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On the Schur Multiplicators and Central Subgroups

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Abstract The aim of the present paper is to give some inequalities for the order of the Schur multiplicator $M(G)=H^2(G, \mathrm{T})$ of a finite group G, where $\mathrm{T}=\mathbf{Q}/\mathbf{Z}$.

1. Introduction

The following notation will be used: e is the identity element of G. If X and Y are subsets of G, $[X, Y]$ denotes a subgroup of G generated by the elements $[x, y]=x^{-1}y^{-1}xy$, where $x \in X$ and $y \in Y$. The lower central series $\{G_i\}$, $i=0, 1, 2, \cdots$, is defined by $G_0=G$, $G_1=G'=[G, G]$, $G_{i+1}=[G_i, G]$. $Z(G)$, or Z, is the center of G. The upper central series $\{Z_i\}$, $i=0, 1, 2, \cdots$, is defined by $Z_0=\{e\}$, $Z_1=Z(G)$, $Z_{i+1}/Z_i=Z(G/Z_i)$. G is nilpotent of class n iff $G_{n-1}\neq G_n=\{e\}$, or equivalently, iff $Z_{n-1}\neq Z_n=G$. Tensor product $\otimes$ is taken over $\mathbf{Z}$.

In [5] Jones gave the following inequality for a central subgroup A of G, using the free presentation:

$$|A\cap G'|\cdot|M(G)|\leqq|M(A)|\cdot|M(G/A)|\cdot|(G/G'A)\otimes A|.$$

As he stated there, the same result can be obtained via the spectral sequence. We shall prove the following theorem as an improvement of the above, using exact sequences by Iwahori-Matsumoto [3] and in Proposition 2 below.

THEOREM 1. *For a central subgroup A of G, we have*

$$|A\cap G'|\cdot|M(G)|\leqq|M(A/A\cap G')|\cdot|M(G/A)|\cdot|(G/G'A)\otimes A|.$$

One can see that moreover if $G=G'$ and if G has a non-trivial central subgroup A, then we have $|A|\cdot|M(G)|=|M(G/A)|$, in particular, G/A has non-trivial multiplicator.

For a nilpotent group G of class n, Vermani [8] used his exact sequence to have the following inequality (See also Gaschütz-Neubüser-Yen [2]):

$$(N_1) \qquad |G'| \cdot |M(G)| \leqq |M(G/G')| \cdot |G'|^{d(G/Z)},$$

where $d(G)$ is the minimum number of generators of G.

Another estimate given by Jones [5] is

$$(N_2) \qquad |G'| \cdot |M(G)| \leqq |M(G/G')| \cdot \prod_{i=1}^{n-1} |(G/G')\otimes(G_i/G_i')|.$$

In the present paper we shall further give the following two inequalities of this nature, one obtained from an exact sequence of Vermani and the other from Theorem 1.

THEOREM 2. *If G is nilpotent of class n, then we have*

$$(N_3) \qquad |G'| \cdot |M(G)| \leqq |M(G/G')| \cdot \prod_{i=1}^{n-1} |(G/G'Z_i)\otimes(G_i/G_{i+1})|.$$

THEOREM 3. *In the same assumption as above, we have*

$$(N_4) \qquad |G'| \cdot |M(G)| \leqq |M(G/Z_{n-1})| \cdot \prod_{i=1}^{n-1} (|M(G'Z_i/G'Z_{i-1})| \cdot |(G/G'Z_i)\otimes(Z_i/Z_{i-1})|).$$

2. Exact Sequences

Let A be an abelian group, written multiplicatively, taken as a $\mathbb{Z}$-module through $r \cdot a = a^r$ for $a \in A$ and $r \in \mathbb{Z}$. Let $\hat{}(A)$ denote the second antisymmetric product of A, i.e. the factor group of $A\otimes A$ by $N(A)$, where $N(A)$ is generated by the elements of form $a\otimes a$ and $(a\otimes b)\cdot(b\otimes a)$. We denote by $a\wedge b$ the coset containing $a\otimes b$.

When A is an abelian subgroup of a group G, we shall define a homomorphism $\varphi_{G,A}$ of $M(G)$ to Hom $(\hat{}(A), \mathrm{T})$ as follows:

First take a 2-cocycle f of G, and define αf: $A\times A \to \mathrm{T}$ by

$$(\alpha f)(a, b) = f(a, b) - f(b, a) \text{ for } a, b \in A.$$

Then αf is bilinear, since we have easily

$$\begin{aligned}(1) \qquad (\alpha f)(a, bc) &= f(a, bc) - f(bc, a) \\ &= -f(b, c) + f(ab, c) + f(a, b) - f(c, a) - f(b, ca) + f(b, c) \\ &= f(a, b) + f(ba, c) - f(b, ac) - f(c, a) \\ &= f(a, b) + f(a, c) - f(b, a) - f(c, a) \\ &= (\alpha f)(a, b) + (\alpha f)(a, c),\end{aligned}$$

and

$$(1)' \qquad \begin{aligned}(\alpha f)(bc, a) &= -(\alpha f)(a, bc)\\ &= -(\alpha f)(a, b)-(\alpha f)(a, c)\\ &= (\alpha f)(b, a)+(\alpha f)(c, a)\end{aligned}$$

for $a, b, c \in A$.

