Sci. Rep. Kanazawa Univ.
Vol. 23 No. 1 pp. 1—6
June 1978

Lipschitz Functions and Convolution on Bounded Vilenkin Groups

Yoshikazu UNO

Department of Mathematics, Faculty of Science, Kanazawa University

(Received May 11, 1978)

Abstract. Let G be a bounded Vilenkin group. Suppose that $1 \leqq p \leqq 2$ and $1/p+1/q=1$. We shall prove that Lip $(\alpha, q) \subset L^p(G) * L^q(G)$ for $\alpha > 1/q$ and Lip $(\alpha, \infty) \cap (q-\varepsilon)-\mathrm{GBF}(G) \subset L^p(G) * L^q(G)$ for $\alpha > 0$ and $\varepsilon > 0$.

1. Introduction.

Let G be an infinite, compact, metrizable, zero-dimensional abelian group. Then the dual group X of G is an infinite, discrete, countable, torsion abelian group. In this paper we shall consider the analogue on G of results obtained for the circle group in [4].

2. Notations and Definitions.

N. Ja. Vilenkin [5] proved that there exist a sequence $\{X_n\}$ of finite subgroups of X and a sequence $\{\varphi_n\}$ of characters in X such that the followings hold.

(i) $X_0 \subset X_1 \subset X_2 \subset \cdots$ and $X = \bigcup_{n=0}^{\infty} X_n$.
(ii) $X_0 = \{\chi_0\}$ where $\chi_0(x)=1$ for all $x \in G$.
(iii) The quotient group X_{n+1}/X_n is of prime order p_{n+1} for each $n \geqq 0$.
(iv) $\varphi_n \in X_{n+1} \setminus X_n$ for all $n \geqq 0$.
(v) $\varphi_n^{p_{n+1}} \in X_n$ for every $n \geqq 0$.

Let $m_0=1$ and let $m_n = p_1 p_2 \cdots p_n$ for all $n \geqq 1$. Every positive integer k has the representation $k = \sum_{j=0}^{s} a_j m_j$ where $0 \leqq a_j < p_{j+1}$ for $0 \leqq j \leqq s$. Then we put $\chi_k = \varphi_0^{a_0} \varphi_1^{a_1} \cdots \varphi_s^{a_s}$. It follows that $X = \{\chi_0, \chi_1, \chi_2, \cdots\}$ and $X_n = \{\chi_j ; 0 \leqq j < m_n\}$. The annihilator of X_n is denoted by G_n. We see that $G = G_0 \supset G_1 \supset G_2 \supset \cdots$, $\bigcap_{n=0}^{\infty} G_n = \{0\}$ and $\{G_n\}$ forms an open neighbourhood base of zero in G. For every $n \geqq 0$, there exists an $x_n \in G_n \setminus G_{n+1}$ such that $\chi_{m_n}(x_n) = \varphi_n(x_n) = \exp(2\pi i/p_{n+1})$ and each $x \in G$ is represented uniquely by $x = \sum_{j=0}^{\infty} b_j x_j$ where $0 \leqq b_j < p_{j+1}$ for all $j \geqq 0$. Then $G_n = \{x \in G ; x = \sum_{j=n}^{\infty} b_j x_j$ where $0 \leqq b_j < p_{j+1}$

for all $j \geqq n\}$ and every coset of G_n is represented by $z+G_n$ where $z=\sum_{j=0}^{n-1} b_j x_j$ for some b_j with $0 \leqq b_j < p_{j+1}$. We denote these z ordered lexiographically by $z_{q,n}$, $0 \leqq q < m_n$.

Each coset of G_n in G_l has the form $z_{q,n}+G_n$ for some $0 \leqq q < m_n/m_l$ where $0 \leqq l < n$.

