{t_0}^{t} \frac{n_{f,\Delta}(s)}{s} ds$$

$$\leqq \frac{n_{f,\Delta}(R_0)}{\omega_\Delta} (\log t - \log t_0)$$

Then

$$\limsup_{r\to\infty} \frac{N_{f,\Delta}(r)}{\log r} \leqq \frac{n_{f,\Delta}(R_0)}{\omega_\Delta} < \infty.$$

Therefore A is algebraic by Theorem 1.

§ 2. Let f(z,w) be an entire function of n+1 complex variables $z = (z_1, z_2, \ldots, z_n)$, w and let $f(o) \neq 0$. Put $A = \{(z, w) \in \mathbf{C}^{n+1} ; f(z, w) = 0\}$. If A is algebraic then there exists a polynomial P(z, w) such that $A = \{(z, w) \in \mathbf{C}^{n+1} ; P(z, w) = 0\}$. Put

$$P(z, w) = P_m(z) w^m + P_{m-1}(z) w^{m-1} + \ldots + P_0(z)$$
$$P_m(z) = Q^{(m)}_t(z) + Q^{(m)}(z) + \ldots + Q^{(m)}(z)$$

and

$$P_0(z) = Q_s^{(0)}(z) + Q^{(0)}(z) + \ldots + Q^{(0)}(z),$$

where $Q_\nu^{(m)}$ and $Q_\mu^{(0)}$ are homogeneous polynomials of degree ν and μ respectively. Since $Q_t^{(m)} \cdot Q_s^{(0)}$ is homogeneous, by Lemma 1, there exists a $\tau=(\tau_1, \tau_2, \ldots, \tau_n) \in \mathbf{R}^n$ such that $Q_t^{(m)}(z) \cdot Q_s^{(0)}(z) \neq 0$ in $\Delta_\tau - \{o\}$. For $\alpha=(\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $|\alpha_j| = \tau_j$ $(1 \leqq j \leqq n)$, $P_m(\alpha_1 r, \alpha_2 r, \ldots, \alpha_n r)$ is the polynomial of degree t with respect to r. Then it is easily seen that there exists an $r_0' \geqq 1$ which is independent of α such that $|P_m(\alpha_1 r, \alpha_2 r, \ldots, \alpha_n r)| \geqq 1$ for all $r \geqq r_0'$. Since $P_\nu(z)$ is a polynomial there exists a positive constant K_ν and a positive integer l_ν such that

$$|P_\nu(\alpha_1 r, \alpha_2 r, \ldots, \alpha_n r)| \leqq K_\nu r^{l_\nu}$$

for all $r \geqq r_0'$. Put

$$K = max\,\{K_\nu;\, 0 \leqq \nu \leqq m-1\}\ ,\ l = max\,\{l_\nu\,;\, 0 \leqq \nu \leqq m-1\}\ .$$

Then

$$|P(\alpha_1 r, \alpha_2 r, \ldots, \alpha_n r, w)| \geqq |w|^{m} - K r^l \sum_{k=1}^{k=m} |w|^{m-k}$$

for all $r \geqq r_0'$. Let $r_0 = \max(K+1, r_0')$. Then it is easily seen that $P(\alpha_1 r, \alpha_2 r, \ldots, \alpha_n r, w) \neq 0$ for all $r \geqq r_0$ and for all w with $|w| \geqq r^{l+1}$. This means that for $z \in \Delta_\tau - B(r_0)$, any root of the equation $f(z, w)=0$ with respect to w is in $\{w \in \mathbf{C}\,;\, |w| < |z|^{l+1}\}$, where $B(r_0) = \{z \in \mathbf{C}^n\,;\, |z| < r_0\}$. Since $Q^{(0)}(z) \neq 0$ in $\Delta_\tau - \{o\}$, for any complex line $\pi \subset \Delta_\tau$ with $o \in \pi$, the number of the zeros of $P_0(z)|_\pi$ is independent of π. Consequently if A is algebraic then f satisfies the following condition (∗).

