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# On the Pointwise "$o$" Saturation Theorem for Positive Convolution Operators

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**Abstract** Saturation problems for positive convolution operators are well-known and widely investigated. In 1971, DeVore proved a pointwise "$o$" saturation theorem. DeVore's method yields somewhat more than the conclusion he obtained. It is shown that any sequence of positive convolution operators, which saturated, is pointwise "$o$" saturated.

## 1. Introduction

$C^*$ denotes the space of $2\pi$-periodic continuous functions with the supremum norm $\|\cdot\|$. Let $(L_n)$ be a sequence of linear operators on $C^*$, given by the convolution formula

(1) $$L_n(f, x)=(f * d\mu_n)(x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)d\mu_n(t)$$

where $d\mu_n$ is a non-negative, even Borel measure on $[-\pi, \pi]$ with $\frac{1}{\pi}\int_{-\pi}^{\pi} d\mu_n(t)=1$.

We consider the saturation of these operators. We say that $(L_n)$ is satureted if there exitsts a sequence of positive numbers $(\phi_n)$ which converges to 0 such that

a) for $f \in C^*$

$$\lim_{n\to\infty}\frac{\|f-L_n(f)\|}{\phi_n}=0 \qquad \text{if and only if } f \text{ is constant;}$$

b) there exists a non-constant function $f_0 \in C^*$ such that $\|f_0-L_n(f_0)\|=O(\phi_n)$.

The sequence $(\phi_n)$ is called the saturation order.

If we define the real Fourier-Stieltjes coefficients of $d\mu_n$ by $\rho_{k,n}=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos kt\ d\mu_n(t)$, the following theorem determines when $(L_n)$ is saturated and its saturation order (DeVore [3], pp.56-58).

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THEOREM A. *Let* $(L_n)$ *be a sequence of operators of the form* (1). *A necessary and sufficient condition that* $(L_n)$ *be saturated is that for some positive integer* $m$

$$\lim_{n\to\infty} \frac{1-\rho_{k,n}}{1-\rho_{m,n}}=\psi_k>0 \qquad \text{for } k=1, 2, 3,\cdots\cdots \tag{2}$$

*In this case* $(1-\rho_{m,n})$ *is a saturation order.*

In paticular, if $(L_n)$ satisfies (2), we obtain that for $f\in C^*$, $\|f-L_n(f)\|=o(1-\rho_{m,n})$ if and only if $f$ is constant. DeVore [2] proved a pointwise "$o$" saturation theorem.

THEOREM B. *Let* $(L_n)$ *be a sequence of positive convolution operators of the form* (1), *which satisfies*

$$\lim_{n\to\infty} \frac{1-\rho_{k,n}}{1-\rho_{1,n}}=\psi_k>0 \qquad \text{for } k=1, 2, 3,\cdots\cdots$$

*If* $f\in C^*$ *then*

$$f(x)-L_n(f, x)=o_x(1-\rho_{1,n}) \qquad \text{for each } x\in[-\pi, \pi]$$

*if and only if* $f$ *is constant on* $[-\pi, \pi]$.

DeVore's method of the proof of THEOREM B yields somewhat more than the conclusion he obtained. In fact we have next theorem.

THEOREM 1. *Let* $(L_n)$ *be a sequence of positive convolution operators of the form* (1), *where the Fourier coefficients of* $d\mu_n$ *satisfy* (2). *If* $f\in C^*$ *then for each* $x\in[-\pi, \pi]$

$$f(x)-L_n(f, x)=o_x(1-\rho_{m,n}) \tag{3}$$

*if and only if* $f$ *is constant on* $[-\pi, \pi]$.

Combining THEOREM A and THEOREM 1, we can easily show next theorem.

THEOREM 2. *Let* $(L_n)$ *be a sequence of positive convolution operators of the form* (1), *which saturated with order* $(\phi_n)$. *If* $f\in C^*$ *then for each* $x\in[-\pi, \pi]$

$$f(x)-L_n(f, x)=o_x(\phi_n)$$

*if and only if* $f$ *is constant on* $[-\pi, \pi]$.

## 2. Proof of THEOREM 1

The "if" part of the theorem is obvious. The proof of the "only if" part is based on a trigonometric analogue of the parabola technique of Bajsanski-Bojanic [1]. For this purpose we must prove two lemmas.