Let dx be the normalized Haar measure on G. The Fourier series of a function f in $L^1(G)$ is the series $S(f)(x)=\sum_{k=0}^{\infty} \hat{f}(k)\chi_k(x)$ where $\hat{f}(k)=\int_G \hat{f}(x)\overline{\chi_k(x)}\, dx$. Let $S_n(f)$ be the n-th partial sum of $S(f)$, then we have the formula

$$S_n(f)(x)=\sum_{k=0}^{n-1} \hat{f}(k)\chi_k(x)=\int_G f(x-y)D_n(y)dy$$

where $D_n(y)=\sum_{k=0}^{n-1} \chi_k(y)$ which is called the Dirichlet kernel of order n.

Let f be a function defined on G. For $H \subset G$, we put

$$\operatorname{osc}(f, H)=\sup\{\,|f(x)-f(y)| : x, y \in H\}\ .$$

If $n \geqq 0$ and $1 \leqq p \leqq \infty$, we write

$$\omega_n^{(p)}(f)=\sup\{\,\|f_y-f\|_p : y \in G_n\}$$

where $f_y(x)=f(x-y)$ for $x, y \in G$. For $r \geqq 1$, we denote

$$V_r(f)=\sup\{[\sum_{q=0}^{m_n-1}\{\operatorname{osc}(f, z_{q,n}+G_n)\}^r]^{1/r};\ n=0,1,2,\cdots\}.$$

Definition 1. For $\alpha > 0$ and $1 \leqq p \leqq \infty$, we define

$$\operatorname{Lip}(\alpha, p)=\{f \in L^p(G)\,;\ \omega_n^{(p)}(f)=O(m_n^{-\alpha})\}\,.$$

Definition 2. A function f with $V_r(f) < \infty$ is called of r-generalized bounded fluctuation and the set of such all functions is denoted by r-GBF (G).

Through the present paper, we assume that $\rho=\sup_n p_n$ is finite.

3. Lemmas.

Lemma 1. *For* $1 \leqq p \leqq \infty$ *and* $1/p+1/q=1$,

$$\|D_n\|_p=O(n^{1/q}(\log n)^{1/p}).$$

Proof. It is trivial that $\|D_n\|_\infty=O(n)$. Let $1 \leqq p < \infty$ and let $n=\sum_{j=0}^{k} a_j m_j$ $(0 \leqq a_j < p_{j+1},\ a_k \neq 0)$. If $x \in G_{s-1} \setminus G_s$ for some $1 \leqq s \leqq k$, then $|D_n(x)| \leqq m_s$ ([5; § 3.61]). Thus we have

$$\|D_n\|_p^p=\int_G |D_n(x)|^p dx$$

$$=\sum_{s=1}^{k}\int_{G_{s-1}\setminus G_s}|D_n(x)|^p dx+\int_{G_k}|D_n(x)|^p\, dx$$

$$\leqq \sum_{s=1}^{k} \frac{m_s^p}{m_{s-1}} + \frac{n^p}{m^k} \leqq \rho\left(\sum_{s=1}^{k} m_s^{p-1} + n^{p-1}\right)$$
$$\leqq \rho(m_k^{p-1}k + n^{p-1}) \leqq 2\rho\, n^{p-1}k \leqq \frac{2\rho}{\log 2} n^{p-1} \log n$$

since $2^k \leqq m_k \leqq \sum_{j=0}^{k} a_j m_j = n$.

LEMMA 2. *Let* $1 \leqq p \leqq \infty$, $1/p + 1/q = 1$ *and let* $\beta > 1/q$, *then there exists a function* g *in* $L^p(G)$ *such that* $\hat{g}(n) = (n+1)^{-\beta}$ $(n = 0,1,2,\cdots)$.

PROOF. The series $\sum_{n=1}^{\infty} (n^{-\beta} - (n+1)^{-\beta})\, D_n(x)$ converges in $L^p(G)$ by Lemma 1. It is easy to see that the sum of this series is a required function.

LEMMA 3. *Let* $0 < \alpha \leqq 1$, $1 \leqq p \leqq \infty$ *and let* $f \in \mathrm{Lip}\,(\alpha, p)$, *then* $\| S_n(f) - f \|_p = O(n^{-\alpha} \log n)$.