There exists a positive number $r_0 \geqq 1$, a positive integer l and a $\tau \in \mathbf{R}^n$ such that

(i) for each $z \in \Delta_\tau - B(r_0)$, any root of the equation $f(z, w)=0$ with respect to w is in $\{w \in \mathbf{C}\,;\, |w| < |z|^l\}$,

(ii) the number of the zeros of $f(z, o)|_\pi$ is independent of π, where π is a complex line passing through the origin and $f(z, o)|_\pi$ is the restriction of $f(z, o)$ to π.

The purpose of this paragraph is to show the converse of the above.

THEORM 2. *Let $f(z, w)$ be an entire function of two complex variables z, w and let $f(o) \neq 0$. Assume that there exists a positive number $r_0 \geqq 1$ and a positive integer l such that for each z with $|z| \geqq r_0$, any root of the equation $f(z, w)=0$ with respect to w is in $\{w \in \mathbf{C}\,;\, |w| < |z|^l\}$. Then* A *is algebraic.*

Proof. If $|z| \geqq r_0$ then the number of the zeros of $f(z, w)$ with respect to w is independent of z.

In fact, for any $r_1 > r_0$, $f(z, w) \neq 0$ in $\{(z, w) \in \mathbf{C}^2\,;\, r_0 \leqq |z| < r_1,\ |w| = r_1{}^l = \rho\}$. Then

$$N(z)=\frac{1}{2\pi i}\int_{|w|=\rho}\frac{\frac{\partial}{\partial w}f(z,w)}{f(z,w)}dw$$

is continuous and integer valued in an annulas $r_0\leqq|z|<r_1$, that is N(z) is constant in $|z|\geqq r_0$. Put $m:=N(z)$ for z with $|z|\geqq r_0$. Let $|z|>r_0$ and let $w_1(z)$, $w_2(z),\ldots,$-$w_m(z)$ be the zeros of f(z, w) with respect to w. Put

$$h_1(z)=-\sum_{j=1}^{j=m}w_j(z),\qquad h_2(z)=\sum_{j\neq k}w_j(z)w_k(z),\ldots$$
$$h_m(z)=(-1)^m\prod_{j=1}^{j=m}w_j(z).$$

Then $h_1(z)$, $h_2(z),\ldots,h_m(z)$ are holomorphic in $|z|>r_0$.

In fact, take a point z_0 with $|z_0|>r_0$. Then for z with $|z-z_0|<|z_0|-r_0$, m zeros of f(z, w) with respect to w are in $\{w\in\mathbf{C};|w|<(2|z_0|-r_0)^l=K\}$. Put

$$\Delta_1=\{z\in\mathbf{C};|z-z_0|<|z_0|-r_0\},\ \Delta_2=\{w\in\mathbf{C};|w|<K\}.$$

Since $A\cap(\Delta_1\times\partial\Delta_2)$ is empty, by Weierstrass Preparation theorem,

$$f(z,w)=(w^m+a_1(z)w^{m-1}+\ldots+a_m(z))e^{H(z,w)}$$

in $\Delta_1\times\Delta_2$, where $a_j(z)$ is holomorphic in Δ_1 and H(z, w) is holomorphic in $\Delta_1\times\Delta_2$. Then

$$A\cap(\Delta_1\times\mathbf{C})=\{(z,w)\in\mathbf{C}^2;w^m+a_1(z)w^{m-1}+\ldots+a_m(z)=0\}\cap(\Delta_1\times\mathbf{C}).$$