Thus αf induces $\overline{\alpha}f \in Hom\ (A \otimes A,\ \mathrm{T})$ such that $(\overline{\alpha}f)\ (a \otimes b)=(\alpha f)\ (a,\ b)$. Since $\overline{\alpha}f$ vanishes on $N(A)$, it induces $\overline{\overline{\alpha}}f \in Hom\ (\wedge(A),\ \mathrm{T})$ such that $(\overline{\overline{\alpha}}f)\ (a \wedge b)=(\alpha f)\ (a,\ b)$.

Finally if f is a coboundary of h, say, then

$$\begin{aligned}(\overline{\overline{\alpha}}f)\ (a \wedge b) &= (\delta h)\ (a,\ b)-(\delta h)\ (b,\ a)\\ &= (h(a)-h(ab)+h(b)\ )-(h(b)-h(ba)+h(a)\ )\\ &= 0.\end{aligned}$$

Now we can define $\varphi_{G,A}\ (\xi)=\overline{\overline{\alpha}}f$ where f is a representative cocycle of $\xi \in M(G)$.

Proposition 1. *For an abelian group A, $\varphi_{A;A}$ is an isomorphism and we have*

$$(2) \qquad M(A) \cong Hom\ (\wedge(A),\ \mathrm{T}).$$

Proof. See Yamazaki [7, corollary 2 to Theorem 2.2].

Remark. Let an abelian group A be decomposed into a direct product $\sum_p A_p$, where A_p is the Sylow p-subgroup of A. It is easy to see $M(A) \cong \wedge(A) \cong \sum_p \wedge(A_p)$. Denote by $\mathbf{Z}(n)$ a cyclic group of order n, and by $\{\mathbf{Z}(n)\}^m$ the direct product of m copies of $\mathbf{Z}(n)$. If $A=\sum_{i=1}^{r} \mathbf{Z}(p^{e_i})$ with $e_1 \geqq e_2 \geqq \cdots \geqq e_r$, then $\wedge(A) \cong \sum_{i=2}^{r} \{\mathbf{Z}(p^{e_i})\}^{i-1}$, which is a well-known result of Schur [6].

The restriction homomorphism $res_{G,A} : M(G) \to M(A)$ for a subgroup A of G induces an exact sequence:

$$(3) \qquad 0 \to M(G)^*_A \to M(G) \to M(A) \quad ,$$

where $M(G)^*_A$ is the kernel of $res_{G,A}$. If A is abelian, then the relation $\varphi_{G,A}=\varphi_{A,A} \circ res_{G,A}$ and Proposition 1 imply $ker(\varphi_{G,A})=ker(res_{G,A})$. Hence we have an exact sequence:

$$(3)' \qquad 0 \to M(G)^*_A \to M(G) \to Hom\ (\wedge(A),\ \mathrm{T}).$$

Proposition 2. *If A is central in G, then we have an exact sequence* :

$$(4) \qquad 0 \to M(G)^*_A \to M(G) \to M(A/A \cap G').$$

Proof. Instead of (4), we shall prove the following exact sequence:

$$(4)' \qquad 0 \to M(G)^*_A \to M(G) \to Hom\ (\wedge(A/A \cap G'),\ \mathrm{T}).$$

In the computation (1), we only used the facts that f is a cocycle on G and that $ab=ba$, $ac=ca$. Since $a\in A\subset Z$ in the present case, we have $ab=ba$ for any $b\in G$. Therefore the extended map to $A\times G$ is bilinear, hence vanishes on $A\times G'$. Thus αf vanishes on $A\times(A\cap G')$ and on $(A\cap G')\times A$, which induces $(\alpha' f)$: $(A/A\cap G')\times(A/A\cap G')\to \mathrm{T}$ such that $(\alpha' f)$ $(a(A\cap G'), b(A\cap G'))=(\alpha f)$ (a, b). Replace A by $A/A\cap G'$, and we have a map $\varphi'_{G,A}$ of $M(G)$ to Hom $(\hat{}(A/A\cap G'), \mathrm{T})$ quite similarly as we obtained $\varphi_{G,A}$. The exactness of (4)′ is obvious, since $\alpha' f=0$ is equivalent to $\alpha f=0$. Q. E. D.

COROLLARY. *We have* $res_{G,A}=0$ *if* (i) $A\subset Z\cap G'$, *i.e.* G *is a stem extension of* A ; *or even if* (ii) $A/A\cap G'$ *is cyclic*.