Let $\mathbf{m}$ be the set of all numbers $x\in[0,2\pi]$, such that, for each neighbourhood $I$ of $x$, we have $\int_I d\mu_n(t)\neq o(1-\rho_{m,n})$. If $\mathbf{m}=\phi$, with a compactness argument, we have

$\int_{-\pi}^{\pi} d\mu_n(t) = o(1-\rho_{m,n})$. This contradicts to the hypothesis of $(d\mu_n)$. So $\mathbf{m}$ is nom-void.

LEMMA 1. *Let $f \in C^*$ be a function which satisfies* (3). *Then for each $x_0 \in [-\pi, \pi]$, with $f(x_0) = \max\ \{f(x)\,;\, x \in [-\pi, \pi]\}$, and each $t \in \mathbf{m}$, we have $f(x_0+t) = f(x_0)$.*

PROOF. Let assume $f(x_0+x) < f(x_0)$ for some $x_0$, with $f(x_0) = \max\{f(x)\,;\, x \in [-\pi, \pi]\}$, and some $x \in \mathbf{m}$, Then $I \equiv \{y \in [-\pi, \pi]\,;\, f(x_0+y) < \frac{1}{2}(f(x_0)+f(x_0+x))\}$ is a neighbourhood of $x$ and

$$\frac{1}{2}(f(x_0)-f(x_0+x)) \int_I d\mu_n(t)$$
$$\leqq \int_I (f(x_0)-f(x_0+t))\, d\mu_n(t)$$
$$\leqq \int_{-\pi}^{\pi} (f(x_0)-f(x_0+t))\, d\mu_n(t)$$
$$= \pi(f(x_0) - L_n(f, x_0))$$
$$= o(1-\rho_{m,n}).$$

This shows that $\int_I d\mu_n(t) = o(1-\rho_{m,n})$ and thus this contradicts to $x \in \mathbf{m}$.

LEMMA 2. *If there exists a non-constant $f \in C^*$, which satisfies* (3), *then* $\mathbf{m}$ *is a finite set. Also, if $x$ is any number in* $\mathbf{m}$ *then $x = 2\pi\alpha$, where $\alpha$ is some rational number.*

PROOF. First of all, we note some properties of numbers, Let $(x)$ be the decimal place of a positive number $x$.

i) Let $\alpha$ be an irrational number. Then the set
$$\{(k\alpha)\,;\, k = 1, 2, 3, \cdots\cdots\}$$
is dense in $[0,1]$.

ii) Let $(\alpha_n)$ be a sequence of distinct numbers in $[0,1]$, and converge to some rational number. Then the set
$$\{(k\alpha_n)\,;\, n = 1, 2, 3, \cdots,\ k = 1, 2, 3, \cdots\}$$
is dense in $[0,1]$.

Suppose that $(x_n)$ is a sequence of distinct numbers, each of which is in $\mathbf{m}$. Choosing a subsequence, if necessary, we can assume $x_n \to x$ where $x \in [0, 2\pi]$. We write $x_n = 2\pi\alpha_n$ and $x = 2\pi\alpha$.

Let $x_0$ be any number in $[-\pi, \pi]$, where $f(x_0) = \max\ \{f(x)\,;\, x \in [-\pi, \pi]\} \equiv M$. Then, by LEMMA 1, for each positive integers $k$ and $n$, we have $f(x_0+kx_n) = M$. By continuity of $f$, $f(x_0+kx) = M$ for each positive integer $k$.

If $\alpha$ is irrational, the set $\{(k\alpha)\,;\, k = 1,2,3,\cdots\cdots\}$ is dense in $[0,1]$ by i). Therefore the set of numbers $kx$ taken modulo $2\pi$, is dense in $[0, 2\pi]$. Thus, in this case, $f = M$ on a set of numbers which is dense in $[x_0, x_0+2\pi]$ and therefore $f = M$ on $[x_0, x_0+2\pi]$. From periodidity we conclude that $f$ is constant.

If $\alpha$ is rational, the set $\{(k\alpha_n)\,;\ n=1,2,3,\cdots,\ k=1,2,3,\cdots\}$ is dense in $[0,1]$ by ii). Therefore, we again obtain that $f$ is constant.

This shows that $\mathfrak{m}$ has no limit point in $[0,2\pi]$ and hence $\mathfrak{m}$ must be a finite set.

Finally, if a number of the form $x=2\pi\alpha$ with $\alpha$ irrational were in $\mathfrak{m}$ then, as we have mentioned before, $f(x_0+kx)=M$ for each positive integer $k$, so that $f=M$ on a set of points which is dense in $[x_0, x_0+2\pi]$. This gives that $f$ is constant.