Since $h_1(z)$, $h_2(z),\ldots,h_m(z)$ are fundamental symmetric functions of $w_1(z)$, $w_2(z)$, ..., $w_m(z)$, it holds that $h_j(z)=a_j(z)$ in Δ_1. That is $h_j(z)$ $(1\leqq j\leqq m)$ are holomorphic in Δ_1. This holds at each point z with $|z|>r_0$, then $h_j(z)$ is holomorphic in $|z|>r_0$. Now, there exists a positive number ρ_0 such that $h_m(z)\neq0$ in $|z|>\rho_0$. In fact, expand $h_m(z)$ into Laurent series $h_m(z)=h_m^-(z)+h_m^+(z)$, where

$$h_m^-(z)=\sum_{\nu=1}^{\nu=\infty}c_{-\nu}z^{-\nu}\ ,\quad h_m^+(z)=\sum_{\nu=0}^{\nu=\infty}c_\nu z^\nu$$

If $h_m(z)$ is identically zero, then f(z, o)=0 in $|z|>r_0$. Then f(z,o) is identically zero and this is impossible. Therefore either h_m^- or h_m^+ is not identically zero. If $|z|\geqq r_0$ then $|w_j(z)|<|z|^l$, so that $|h_m(z)|<|z|^{ml}$. Then

$$|c_\nu|\leqq\frac{1}{2\pi}\int\frac{r^{ml}}{r^{\nu+1}}|d\zeta|=r^{ml-\nu}.$$

Since $r>r_0$ is arbitrary, $c_\nu=0$ if $\nu>ml$, so that $h_m^+(z)$ is a polynomial. Then it is easily seen that there exists a positive number $\rho_0>r_0$ such that $h_m(z)\neq0$ in $|z|>\rho_0$. Let r $>\rho_0$ and put

$$N_f(r,r^l)=\left(\frac{1}{2\pi}\right)^2\int_0^{2\pi}\int_0^{2\pi}\log|f(re^{i\theta},r^le^{i\phi})|\,d\theta d\phi$$

and

$$I(z)=\frac{1}{2\pi}\int_0^{2\pi} log\mid f(z,\ r^l e^{i\phi})\mid d\,\phi.$$

Then

$$N_f(r,\ r^l)=\frac{1}{2\pi}\int_0^{2\pi} I(re^{i\theta})\,d\,\theta.$$

Since $h_m(z)\neq 0$ in $\mid z\mid >\rho_0$, it holds that $f(z,\ o)\neq 0$ if $\mid z\mid =r$. Then

$$I(z)=log\mid f(z,\ o)\mid-\sum_{\nu=1}^{\nu=m} log\mid\frac{w_\nu(z)}{r^l}\mid=ml\ log\ r$$

$$+log\mid f(z,\ o)\mid-log\mid h_m(z)\mid.$$

The function $\log\mid h_m(z)\mid$ is harmonic in $\mid z\mid >\rho_0$, so that by the well known formula in an annulas $\rho_0<\mid z\mid <r_1$,

$$\frac{1}{2\pi}\int_0^{2\pi} log\mid h_m(te^{i\theta})\mid d\,\theta=a log\ t+b,$$

where $\rho_0<t<r_1$ and a, b are constants independent of t. This holds for all $r_1(>\rho_0)$, so that we have

$$\frac{1}{2\pi}\int_0^{2\pi} log\mid h_m(re^{i\theta})\mid d\,\theta=a log\ r+b$$

for all $r>\rho_0$. Since $f(z,\ o)\neq 0$ in $\mid z\mid >\rho_0$,

$$\frac{1}{2\pi}\int_0^{2\pi} log\mid f(re^{i\theta},\ o)\mid d\,\theta=k_1'\ log\ r+k_2'$$

for all $r>\rho_0$. Therefore we have

$$N_f(r,\ r^l)=ml\ log\ r+\frac{1}{2\pi}\int_0^{2\pi} log\mid f(re^{i\theta},\ o)\mid d\,\theta-a log\ r$$

$$-b=k_1 log\ r+k_2$$

where k_1 and k_2 are constants independent of $r>\rho_0$. Here we need the following