REMARK. The case (ii) is Theorem (3. 1)′ of Vermani [9], and the case (i) is generalized to the case of a weak stem extension by Vermani [10, Theorem 4. 2].

3. Proof of Theorems

PROOF OF THEOREM 1. It follows from Proposition 2 that

$$|M(G)|\leqq|M(A/A\cap G')|\cdot|M(G)^*_A|.$$

Moreover the following exact sequence of Iwahori-Matsumoto [3]:

$$\begin{aligned}0&\to Hom\ (G/A, \mathrm{T})\to Hom\ (G, \mathrm{T})\to Hom\ (A, \mathrm{T})\\ &\to M(G/A)\to M(G)^*_A\to Hom\ ((G/G'A)\otimes A, \mathrm{T})\end{aligned}$$

implies

$$\begin{aligned}|M(G)^*_A|&\leqq\frac{|(G/G'A)\otimes A|\cdot|M(G/A)|\cdot|G/G'|}{|A|\cdot|(G/A)/(G/A)'|}\\ &=\frac{|(G/G'A)\otimes A|\cdot|M(G/A)|}{|A\cap G'|}.\end{aligned}$$

We obtain the result by combining the above two inequalities. Q. E. D.

PROOF OF THEOREM 3. Put $G=G/Z_{i-1}$ and $A=Z(G/Z_{i-1})=Z_i/Z_{i-1}$ in Theorem 1. Since we have

$$\begin{aligned}(Z_i/Z_{i-1})\cap(G'Z_{i-1}/Z_{i-1})&=Z_{i-1}\cdot(Z_i\cap G')/Z_{i-1}\\ &\cong(Z_i\cap G')/(Z_{i-1}\cap G')\end{aligned}$$

and

$$\begin{aligned}(Z_i/Z_{i-1})/((Z_i\cap G'Z_{i-1})/Z_{i-1})&\cong Z_i/(Z_i\cap G'Z_{i-1})\\ &\cong G'Z_i/G'Z_{i-1},\end{aligned}$$

we have

$$\left|\frac{Z_i \cap G'}{Z_{i-1} \cap G'}\right| \cdot |M(G/Z_{i-1})| \leqq |M(G/Z_i)| \cdot |M(G'Z_i/G'Z_{i-1})| \cdot |(G/G'Z_i) \otimes (Z_i/Z_{i-1})|.$$

The product of the above inequalities for $i=1, 2, \cdots, n-1$ implies the result. Q. E. D.

Proof of Theorem 2. The exact sequence of Vermani [8]

$$0 \to Hom\ (G_{n-1}, \mathrm{T}) \to M(G/G_{n-1}) \to M(G) \to Hom\ ((G/Z_{n-1}) \otimes G_{n-1}, \mathrm{T})$$

implies

$$|G_{n-1}| \cdot |M(G)| \leqq |M(G/G_{n-1})| \cdot |(G/Z_{n-1}) \otimes G_{n-1}|.$$

If we replace G by G/G_i, then G_{n-1} becomes G_{i-1}/G_i. Though we do not have an appropriate expression X/G_i of Z_{i-1} (G/G_i) by a subgroup X of G, we can use the subgroup $G'Z_{i-1}/G_i$ of Z_{i-1} (G/G_i). Suceeding steps are similar to those of Theorem 3. Q. E. D.

4. Examples

Finally we present a couple of examples to compare the upper bounds of $M(G)$ given by (N_1), (N_2), (N_3) and (N_4).

(i) Let G be generated by a, b and c with defining relations:

$$a^{p^\alpha} = b^{p^\beta} = c^{p^\gamma} = [a, b] = [a, c] = e,$$
$$[b, c] = a^{p^\delta},$$

where p is a prime and α, β, γ and δ are positive integers such that $\alpha \geqq \beta \geqq \gamma$ and $\alpha > \delta \geqq \alpha - \gamma > 0$.

Then G is of class 2, and (N_1), (N_2), (N_3) and (N_4) imply the bounds p^4, p^5, p^4 and p^2 respectively in the case of $(\alpha, \beta, \gamma, \delta) = (2, 1, 1, 1)$, and p^4, p^5, p^4, p^5 in the case of $(\alpha, \beta, \gamma, \delta) = (2, 2, 1, 1)$.

(ii) Let G be generated by $a_1, a_2, \cdots, a_n$ with defining relations:

$$a_i^{p^i} = e, \ [a_i, a_j] = a_j^{-p} \text{ for } i<j,$$

where $n \geqq 2$ and p is a prime.

Then G is of class n, and, if p is odd, all the four inequalities imply the same bound $p^{\frac{1}{2}(n^3-n^2)}$. If $p=2$, (N_1) and (N_2) imply the bound $2^{\frac{1}{2}(n^3-n^2)}$, while (N_3) and (N_4) imply the bound $2^{\frac{1}{2}(n^3-n^2-2n+4)}$.

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