**Proof Of Theorem B.**

Let $f$ be a function which satisfies (3) and suppose $f$ is not constant. By subtracting a constant and considering the function $-f$ instead of $f$, if necessary, we can suppose that $f(-\pi)=f(\pi)=0$ and $M\equiv \max\ \{f(x)\,;\ x\in[0,2\pi]\}>0$. Then, it follows from Lemma 2 that there is a positive integer $N$ such that $\mathfrak{m}\subseteq\{\frac{2k\pi}{N}\,;\ k=0,1,2,\cdots,N\}$. In addition let $N$ be the smallest positive integer which satisfies the above. In this case, by using Lemma 1, we see that for any real number $x_0$, with $f(x_0)=M$, and any integer $k$,

$$f(x_0)=M=f(x_0+\frac{2k\pi}{N}). \tag{4}$$

The function $h_\alpha(x)\equiv-\alpha\sin^2\frac{N}{2}x+2M$, with $0<\alpha\leqq M$, is $\geqq f(y)$ for any $y\in[-\pi,\pi]$. For each $0<\varepsilon,\ \delta<\frac{\pi}{2N}$, we define

$$I_{\varepsilon,\delta}\equiv\bigcup_{k=-N}^{N}[\frac{2k\pi}{N}-\varepsilon,\ \frac{2k\pi}{N}+\delta]\cap[-\pi,\pi]$$

$$S_{\varepsilon,\delta}\equiv[-\pi,\pi]\backslash I_{\varepsilon,\delta}.$$

Now, we fix $x_0$. Then either

a) there exist some $\varepsilon$ and some $\delta$ such that

$$A\equiv\max\ \{f(x_0+\frac{2k\pi}{N}-\varepsilon),\ f(x_0+\frac{2k\pi}{N}+\delta)\,;\ k=0,1,\cdots,N\}<M,$$

or

b) for each $\delta$ (or $\varepsilon$), there exists some integer $k_0$ such that

$$f(x_0+\frac{2k_0\pi}{N}+\delta)=M\ \ (\text{or } f(x_0+\frac{2k_0\pi}{N}-\varepsilon)=M).$$

CASE a)

Let $\alpha\equiv\min\ \{\frac{M-A}{\sin^2\frac{N}{2}\varepsilon+\sin^2\frac{N}{2}\delta},\ M\}$. By (4),

$$h_\alpha(\frac{2k\pi}{N})-f(x_0+\frac{2k\pi}{N})=2M-M=M.$$

On the other hand, by the definitions of $A$ and $\alpha$,

$$h_\alpha(\frac{2k\pi}{N}+\delta)-f(x_0+\frac{2k\pi}{N}+\delta)$$

$$\begin{aligned}&= -\alpha \sin^2 \frac{N}{2}\delta + 2M - f(x_0 + \frac{2k\pi}{N} + \delta)\\&\geqq -\frac{M-A}{\sin^2 \frac{N}{2}\varepsilon + \sin^2 \frac{N}{2}\delta} \sin^2 \frac{N}{2}\delta + 2M - A\\&> -(M-A) + 2M - A\\&= M.\end{aligned}$$

Similarly

$$h_\alpha(\frac{2k\pi}{N} - \varepsilon) - f(x_0 + \frac{2k\pi}{N} - \varepsilon) > M.$$

These show that $C \equiv \min \{h_\alpha(x) - f(x_0 + x) ; x \in I_{\varepsilon,\delta}\}$ is assumed at a point $y$ in the interior of $I_{\varepsilon,\delta}$. Therefore, for each $x \in I_{\varepsilon,\delta}$

$$h_\alpha(x) - C \geqq f(x_0 + x)$$

and

$$h_\alpha(y) - C = f(x_0 + y).$$

Then

(5) $$\begin{aligned}&\int_I [h_\alpha(x) - h_\alpha(y)]\, d\mu_n(x-y)\\&\geqq \int_{I_{\varepsilon,\delta}} [f(x_0+x) - f(x_0+y)]\, d\mu_n(x-y).\end{aligned}$$

Since the interior of $I_{\varepsilon,\delta} - y$ contains $\mathbf{m}$, we obtain by the use of the compactness argument that

(6) $$\int_{S_{\varepsilon,\delta}} d\mu_n(x-y) = \int_{S_{\varepsilon,\delta} - y} d\mu_n(x) = o(1 - \rho_{m,n}).$$