LEMMA 4(Ronkin [5] page 125 and 138). *Let f be an entire function of n complex variables and let $f(o)\neq 0$. Then for any $\tau=(\tau_1,\ \tau_2,\dots,\ \tau_n)\in\mathbf{R}_+^n$ with $\mid\tau\mid=1$, it holds that*

$$N_f(\tau_1 r,\ \tau_2 r,\dots,\ \tau_n r):=\left(\frac{1}{2\pi}\right)^n\int_0^{2\pi}\cdots\int_0^{2\pi} log\mid f(\tau_1 re^{i\theta_1},$$

$$\tau_2 re^{i\theta_2},\dots,\ \tau_n re^{i\theta_n})\,d\,\theta_1 d\,\theta_2\dots d\,\theta_n=N_{f,\Delta}(r).$$

Since $N_f(r_1,\ r_2)$ is monotone increasing with respect to r_1 and r_2, we have

$$N_f(\tau_1 r,\ \tau_2 r)\leqq N_f(r,\ r^l)\leqq k_1 log\ r+k_2$$

for all $r>\rho_0$ and for any fixed τ with $\mid\tau\mid=1$. Then A is algebraic by Theorem 1. q.e.d.

THEOREM 3. *Let* $f(z, w)$ *be an entire function of* $n+1$ *complex variables* $z=(z_1, z_2, \ldots, z_n)$ *and* w. *Let* $f(o, o) \neq 0$. *If* $f(z, w)$ satisfies the condition $(*)$, *then* $A= \{(z, w) \in \mathbf{C}^{n+1} ; f(z, w)=0\}$ *is algebraic.*

Proof. Put

$$t_1 = \frac{z_1}{\tau_1},\ t_2 = \frac{z_2}{\tau_2}, \ldots, t_n = \frac{z_n}{\tau_n},\ |\tau| =1,$$

and

$$g(t, w) = f(\tau_1 t_1, \tau_2 t_2, \ldots, \tau_n t_n, w).$$

If $g(t, w)=0$ for $t \in \Delta_0 \cap \{t \in \mathbf{C}^n;\ |t| \geqq \sqrt{n} \cdot r_0\}$, then $\Sigma |\tau_i t_i|^2 = \frac{|t|^2}{n} \geqq r_0^2$, so that $|w| < |t|^l$, where $\Delta_0= \{t \in \mathbf{C}^n; |t_1| = |t_2| =\ldots= |t_n|\}$. Since (ii) in the condition $(*)$ holds for $g(t, o)$ and Δ_0, we may assume that

$$\tau=(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}).$$

Put

$$\tilde{T}_n = \{\alpha \in \mathbf{C}^n ; |\alpha_1| = |\alpha_2| = \ldots = |\alpha_n| = \frac{1}{\sqrt{n}}\}$$

and

$$f_\alpha(v, w) = f(\alpha_1 v, \alpha_2 v, \ldots, \alpha_n v, w).$$

Then $f_\alpha(v, w)$ is an entire function of two complex variables v, w which is not identically zero. If $z \in \Delta_0$, then $z_j=\alpha_j v$ for some $\alpha \in \tilde{T}_n$ and v with $|v| = |z|$, so that any zero w of $f_\alpha(v, w)$ with respect to w satisfies the condition $|w| < |v|^l$ for all v with $|v| > r_0$. Then by Theorem 2, $A= \{(v, w) \in \mathbf{C}^2 ; f_\alpha(v, w)=0\}$ is algebraic. Put

$$f_\alpha(v_0, w) = \sum_{\nu=0}^{\nu=m(\alpha)} P_\nu^{(\alpha)}(v) exp(H_\alpha(v, w))$$