PROOF. N. Ja. Vilenkin ([5; § 3.5]) showed that

$$| S_n(f)(x) - f(x) |$$
$$\leqq \sum_{l=0}^{k-1} a_l m_l \sum_{q=0}^{m_k/m_l - 1} \left| \int_{z_{q,k} + G_k} \{\theta(x,y) - \theta(x, z_{q,k})\}\ \chi_n(y)dy \right|$$
$$+ a_k m_k \int_{G_k} | \theta(x, y) |\, dy.$$

where $n = \sum_{l=0}^{k} a_l m_l$ $(0 \leqq a_l < p_{l+1},\ a_k \neq 0)$ and $\theta(x, y) = f(x) - f(x-y)$. We therefore obtain that for $f \in \mathrm{Lip}\,(\alpha, p)$,

$$\| S_n(f) - f \|_p \leqq K\left(\sum_{l=0}^{k-1} a_l m_l m_k^{-\alpha} m_k^{-1} \frac{m_k}{m_l} + a_k m_k m_k^{-\alpha} m_k^{-1}\right)$$
$$\leqq K\left(m_k^{-\alpha} \sum_{l=0}^{k} a_l\right) \leqq K\rho\,(m_k^{-\alpha} k)$$
$$= K \frac{\rho^{1+\alpha}}{\log 2} n^{-\alpha} \log n$$

since $a_l \leqq \sup p_n = \rho$ and $n \geqq 2^k$ where K is some constant.

4. Main results.

THEOREM 1. *Let* $1 \leqq p \leqq \infty$, $1/p + 1/q = 1$, $1 \leqq r \leqq \infty$ *and let* $f \in L^r(G)$. *Assume that there exists a* $\beta > 1/q$ *such that* $\sum_{n=1}^{\infty} n^{\beta-1} \| S_n(f) - f \|_r < \infty$ *and* $\lim_{n\to\infty} n^{\beta} \| S_n(f) - f \|_r = 0$. *Then* $f \in L^p(G) * L^r(G)$.

PROOF. There exists a function g in $L^p(G)$ such that $\hat{g}(n) = (n+1)^{-\beta}$ $(n = 0,1,2,\cdots)$ by Lemma 2. We put $\mu_n = \hat{g}(n)^{-1}$ $(n = 0,1,2,\cdots)$ and we define

$$T_k(x) = \sum_{n=0}^{k} \mu_n \hat{f}(n) \chi_k(x).$$

By Abel transformation we have

$$T_k(x) = \sum_{n=0}^{k-1}(\mu_n - \mu_{n+1})\, S_{n+1}(f)\,(x) - \mu_k S_k(f)\,(x)$$
$$= \sum_{n=0}^{k-1}(\mu_n - \mu_{n+1})\,(S_{n+1}(f)\,(x) - f(x)) + \mu_k(S_k(f)\,(x) - f(x)) + \mu_0 f(x).$$

Since $\mu_n - \mu_{n+1} = O(n^{\beta-1})$, T_k converges to some function h in $L^r(G)$ by assumptions. It follows that $\hat{f}(n) = \hat{g}(n)\hat{h}(n)$ for all $n \geqq 0$ and consequently $f = g * h \in L^p(G) * L^r(G)$.

COROLLARY 1. *Let* $1 \leqq p \leqq 2$ *and* $1/p + 1/q = 1$. *Then*

$$\mathrm{Lip}\,(\alpha, q) \subset L^p(G) * L^q(G) \quad \textit{for } \alpha > 1/q.$$

PROOF. If $f \in \mathrm{Lip}\,(\alpha, q)$, then $\| S_n(f) - f \|_q = O(n^{-\alpha} \log n)$ by Lemma 3. We choose a β with $\alpha > \beta > 1/q$. Then $\sum_{n=1}^{\infty} n^{\beta-1} \|S_n(f) - f\|_q < \infty$ and $\lim_{n\to\infty} n^\beta \| S_n(f) - f\|_q = 0$. Thus $f \in L^p(G) * L^q(G)$ by Theorem 1.

