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Exact Solutions with Arbitrary Integer Deltas for Gravitational Fields of Spinning Masses

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Abstract A simple and systematic recipe is presented which gives for any integer δ (distortion parameter) the family of exact solutions for gravitational fields of spinning masses and which reduces to give the famous Kerr solution for $\delta=1$ and the Tomimatsu-Sato solutions for $\delta=2$, 3, and 4. Our family of solutions reduces to that of the Weyl metrics in the case of no rotation where the parameter q vanishes.

Kerr[1] has discovered a solution ($\delta=1$ solution) for the gravitational field of a spinning mass. Ernst[2] has formulated the axially symmetric gravitational field problem, obtained the differential equation

$$(\xi\xi^*-1)\nabla^2\xi = 2\xi^* \nabla\xi \cdot \nabla\xi,$$

and showed that the Kerr solution satisfies this equation. Tomimatsu and Sato[3] have discovered a series of solutions ($\delta=2$, 3, and 4) for gravitational fields of spinning masses which reduce to the series of Weyl[4] metrics in the limit of angular momentum parameter $q=0$.

We shall present a simple and systematic recipe to give exact solutions with arbitrary integer δ(distortion parameter) which are members of the series of Kerr and Tomimatsu–Sato solutions and which therefore reduce to the family of Weyl metrics[4] in the limit of the parameter $q=0$.

We use notations in reference 3). We use prolate spheroidal coordinates x, y in place of cylindrical coordinates ρ, z and the notation $a=x^2-1$ and $b=y^2-1$. Ernst's complex functions ξ are written as $\xi=(u+iv)/(m+in)$. We use the notations $A=u^2+v^2-m^2-n^2$, $G=m^2+n^2$, $H=um+vn$, and $I=vm-un$. There are the distortion parameter δ and the angular momentum parameter q (and p such as $p^2+q^2=1$).

Our family of exact solutions with any integer δ can be expressed, besides px, qy, a^j, and b^j ($j=1, 2, 3, \cdots$), by functions $F(i)$ with $i=\delta^2-k_\delta$ and $k_\delta=0, 1, 2, \cdots, \delta$. The functions $F(i)$ are polynomials of a, b, p^2, and q^2. Polynomials $F(i)$ are homogeneous with degree i over a and b and simultaneously homogeneous with degree δ over p^2 and

q^2 for $k_\delta=0$ and with degree $\delta-1$ over p^2 and q^2 for $k_\delta=1, 2, \cdots, \delta$. Polynomials $F(i)$ are symmetrical about simultaneous exchanges of a and b and of p^2 and q^2 ($F(i)=F_i(a,b;p^2,q^2)=F_i(b,a;q^2,p^2)$). Polynomials $F(i)$ can be expressed completely by the functions

$$f(j) = p^2a^j+q^2b^j \qquad (j=1, 2, 3, \cdots).$$

If we suppose a virtual case $a=b=1$, polynomials $F(i)=F(\delta^2-k_\delta)$ become $(p^2+q^2)^\delta$ for $k_\delta=0$ and $(p^2+q^2)^{\delta-1}$ for $k_\delta=1, 2, \cdots, \delta$. In other words coefficients in $F(i)$, when $a=b=1$, become the binomial coefficient ${}_nC_r$ ($(p^2+q^2)^n=\Sigma\, {}_nC_r(p^2)^{n-r}(q^2)^r$). This can be used as a powerful check for $F(i)$ in actual computations. Actual computations of polynomials $F(i)$ in terms of functions $f(j)$ can be elementarily, though tediously, carried out with the help of the identity $H^2+I^2=(A+G)G$.