where $P_\nu^{(\alpha)}(v)$ is a polynomial and $P_{m(\alpha)}^{(\alpha)}(v) \not\equiv 0$. By the proof of Theorem 2, the number of the zeros of $f_\alpha(v, w)$ with respect to w is independent of v for each $\alpha \in \tilde{T}_n$ whenever $|v| > r_0$. Then $P_{m(\alpha)}^{(\alpha)}(v) \neq 0$ in $|v| > r_0$. Now, $m(\alpha)$ is independent of $\alpha \in \tilde{T}_n$. In fact, take any v_0 with $|v_0| > r_0$ and fix it. Then $m(\alpha)$ is the number of the zeros of $f_\alpha(v_0, w)$ for $\alpha \in \tilde{T}_n$. Take any $\beta \in \tilde{T}_n$. Since $f_\beta(v_0, w) \neq 0$ in $|w| \geqq |v_0|^l$, there exists a positive number ε such that

$$\varepsilon < min \{ |f_\beta(v, w)| ; |w| = |v_0|^l\} .$$

Since $f_\alpha(v_0, w)$ is continuous with respect to α and w, there exists an open neighbourhood $V \subset \tilde{T}_n$ of β such that $|f_\alpha(v_0, w)-f_\beta(v_0, w)| < \varepsilon$ if $\alpha \in V$ and $|w| = |v_0|^l$. Then by the theorem of Rouché, the number of the zeros of $f_\alpha(v_0, w)$ is the same as that of $f_\beta(v_0, w)$ for all $\alpha \in V$. Therefore $m(\alpha)=m(\beta)$ if $\alpha \in V$, so that by the connectedness of $\tilde{T}_n$, $m(\alpha)$ is independent of $\alpha \in \widetilde{T}_n$. put m: $=m(\alpha)$ and let $w_1(\alpha: v)$, $w_2(\alpha: v), \ldots, w_{k_\alpha}(\alpha: v)$ be the distinct zeros of $f_\alpha(v, w)$ with respect to w, where $|v|>r_0$. Let $m_1(\alpha)$, $m(\alpha), \ldots, m_{k_\alpha}(\alpha)$ be their multiplicities and put

$$h_\alpha(v)=[w_1(\alpha: v)]^{m_1(\alpha)}[w_2(\alpha: v)]^{m_2(\alpha)} \ldots [w_{k_\alpha}(\alpha: v)]^{m_{k_\alpha}(\alpha)}.$$

Then $h_\alpha(v)$ is continuous with respect to α and v in $\widetilde{T}_n \times \{v \in \mathbf{C} ; |v|>r_0\}$.

In fact, take any $\beta \in \widetilde{T}_n$ and any v_0 with $|v_0|>r_0$. Let δ_1 be a positive number such that $|w_i(\beta: v_0)-w_j(\beta: v_0)|>3 \delta_1$ if $i \neq j$. Then $f_\beta(v_0, w) \neq 0$ in $|w-w_j(\beta: v_0)|=\delta_1$ for any j, so that there exists a positive number δ_2 such that

$$\delta_2<\min_{1 \leqq j \leqq k_\beta} \min \{|f_\beta(v_0, w)| ; |w-w_j(\beta: v_0)|=\delta_1\}.$$

Now, for any given $\varepsilon>0$, we may assume that $\delta_1<\varepsilon$. Take a positive number $\delta<|v_0|-r_0$ such that $|f_\alpha(v, w)-f_\beta(v_0, w)|<\delta_2$ whenever $|\alpha-\beta|<\varepsilon$, $|v-v_0|<\delta$ and $|w-w_j(\beta: v_0)|<\delta_1$. Then the number of the zeros of $f_\alpha(v, w)$ in $|w-w_j(\beta: v_0)|<\delta_1$ conincides with $m_j(\beta: v_0)$, the multiplicity of $w_j(\beta: v_0)$, for each j whenever $|\alpha-\beta|<\delta$ and $|v-v_0|<\delta$. Since