COROLLARY 2. *If* $1 \leqq p \leqq 2$ *and* $1/p + 1/q = 1$, *then*

$$\mathrm{Lip}\,(\alpha, \infty) \cap (q-\varepsilon)-\mathrm{GBF}(G) \subset L^p(G) * L^q(G)$$

for all $\alpha > 0$ *and* $\varepsilon > 0$.

PROOF. If $f \in r-\mathrm{GBF}(G)$, it holds that for $y \in G_n$,

$$\int_G | f(x-y) - f(x) |^r\, dx = \sum_{q=0}^{m_n - 1} \int_{z_{q,n} + G_n} | f(x-y) - f(x) |^r\, dx$$
$$\leqq \sum_{q=0}^{m_n-1} \mathrm{osc}(f, z_{q,n} + G_n)^r m_n^{-1} \leqq V_r(f)^r\, m_n^{-1}.$$

Thus $\omega_n^{(r)}(f) = O(m_n^{-1/r})$. If $f \in \mathrm{Lip}\,(\alpha, \infty) \cap (q-\varepsilon)-\mathrm{GBF}(G)$ for some $\alpha > 0$ and $\varepsilon > 0$, then for $y \in G_n$ we have

$$\| f_y - f\|_q = \{\int_G | f(x-y) - f(x) |^q dx\}^{1/q}$$
$$\leqq \{\omega_n^{(\infty)}(f)^\varepsilon \| f_y - f\|_{q-\varepsilon}^{q-\varepsilon}\}^{1/q}$$
$$= O(m_n^{-\alpha\varepsilon} m_n^{-1})^{1/q} = O(m_n^{-(1+\alpha\varepsilon)/q}).$$

Therefore $f \in \mathrm{Lip}\,((1+\alpha\varepsilon)/q,\, q)$ and then $f \in L^p(G) * L^p(G)$ by Corollary 1.

5. Remark.

S. V. Bochkarev [1, Theorem 1.4] proved that for $\alpha \in (0, 1)$ and a complete orthonormal system $\{\psi_n\}$ on $[0, 1]$, there exists a function F belonging to Lip α on $[0, 1]$ such that

$$\sum_{n=0}^{\infty} | \int_0^1 F(t)\, \overline{\psi_n(t)} dt |^{2/(2\alpha+1)} = \infty$$

Let λ be the mapping of G onto $[0, 1]$ defined by

$$\lambda(x)=\sum_{j=0}^{\infty}\frac{b_j}{m_{j+1}} \qquad \text{for } x=\sum_{j=0}^{\infty} b_j x_j \in G$$

where the representation of x is as in § 2. The mapping λ is injective except for a countable set $E\subset G$. Thus we can consider the inverse mapping λ^{-1} of $[0, 1]$ onto $G\backslash E$. N. Ja. Vilenkin [5, § 3.21] proved that these mappings are of measure preserving. Therefore if $\psi_n(t)=\chi_n(\lambda^{-1}(t))$ for $t\in[0, 1]$, then $\{\psi_n\}$ is a complete orthonormal system on $[0, 1]$. For $1\leqq p\leqq 2$, $1/p+1/q=1$, by the above S. V. Bochkarev's result, there exists a function F belonging to Lip $1/q$ on $[0, 1]$ such that

$$\sum_{n=0}^{\infty}\left|\int_0^1 F(t)\overline{\psi_n(t)}dt\right|^{2p/(3p-2)}=\infty.$$

We define $f(x)=F(\lambda(x))$ for $x\in G$. It is easy to see that $f\in \mathrm{Lip}\,(1/q, \infty)$ and $\hat{f}(n)=\int_G f(x)\overline{\chi_n(x)}dx=\int_0^1 F(t)\overline{\psi_n(t)}\,dt$. Thus we have

$$\sum_{n=1}^{\infty}|\hat{f}(n)|^{2p/(3p-2)}=\infty.$$

If $f\in L^p(G) * L^2(G)$ for $1\leqq p\leqq 2$, then Hausdorff-Young and Hölder's inequalities imply that $\sum_{n=0}^{\infty}|\hat{f}(n)|^{2p/(3p-2)}<\infty$. Therefore the above function f does not belong to $L^p(G) * L^2(G)$ and so Corollary 1 does not hold as $\alpha=1/q$. Since Lip $(1/q, \infty)$ is contained in q-GBF(G), Corollary 2 does not hold as $\alpha=1/q$ and $\varepsilon=0$.