We have the following relations:

$$\begin{aligned}\mathrm{Re}\xi &= [px \sum_{r_\delta=1}^{\delta} d(r_\delta)\,(-a)^{r_\delta-1} \sum_{r'_\delta=r_\delta}^{\delta} c(\delta,r'_\delta)F(\delta^2-r'_\delta)]/G \\ &= H/G \end{aligned} \tag{1}$$

$$\begin{aligned}\mathrm{Im}\xi &= [-qy \sum_{r_\delta=1}^{\delta} d(r_\delta)\,(-b)^{r_\delta-1} \sum_{r'_\delta=r_\delta}^{\delta} c(\delta,r'_\delta)F(\delta^2-r'_\delta)]/G \\ &= I/G \end{aligned} \tag{2}$$

$$G = \sum_{r_\delta=1}^{\delta} c(\delta,r_\delta)F(\delta^2-r_\delta) \tag{3}$$

$$A = F(\delta^2) = \sum_{r_\delta=1}^{\delta} e(r_\delta)c(\delta,r_\delta)f(r_\delta)F(\delta^2-r_\delta), \tag{4}$$

where $F(i)=F(\delta^2-k_\delta)$, $k_\delta=0, 1, 2, \cdots, \delta$ and, when we want summations from 1 to δ, we use $r_\delta=1, 2, \cdots, \delta$ instead of k_δ. We have in eqs. (1)~(4) three kinds of numerical coefficients $c(\delta,r_\delta)$, $d(s)$, and $e(s)$, which are shown in Table 1 and 2. Coefficients $c(\delta,r_\delta)$ are dependent on both δ and r_δ, but $d(s)$ and $e(s)$ are both independent of δ. We can determine coefficients $c(\delta,r_\delta)$ and $d(s)$ from the requirement that our family of solutions must reduce to the family of Weyl metrics[4)]

$$\xi = \frac{\{(x+1)^\delta+(x-1)^\delta\}\ a^{\delta(\delta-1)/2}/2}{\{(x+1)^\delta-(x-1)^\delta\}\ a^{\delta(\delta-1)/2}/2}$$

in the limit of $q=0$. We can determine $e(s)$ from the requirement that the following relation must hold:

$$\sum_{r_\delta=1}^{\delta} e(r_\delta)c(\delta, r_\delta)=1.$$

Eqs. (1)~(4) are our basic relations from which one can obtain in principle exact solutions with every integer δ. In practice actual computations become tremendously lengthy as δ increases.

In the following we tabulate our solutions obtained from our recipe eqs. (1)~(4).

Table 1. Coefficients $c(\delta,\ r_\delta)$

r_δ \ δ	1	2	3	4	5	6	7	8
1	1	4	9	16	25	36	49	64
2		4	24	80	200	420	784	1344
3			16	128	560	1792	4704	10752
4				64	640	3456	13440	42240
5					256	3072	19712	90112
6						1024	14336	106496
7							4096	65536
8								16384

Table 2. Coefficients $d(s)$ and $e(s)$

s	$d(s)$	$e(s)$
1	1	1
2	1/2	−3/4
3	3/8	5/8
4	5/16	−35/64
5	35/128	63/128
6	63/256	−231/512
7	231/1024	429/1024
8	429/2048	−6435/16384

$\delta = 1$ (Kerr solution)

$$H = px[(F(0))],\ I = -qy[(F(0))],\ G = F(0),$$
$$A = F(1) = f(1)F(0),\ F(0) = 1$$

$\delta = 2$ (T.–S. solution)

$$H = px[(4F(3)+4F(2))-a(2F(2))]$$
$$I = -qy[(4F(3)+4F(2))-b(2F(2))]$$
$$G = 4F(3)+4F(2),\ A = F(4) = 4f(1)F(3)-3f(2)F(2)$$
$$F(3) = f(3)F(0),\ F(2) = f(2)F(0)$$

$\delta = 3$ (T.–S. solution)

$$H = px[(9F(8)+24F(7)+16F(6))-a(12F(7)+8F(6))+a^2(6F(6))]$$
$$I = -qy[(9F(8)+24F(7)+16F(6))-b(12F(7)+8F(6))+b^2(6F(6))]$$
$$G = 9F(8)+24F(7)+16F(6)$$
$$A = F(9) = 9f(1)F(8)-18f(2)F(7)+10f(3)F(6)$$

$$F(8) = 16f(5)F(3)-15f(4)f(4)$$
$$F(7) = -5f(4)F(3)+6f(5)F(2)$$
$$F(6) = -8f(3)F(3)+9f(4)F(2)$$