$$m=\sum_{j=1}^{j=k_\beta} m_j(\beta: v_0),$$

any zero of $f_\alpha(v, w)$ is in some disc $\{w \in \mathbf{C} ; |w-w_j(\beta: v_0)|<\delta_1\}$ whenever $|\alpha-\beta|<\delta$ and $|v-v_0|<\delta$. That is to say, any zero $w(\alpha: v)$ of $f_\alpha(v, w)$ with respect to w satisfies the condition $|w(\alpha: v)-w_j(\beta: v_0)|<\delta_1<\varepsilon$ for some j whenever $|\alpha-\beta|<\delta$ and $|v-v_0|<\delta$. Then $h_\alpha(v)$ is continuous in $\widetilde{T}_n \times \{v \in \mathbf{C} ; |v|>r_0\}$.

Moreover, there exists a $\rho_0 \geqq r_0$ such that $h_\alpha(v) \neq 0$ if $|v|>\rho_0$.

In fact, by the proof of Theorem 2, for each $\alpha \in \widetilde{T}_n$, there is a $\rho_0(\alpha) \geqq r_0$ such that $h_\alpha(v) \neq 0$ if $|v| \geqq \rho_0(\alpha)$. Take any $\beta \in \widetilde{T}_n$ and any $\varepsilon>0$ such that $\varepsilon<\min \{|f_\beta(v, o)| ; |v|=\rho_0(\beta)\}$. Then there exists an open neighbourhood V of β such that $|f_\alpha(v, o)-f_\beta(v, o)|<\varepsilon$ if $\alpha \in V$ and $|v|=\rho_0(\beta)$. Then $f_\alpha(v, o)$ has the same number of the zeros as that of $f_\beta(v, o)$ in $|v|<\rho_0(\beta)$ for all $\alpha \in V$. Then by the condition (ii) of (*), $f_\alpha(v, o) \neq 0$ if $\alpha \in V$ and $|v| \geqq \rho_0(\beta)$. Therefore by the compactness of $\widetilde{T}_n$, we can choose a $\rho_0 \geqq r_0$ such that $h_\alpha(v) \neq 0$ if $|v| \geqq \rho_0$. Thus we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log |h_\alpha(re^{i\theta})| \, d\theta = k_1(\alpha) \log r + k_2(\alpha),$$

and $k_1(\alpha) \log r + k_2(\alpha)$ is continuous in $\widetilde{T}_n$ for each $r \geqq \rho_0$. Let $r_1>r_2>\rho_0$, then $k_1(\alpha)\log r_1 - k_1(\alpha)\log r_2$ is continuous in $\widetilde{T}_n$. Then $k_1(\alpha)$ and $k_2(\alpha)$ are continuous in $\widetilde{T}_n$. Put $k_i'=\min \{k_i(\alpha); \alpha \in \widetilde{T}_n\}$ $(i=1, 2)$. Then since we may assume that $\rho_0>e$,

$$\frac{1}{2\pi} \int_0^{2\pi} \log |h_\alpha(re^{i\theta})| \, d\theta > k_1' \log r + k_2' \qquad (3).$$

Because $h_\alpha(v) \neq 0$ in $|v| \geqq \rho_0$, it holds that $f_\alpha(v, o) \neq 0$ if $|v| \geqq \rho_0$. Then by the same method, the number of the zeros of $f_\alpha(v, o)$ with respect to v is independent of $\alpha \in \widetilde{T}_n$.

Let $\overline{m}$ be its number. By the same calculation as that in the proof of Theorem 2,

$$I(\alpha, r) := \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} log \mid f_\alpha(re^{i\theta}, r^l e^{i\phi}) \mid d\theta d\phi$$

$$= log \mid f_\alpha(o, o) \mid - \sum_{\nu=1}^{\nu=m} log \mid \frac{a_\nu}{r} \mid + ml \, log \, r$$