If G is primary, these facts are also proved by making use a function due to C. W. Onneweer [2, Theorem 6].

This observation implies that our results contain some ones of C. W. Onneweer [2,3].

6. A generalization of Theorem 1.

Let $1\leqq p_1\leqq \cdots \leqq p_m\leqq\infty$, $1/p_1+\cdots+1/p_m=m-1$ $(m\geqq 2)$ *and* $1/p_j+1/p_j'=1(1\leqq j\leqq$ If $\beta>1/p_m$, then we can write $\beta=\beta_1+\cdots+\beta_{m-1}$ with $\beta_j>1/p_j'$ $(1\leqq j\leqq m-1)$ since $1/p_1'+\cdots+1/p_{m-1}'=1/p_m$.

We can find functions $g_j\in L^{p_j}(G)$ such that $\hat{g}_j(n)=(n+1)^{-\beta_j}$ for $n=0,1,2,\cdots$ and $1\leqq j\leqq m-1$ by Lemma 2. If g is the function in the proof of Theorem 1, then $g=g_1 * \cdots * g_{m-1}$. Thus we obtain a following generalization of Theorem 1.

THEOREM 2. *Let $\{p_j\}_{j=1}^m$ be as in above and $f\in L^r(G)$ $(1\leqq r\leqq\infty)$. If there exists a $\beta>1/p_m$ such that*

$$\sum_{n=1}^{\infty} n^{\beta-1}\|S_n(f)-f\|_r<\infty \quad \textit{and} \quad \lim_{n\to\infty} n^{\beta}\|S_n(f)-f\|_r=0,$$

then $f\in L^{p_1}(G) * \cdots * L^{p_{m-1}}(G) * L^r(G)$.

COROLLARY 3. *Let $\{p_j\}_{j=1}^m$ be as in Theorem 2. We have*

$$\mathrm{Lip}\,(\alpha,\ p_m) \subset L^{p_1}(G) * \cdots * L^{p_m}(G) \qquad \textit{for } \alpha > 1/p_m.$$

COROLLARY 4. *Let* $\{p_j\}_{j=1}^{m}$ *be as in Theorem* 2. *Then*

$$\mathrm{Lip}\,(\alpha,\ \infty) \cap (p_m - \varepsilon)\text{-}\mathrm{GBF}(G) \subset L^{p_1}(G) * \cdots * L^{p_m}(G)$$

for all $\alpha > 0$ *and* $\varepsilon > 0$.

Here we note that $L^{p_1}(G) * \cdots * L^{p_{m-1}}(G) \subset L^{p'_m}(G)$ in Theorem 2 (see [6, p. 38]).

References

[1] S. V. Bochkarev, Absolute convergence of Fourier series with respect to complete orthonormal systems, Russian Math. Surveys, **27** (1972), 58–81.

[2] C. W. Onneweer, Absolute convergence of Fourier series on certain groups, Duke Math. Journ., **39** (1972), 599–609.

[3] C. W. Onneweer, Absolute convergence of Fourier series on certain groups, II, Duke Math. Journ., **41** (1974), 679–688.

[4] Y. Uno, Lipschitz functions and convolution, Proc. Japan Acad., **50** (1974), 785–788.

[5] N. Ja. Vilenkin, On a class of complete orthonormal system, Amer. Math. Soc. Transl., **28** (1963), 1–35.

[6] A. Zygmund, *Trigonometric series*, vol I, Cambridge Univ. Press, 1959.