$\delta = 4$ (T.–S. solution)

$$H = px[(16F(15)+80F(14)+128F(13)+64F(12))-a(40F(14)+64F(13)+32F(12))+a^2(48F(13)+24F(12))-a^3(20F(12))]$$
$I = -qy$[*a is substituted by b in the above*]
$$G = 16F(15)+80F(14)+128F(13)+64F(12)$$
$$A = F(16) = 16f(1)F(15)-60f(2)F(14)+80f(3)F(13)-35f(4)F(12)$$

$$F(15) = 225f(7)F(8)-3500f(3)f^2(6)+6300f(4)f(5)f(6)-3024f^3(5)$$
$$F(14) = 35f(6)F(8)+120f(7)F(7)-700f(2)f^2(6)-504f(4)f^2(5)+1050f^2(4)f(6)$$
$$F(13) = 21f(5)F(8)-70f(6)F(7)+50f(7)F(6)$$
$$F(12) = 45f(4)F(8)-144f(5)F(7)+100f(6)F(6)$$

$\delta = 5$

$$H = px[(25F(24)+200F(23)+560F(22)+640F(21)+256F(20))-a(100F(23)+280F(22)+320F(21)+128F(20))+a^2(210F(22)+240F(21)+96F(20))-a^3(200F(21)+80F(20))+a^4(70F(20))]$$
$I = -qy$[*a is substituted by b in the above*]

$$G = 25F(24)+200F(23)+560F(22)+640F(21)+256F(20)$$

$$A = F(25)=25f(1)F(24)-150f(2)F(23)+350f(3)F(22)-350f(4)F(21)+126f(5)F(20)$$

$$F(24) = 3136f(9)F(15)-11113200f(3)f(5)f^2(8)+21168000f(3)f(6)f(7)f(8)-10368000f(3)f^3(7)+10418625f^2(4)f^2(8)-19051200f(4)f(5)f(7)f(8)-18522000f(4)f^2(6)f(8)+18144000f(4)f(6)f^2(7)+17781120f^2(5)f(6)f(8)+8709120f^2(5)f^2(7)-25401600f(5)f^2(6)f(7)+8232000f^4(6)$$

$$F(23) = 1960f(9)F(14)-1389150f(2)f(5)f^2(8)+2646000f(2)f(6)f(7)f(8)-1296000f(2)f^3(7)+1157625f(3)f(4)f^2(8)-1058400f(3)f(5)f(7)f(8)-1029000f(3)f^2(6)f(8)+1008000f(3)f(6)f^2(7)-992250f^2(4)f(7)f(8)+1814400f(4)f(5)f^2(7)-882000f(4)f^2(6)f(7)+889056f^3(5)f(8)-1693440f^2(5)f(6)f(7)+823200f(5)f^3(6)$$

$$F(22) = 1120f(9)F(13)-496125f(2)f(4)f^2(8)+453600f(2)f(5)f(7)f(8)+441000f(2)f^2(6)f(8)-432000f(2)f(6)f^2(7)-378000f(3)f(4)f(7)f(8)-1058400f(3)f(5)f(6)f(8)+345600f(3)f(5)f^2(7)+336000f(3)f^2(6)f(7)+441000f^2(3)f^2(8)+317520f(4)f^2(5)f(8)+330750f^2(4)f(6)f(8)-294000f(4)f^3(6)-290304f^3(5)f(7)+282240f^2(5)f^2(6)$$

$$F(21) = -84f(6)F(15)+540f(7)F(14)-945f(8)F(13)+490f(9)F(12)$$

$$F(20) = -224f(5)F(15)+1400f(6)F(14)-2400f(7)F(13)+1225f(8)F(12)$$

$\delta = 6$

$$H = px[(36F(35)+420F(34)+1792F(33)+3456F(32)+3072F(31)+1024F(30))-a(210F(34)+896F(33)+1728F(32)+1536F(31)+512F(30))+a^2(672F(33)+1296F(32)+1152F(31)+384F(30))-a^3(1080F(32)+960F(31)+320F(30))+a^4(840F(31)+280F(30))-a^5(252F(30))]$$