$$- \frac{1}{2\pi}\int_0^{2\pi} log \mid h_\alpha(re^{i\theta}) \mid d\theta$$

for all $r > \rho_0$, where a_ν is the zero of $f_\alpha(v, o)$. Since $f_\alpha(o, o) \neq 0$, there exists a positive number δ which is independent of $\alpha \in \widetilde{T}_n$ such that $\mid a_\nu \mid > \delta$ for all ν, so that $\log \mid a_\nu \mid > \log \delta$. Then by (3)

$$I(\alpha, r) \leqq (ml + \overline{m} - k_1') log \, r - \overline{m} log \, \delta + log \mid f_\alpha(o, o) \mid$$
$$- k_2' = k_1 log \, r + k_2$$

where k_1 and k_2 are constants independent of r. Put

$$\alpha_1 = \frac{1}{\sqrt{n}} e^{i\theta_1}, \; \alpha_2 = \frac{1}{\sqrt{n}} e^{i\theta_2}, \ldots, \alpha_n = \frac{1}{\sqrt{n}} e^{i\theta_n}, \; v = re^{i\theta},$$

then

$$\left(\frac{1}{2\pi}\right)^{n+2} \int_0^{2\pi} \cdots \int_0^{2\pi} log \mid f(\frac{1}{\sqrt{n}} re^{i(\theta_1+\theta)}, \frac{1}{\sqrt{n}} re^{i(\theta_1+\theta)}, \ldots,$$

$$\frac{1}{\sqrt{n}} re^{i(\theta_n+\theta)}, r^l e^{i\phi}) \mid d\theta d\phi d\theta_1 d\theta_2 \ldots d\theta_n$$

$$= \left(\frac{1}{2\pi}\right)^{n+1} \int_0^{2\pi} \cdots \int_0^{2\pi} log \mid f(\frac{1}{\sqrt{n}} re^{i\phi_1}, \frac{1}{\sqrt{n}} re^{i\phi_1}, \ldots, \frac{1}{\sqrt{n}} re^{i\phi_n},$$

$$r^l e^{i\phi}) \mid d\phi_1 d\phi_2 \ldots d\phi_n d\phi = N_f(\frac{r}{\sqrt{n}}, \ldots, \frac{r}{\sqrt{n}}, r^l).$$

Now,

$$\left(\frac{1}{2\pi}\right)^{n+2} \int_0^{2\pi} \cdots \int_0^{2\pi} log \mid f(\frac{1}{\sqrt{n}} re^{i(\theta_1+\theta)}, \ldots, \frac{1}{\sqrt{n}} re^{i(\theta_n+\theta)},$$

$$r^l e^{i\phi}) \, d\theta_1 d\theta_2 \ldots d\theta_n \, d\theta d\phi = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} \left[\left(\frac{1}{2\pi}\right)^2\right.$$

$$\left. log \mid f_\alpha(re^{i\theta}, r^l e^{i\phi}) \mid d\theta d\phi \right] d\theta_1 d\theta_2 \ldots d\theta_n$$

$$= \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} I(\alpha, r) d\theta_1 d\theta_2 \ldots d\theta_n \leqq k_1 \log r + k_2.$$

Therefore we have

$$N_f(\frac{r}{\sqrt{n}}, \ldots, \frac{r}{\sqrt{n}}, r) \leqq k_1 log \, r + k_2.$$

Put

$$\tau=(\frac{1}{\sqrt{n+1}},\ldots,\frac{1}{\sqrt{n+1}})\in \mathbf{R}_{+}^{n+1},$$

then it holds that

$$N_{f,\Delta_\tau}(r)=N_f(\frac{1}{\sqrt{n+1}},\ldots,\frac{1}{\sqrt{n+1}},\frac{r}{\sqrt{n+1}})\leqq N_f(\frac{r}{\sqrt{n}},\ldots,\frac{r}{\sqrt{n}},r^l)$$
$$\leqq k_1 \log r+k_2$$

for all $r>\rho_0$. Thus A is algebraic by Theorem 1. q.e.d.