By (5), (6) and the fact

$$\begin{aligned}h_\alpha(x) - h_\alpha(y) = &-\alpha \cos Ny \sin^2 \frac{N}{2}(x-y)\\&-\frac{\alpha}{2} \sin Ny \sin N(x-y)\end{aligned}$$

we have

$$\begin{aligned}&L_n(f, x_0+y) - f(x_0+y)\\&= \frac{1}{\pi}\int_{I_{\varepsilon,\delta}} [f(x_0+x) - f(x_0+y)]\, d\mu_n(x-y)\\&\quad + \frac{1}{\pi}\int_{S_{\varepsilon,\delta}} [f(x_0+x) - f(x_0+y)]\, d\mu_n(x-y)\end{aligned}$$

$$\leqq \frac{1}{\pi}\int_{I_{\varepsilon,\delta}}[h_\alpha(x)-h_\alpha(y)]\ d\mu_n(x-y)+o(1-\rho_{m,n})$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}[h_\alpha(x)-h_\alpha(y)]\ d\mu_n(x-y)+o(1-\rho_{m,n})$$

$$=-\frac{\alpha}{\pi}\cos Ny\int_{-\pi}^{\pi}\sin^2\frac{N}{2}(x-y)\ d\mu_n(x-y)+o(1-\rho_{m,n})$$

$$=-\frac{\alpha}{2}\cos Ny\ (1-\rho_{N,n})+o(-\rho_{m,n}).$$

Finally, from (2), there exists some sequence $\{n_i\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty}\frac{L_{n_i}(f,x_0+y)-f(x_0+y)}{1-\rho_{m,n_i}}\leqq-\frac{\alpha}{2}\lim_{i\to\infty}\cos Ny\ \frac{1-\rho_{N,n_i}}{1-\rho_{m,n_i}}$$

$$=-\frac{\alpha}{2}\cos Ny\ \psi_N.$$

Since $\cos Ny>\cos\frac{\pi}{2}=0, \alpha>0$ and $\psi_N>0$, we have

$$L_n(f,x_0+y)-f(x_0+y)\neq o(1-\rho_{m,n}).$$

This contradicts to (3) at the point $x=x_0+y$.

CASE b)

From (4), we have for each integer $k$ and each $\delta<\frac{\pi}{2N}$

$$f(x_0+\frac{2k\pi}{N}+\delta)=M.$$

Therefore there exists some $x_1$ such that

$$f(x_1+x)=M \qquad \text{for each } x\in\bigcup_{k=0}^{N-1}[\frac{2k\pi}{N},\frac{2k\pi}{N}+\frac{\pi}{2N}].$$

The other case of b), that is $f(x_0+\frac{2k_0\pi}{N}-\varepsilon)=M,$ we can also obtain the same conclusion.

Then we use $x_1+\frac{\pi}{4N}$ in place of $x_0$ in the armument above. In the case a), we have contradiction. In the case b), there exists some $x_2$ such that

$$f(x_2+x)=M \qquad \text{for each } x\in\bigcup_{k=0}^{N-1}[\frac{2k\pi}{N},\frac{2k\pi}{N}+\frac{3\pi}{4N}].$$

Now, we use $x_2+\frac{3\pi}{8N}$ in place of $x_0$.

In generally, if case b) continues $n$ times, there exists some $x_n$ such that

$$f(x_n+x)=M \qquad \text{for each } x\in\bigcup_{k=0}^{N-1}[\frac{2k\pi}{N},\frac{(2k+1)\pi}{N}-\frac{\pi}{2^nN}].$$

Therefore, if case b) continues infinitely, there exists some $z\in[0,\frac{\pi}{N}]$ such that

$$f(x)=M \qquad \text{for each } x\in I_0\equiv\bigcup_{k=-N}^{N}\left[z+\frac{2k\pi}{N},\ z+\frac{(2k+1)\pi}{N}\right]\cap[-\pi,\pi].$$

If we let $g(x)\equiv M-f(x)$,

$$\begin{aligned} &g(x)\geqq 0 && \text{on } [-\pi,\pi] \\ &g(x)=0 && \text{on } I_0 \\ &g(\pm\pi)=M>0. \end{aligned}$$

Let $x'\in[-\pi,\pi]$ be $M'\equiv\max\{g(x);x\in[-\pi,\pi]\}=g(x')$. Now we can wark with $g$, $M'$ and $x'$, in places of $f$, $M$ and $x_0$ in the argument above. Then either case a) is reached, or case b) continues infinitely. In the latter case, we can conclude

$$\begin{aligned} &g(x)=0 && \text{on } I_0 \\ &g(x)=M' && \text{on } [-\pi,\pi]\setminus I_0. \end{aligned}$$

This contradicts to the continuity of $g$.

From the argument above, in any cases, we have the desired contradiction. So (3) implies $f\equiv$ constant.

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