$I = -qy$[*a is substituted by b in the above*]

$$G = 36F(35)+420F(34)+1792F(33)+3456F(32)+3072F(31)+1024F(30)$$

$$A = F(36)=36f(1)F(35)-315f(2)F(34)+1120f(3)F(33)-1890f(4)F(32)+1512f(5)F(31)-462f(6)F(30)$$

$\delta = 7$

$$H = px[(49F(48)+784F(47)+4704F(46)+13440F(45)+19712F(44)+14336F(43)+4096F(42))-a(392F(47)+2352F(46)+6720F(45)+9856F(44)+7168F(43)+2048F(42))+a^2(1764F(46)+5040F(45)+7392F(44)+5376F(43)+1536F(42))-a^3(4200F(45)+6160F(44)+4480F(43)+1280F(42))+a^4(5390F(44)+3920F(43)+1120F(42))-a^5(3528F(43)+1008F(42))+a^6(924F(42))]$$

$I=\ -qy[a$ *is substituted by* b *in the above*$]$

$G=\ 49F(48)+784F(47)+4704F(46)+13440F(45)+19712F(44)+14336F(43)$
$+4096F(42)$

$A=\ F(49)=49f(1)F(48)-588f(2)F(47)+2940f(3)F(46)-7350f(4)F(45)$
$+9702f(5)F(44)-6468f(6)F(43)+1716f(7)F(42)$

$\delta=8$

$H=\ px[(64F(63)+1344F(62)+10752F(61)+42240F(60)+90112F(59)+$
$106496F(58)+65536F(57)+16384F(56))-a(672F(62)+5376F(61)$
$21120F(60)+45056F(59)+53248F(58)+32768F(57)+8192F(56))+$
$a^2(4032F(61)+15840F(60)+33792F(59)+39936F(58)+24576F(57)$
$+6144F(56))-a^3(13200F(60)+28160F(59)+33280F(58)+20480F(57)$
$+5120F(56))+a^4(24640F(59)+29120F(58)+17920F(57)+4480F(56))$
$-a^5(26208F(58)+16128F(57)+4032F(56))+a^6(14784F(57)+3696F(56))$
$-a^7(3432F(56))]$

$I=\ -qy[a$ *is substituted by* b *in the above*$]$

$G=\ 64F(63)+1344F(62)+10752F(61)+42240F(60)+90112F(59)+106496$
$F(58)+65536F(57)+16384F(56)$

$A=\ F(64)=64f(1)F(63)-1008f(2)F(62)+6720f(3)F(61)-23100f(4)$
$F(60)+44352f(5)F(59)-48048f(6)F(58)+27456f(7)F(57)-6435f(8)F(56)$

$\delta=1\quad F(0)=1,\qquad F(1)=p^2a+q^2b$

$\delta=2\quad F(2)=p^2a^2+q^2b^2,\quad F(3)=p^2a^3+q^2b^3,\quad F(4)=p^4a^4+p^2q^2(4a^3b$
$-6a^2b^2+4ab^3)+q^4b^4$

$\delta=3$

$F(6)=p^4a^6+p^2q^2(9a^4b^2-16a^3b^3+9a^2b^4)+q^4b^6$

$F(7)=p^4a^7+p^2q^2(6a^5b^2-5a^4b^3-5a^3b^4+6a^2b^5)+q^4b^7$

$F(8)=p^4a^8+p^2q^2(16a^5b^3-30a^4b^4+16a^3b^5)+q^4b^8$

$F(9)=p^6a^9+p^4q^2(9a^8b-36a^7b^2+84a^6b^3-90a^5b^4+36a^4b^5)+p^2q^4(36a^5b^4$
$-90a^4b^5+84a^3b^6-36a^2b^7+9ab^8)+q^6b^9$