REMARK. *Let* $f(z, w)$ *be a pseudopolynomial given by*

$$f(z,w)=a_0(z)w^m+a_1(z)w^{m-1}+\ldots+a_m(z),$$

where $a_0(o)\neq 0$ *and* $a_j(z)$ $(0\leqq j\leqq m)$ *are entire functions of n complex variables* $z=(z_1, z_2,\ldots,z_n)$. *If f satisfies the condition* (i) *of* $(*)$, *then* $A=\{(z,w)\in C^{n+1};f(z,w)=0\}$ *is algebraic.*

Proof. From the assumption $a_0(z)\neq 0$ if $z\in\Delta_\tau$ and $|z|>r_0$. In fact, let $a_0(z^0)=0$ for some $z^0\in\Delta_\tau$ with $|z^0|>r_0$. Then since $a_0(o)\neq 0$, it is easily seen that $a_0(z)$ is not identically zero in $\Omega\cap\Delta_\tau$ for any open neighbourhood Ω of z_0. By the condition (i), there exists a j such that $a_j(z^0)\neq 0$. Let $a_1(z_0)\neq 0$ for simplicity. Take a bounded neighbourhood V of z^0 such that $V\cap\Delta_\tau\subset\Delta_\tau-\overline{B(r_0)}$ and that $|a_1(z)|\geqq k>0$ in V. Then there exists a constant $K>0$ such that any zero w of f(z, w) satisfies the inequality $|w|<K$ if $z\in\Delta_\tau\cap V$. Then

$$\left|\frac{a_1(z)}{a_0(z)}\right|<mK$$

if $z\in\Delta_\tau\cap V$ and $a_0(z)\neq 0$. But this is a contradiction since we can choose a sequence $\{z^{(\nu)}\}$ in Δ_τ such that $a_0(z^{(\nu)})\neq 0$ and that $z^{(\nu)}\to z_0$. That is $a_0(z)\neq 0$ if $z\in\Delta_\tau$ and $|z|>r_0$. Then since the analytic set $\{z\in C^n; a_0(z)=0\}$ is algebraic, there exists an entire function H(z) and a polynomial P(z) such that $a_0(z)=P(z)\exp(H(z))$. We shall show that $a_j(z)\exp(-H(z))$ is a polynomial. Take any $\alpha\in\widetilde{T}_n$ and fix it. Let $|z|>r_0$ and let $w_1(v), w_2(v),\ldots,w_m(v)$ be the zeros of $f_\alpha(v,w)$ with respect to w. Put

$$h_\alpha(v)=-\sum_{j=1}^{j=m}w_j(v),\ \widetilde{a}_j(z)=a_j(z)exp(-H(z)).$$

Then by the proof of Theorem 2, $h_\alpha(v)$ is holomorphic in $|v|>r_0$ and its Laurent series is given by

$$h_\alpha(v)=Q(v)+\sum_{\nu=1}^{\nu=\infty}c_{-\nu}v^{-\nu},$$

where Q(v) is a polynomial of degree at most l.

On the other hand, since $a_1(\alpha v)=a_0(\alpha v)h_\alpha(v)$ in $|v|>r_0$,

$$\tilde{a}_1(\alpha v) = R(v) + \sum_{\nu=1}^{\nu=\infty} d_{-\nu} v^{-\nu},$$

where R(v) is a polynomial of degree at most μ which is independent of $\alpha \in T_n$. Then $d_{-\nu} = 0$ for all ν since $a_1(z)$ is an entire function, that is $\tilde{a}_1(\alpha v)$ is a polynomial of degree at most μ for all $\alpha \in \tilde{T}_n$. Therefore $\tilde{a}_1(z)$ is a polynomial because $\tilde{T}_n$ is a set of uniqueness for holomorphic functions. Entire functions $\tilde{a}_2(z), \tilde{a}_3(z), \ldots, \tilde{a}_m(z)$ are also polynomials by the same manner. Q.E.D.

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