$\delta=4$

$F(12)=p^6a^{12}+p^4q^2(36a^{10}b^2-160a^9b^3+315a^8b^4-288a^7b^5+100a^6b^6)+$

$p^2q^4(100a^6b^6-288a^5b^7+315a^4b^8-160a^3b^9+36a^2b^{10})+q^6b^{12}$

$F(13)=p^6a^{13}+p^4q^2(30a^{11}b^2-114a^{10}b^3+170a^9b^4-63a^8b^5-70^7b^6+50a^6b^7)$
$+p^2q^4(50a^7b^6-70a^6b^7-63a^5b^8+170a^4b^9-114a^3b^{10}+30a^2b^{11})+q^6b^{13}$

$F(14)=p^6a^{14}+p^4q^2(20a^{12}b^2-40a^{11}b^3-54a^{10}b^4+272a^9b^5-315a^8b^6+120a^7b^7)$
$+p^2q^4(120a^7b^7-315a^6b^8+272a^5b^9-54a^4b^{10}-40a^3b^{11}+20a^2b^{12})+q^6b^{14}$

$F(15)=p^6a^{15}+p^4q^2(100a^{12}b^3-450a^{11}b^4+828a^{10}b^5-700a^9b^6+225a^8b^7)+$
$p^2q^4(225a^7b^8-700a^6b^9+828a^5b^{10}-450a^4b^{11}+100a^3b^{12})+q^6b^{15}$

$F(16)=p^8a^{16}+p^6q^2(16a^{15}b-120a^{14}b^2+560a^{13}b^3-1420a^{12}b^4+1968a^{11}b^5$
$-1400a^{10}b^6+400a^9b^7)+p^4q^4(400a^{12}b^4-2400a^{11}b^5+6608a^{10}b^6-$
$11040a^9b^7+12870a^8b^8-11040a^7b^9+6608a^6b^{10}-2400a^5b^{11}+400a^4b^{12})$
$+p^2q^6(400a^7b^9-1400a^6b^{10}+1968a^5b^{11}-1420a^4b^{12}+560a^3b^{13}-$
$120a^2b^{14}+16ab^{15})+q^8b^{16}$

$\delta=5$

$F(20)=p^8a^{20}+p^6q^2(100a^{18}b^2-800a^{17}b^3+3075a^{16}b^4-6496a^{15}b^5+7700a^{14}b^6$
$-4800a^{13}b^7+1225a^{12}b^8)+p^4q^4(2500a^{14}b^6-168000a^{13}b^7+51275a^{12}b^8$
$-93600a^{11}b^9+113256a^{10}b^{10}-93600a^9b^{11}+51275a^8b^{12}-16800a^7b^{13}$
$+2500a^6b^{14})+p^2q^6(1225a^8b^{12}-4800a^7b^{13}+7700a^6b^{14}-6496a^5b^{15}$
$+3075a^4b^{16}-800a^3b^{17}+100a^2b^{18})+q^8b^{20}$

$F(21)=p^8a^{21}+p^6q^2(90a^{19}b^2-670a^{18}b^3+2340a^{17}b^4-4257a^{16}b^5+3766a^{15}b^6$
$-810a^{14}b^7-945a^{13}b^8+490a^{12}b^9)+p^4q^4(1750a^{15}b^6-10170a^{14}b^7$
$+24885a^{13}b^8-30970a^{12}b^9+14508a^{11}b^{10}+14508a^{10}b^{11}-30970a^9b^{12}$
$+24885a^8b^{13}-10170a^7b^{14}+1750a^6b^{15})+p^2q^6(490a^9b^{12}-945a^8b^{13}$
$-810a^7b^{14}+3766a^6b^{15}-4257a^5b^{16}+2340a^4b^{17}-670a^3b^{18}+90a^2b^{19})$
$+q^8b^{21}$

$F(22)=p^8a^{22}+p^6q^2(75a^{20}b^2-480a^{19}b^3+1295a^{18}b^4-1152a^{17}b^5-1570a^{16}b^6$
$+4496a^{15}b^7-3780a^{14}b^8+1120a^{13}b^9)+p^4q^4(875a^{16}b^6-2800a^{15}b^7$
$-2610a^{14}b^8+28640a^{13}b^9-67782a^{12}b^{10}+87360a^{11}b^{11}-67782a^{10}b^{12}$
$+28640a^9b^{13}-2610a^8b^{14}-2800a^7b^{15}+875a^6b^{16})+p^2q^6(1120a^9b^{13}$
$-3780a^8b^{14}+4496a^7b^{15}-1570a^6b^{16}-1152a^5b^{17}+1295a^4b^{18}-480a^3b^{19}$
$+75a^2b^{20})+q^8b^{22}$

$F(23)=p^8a^{23}+p^6q^2(50a^{21}b^2-175a^{20}b^3-315a^{19}b^4+3458a^{18}b^5-9240a^{17}b^6$
$+11910a^{16}b^7-7644a^{15}b^8+1960a^{14}b^9)+p^4q^4(3675a^{16}b^7-22050a^{15}b^8$
$+55520a^{14}b^9-70812a^{13}b^{10}+33670a^{12}b^{11}+33670a^{11}b^{12}-70812a^{10}b^{13}$
$+55520a^9b^{14}-22050a^8b^{15}+3675a^7b^{16})+p^2q^6(1960a^9b^{14}-7644a^8b^{15}$
$+11910a^7b^{16}-9240a^6b^{17}+3458a^5b^{18}-315a^4b^{19}-175a^3b^{20}+50a^2b^{21})$
$+q^8b^{23}$

$F(24)=p^8a^{24}+p^6q^2(400a^{21}b^3-3150a^{20}b^4+11088a^{19}b^5-21280a^{18}b^6+$
$23040a^{17}b^7-13230a^{16}b^8+3136a^{15}b^9)+p^4q^4(11025a^{16}b^8-78400a^{15}b^9$
$+250848a^{14}b^{10}-473760a^{13}b^{11}+580580a^{12}b^{12}-473760a^{11}b^{13}+$

$$250848a^{10}b^{14}-78400a^{9}b^{15}+11025a^{8}b^{16})+p^{2}q^{6}(3136a^{9}b^{15}-13230a^{8}b^{16}+23040a^{7}b^{17}-21280a^{6}b^{18}+11088a^{5}b^{19}-3150a^{4}b^{20}+400a^{3}b^{21})+q^{8}b^{24}$$

$$F(25)=p^{10}a^{25}+p^{8}q^{2}(25a^{24}b-300a^{23}b^{2}+2300a^{22}b^{3}-10150a^{21}b^{4}+26880a^{20}b^{5}-43400a^{19}b^{6}+41800a^{18}b^{7}-22050a^{17}b^{8}+4900a^{16}b^{9})+p^{6}q^{4}(2500a^{21}b^{4}-26250a^{20}b^{5}+133700a^{19}b^{6}-438900a^{18}b^{7}+1059525a^{17}b^{8}-2007450a^{16}b^{9}+3023760a^{15}b^{10}-3553200a^{14}b^{11}+3158400a^{13}b^{12}-2041900a^{12}b^{13}+904200a^{11}b^{14}-245000a^{10}b^{15}+30625a^{9}b^{16})+p^{4}q^{6}(30625a^{16}b^{9}-245000a^{15}b^{10}+904200a^{14}b^{11}-2041900a^{13}b^{12}+3158400a^{12}b^{13}-3553200a^{11}b^{14}+3023760a^{10}b^{15}-2007450a^{9}b^{16}+1059525a^{8}b^{17}-438900a^{7}b^{18}+133700a^{6}b^{19}-26250a^{5}b^{20}+2500a^{4}b^{21})+p^{2}q^{8}(4900a^{9}b^{16}-22050a^{8}b^{17}+41800a^{7}b^{18}-43400a^{6}b^{19}+26880a^{5}b^{20}-10150a^{4}b^{21}+2300a^{3}b^{22}-300a^{2}b^{23}+25ab^{24})+q^{10}b^{25}$$

If we suppose a virtual case $p=q=1$, coefficients in $F(\delta^2)$ become, regardless of signs, binomial coefficients of $(a+b)^{\delta^2}$.

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