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Genetic Diagrams for Groups of Orthogonal Matrices O(n)

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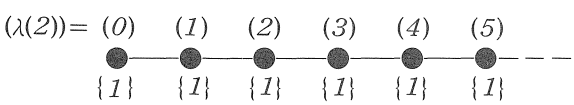
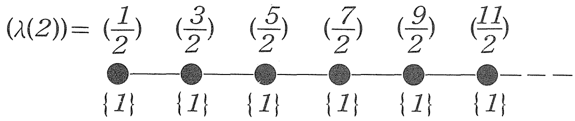
Abstract A simple and systematic rule for preparing of diagrams is presented which are convenient to a perspective survey of and a step-by-step construction of any irreducible representations for groups of rotations in space with arbitrary dimension.

It will be shown in this note that, starting from the trivial array of irreducible representations (abbreviated as IR's) of trivial two-dimensional group of rotations $O(2)$, one can prepare, with the help of a very simple and systematic rule, diagrams-genetic diagrams- which are convenient to a perspective survey of and a step-by-step construction of any IRs for n-dimensional groups of rotations[1),2)], or what comes to the same things, for groups of orthogonal matrices of any orders $O(n)$.

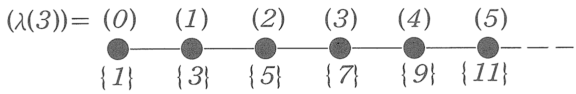
we can see essences of our rule when we figure and look at trivial $O(2)$ and $O(3)$ diagrams. The rank of groups $O(2)$ and $O(3)$ is both one and IRs of groups $O(2)$ and $O(3)$ are labelled by $(\lambda(2))$ and $(\lambda(3))$, respectively. Numbers $\lambda(2)$ and $\lambda(3)$ are integers for vector IRs and half-integers for spinor IRs. Numbers $\lambda(3)$ is positive and $\lambda(2)$ is positive or negative. The IRs of group $O(2)$ are arranged into a one dimensional array



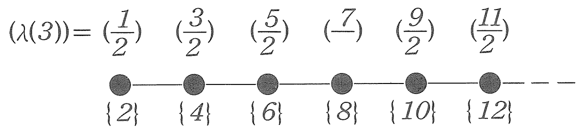
for vector IRs, and into another one



for spinor IRs. The IRs are all one-dimensional or singlets $\{1\}$. For $\lambda(2)\neq 0$ points (IRs) on diagrams are doubly degenerated because there exist conjugate IRs $(\lambda(2))$ and $(-\lambda(2))$ (complex IRs). The array is one-dimensional because the rank of group is one. The IRs of group $O(3)$ are arranged into the other one-dimensional array



for vector IRs, and also into another one



for spinor IRs. points (IRs) on diagrams are not degenerated because there exist no complex IRs. Here also the array is one-dimensional because the rank of group is one. When we look at two diagrams now constructed for groups $O(2)$ and $O(3)$, we can see, for example, IR {7} with label (3) for group $O(3)$ is composed of IRs (and bases) {1} with label (0), {1} with (1), {1} with (-1), {1} with (2), {1} with (-2), {1} with (3), and {1} with (-3) for group $O(2)$. We can also see, for example, IR (and bases) {6} with label (5/2) for group $O(3)$ is composed of IRs {1} with label (1/2), {1} with (-1/2), {1} with (3/2), {1} with (-3/2), {1} with (5/2), and {1} with (-5/2) for group $O(2)$. Any bases of IR ($\lambda(3)$) for group $O(3)$ are designated by two labels $\lambda(3)$ and $\lambda(2)$, and can be written as $|\lambda(3),\lambda(2)>$, that is $|l,m>$ in usual notations. There exist the inequality $\lambda(3)\geqq\lambda(2)\geqq-\lambda(3)$. Thus we can obtain basis vectors $|\lambda(3),\lambda(2)>$ with $\lambda(3)\geqq\lambda(2)\geqq-\lambda(3)$ for group $O(3)$ from two diagrams for groups $O(2)$ and $O(3)$.

The above stated recipe to obtain IRs (and bases thereof) for group $O(3)$ from $O(2)$ diagram can be generalized to a set of simple and systematic rules for preparing of diagrams for group of rotations in space with arbitrary dimension. The general rules are:

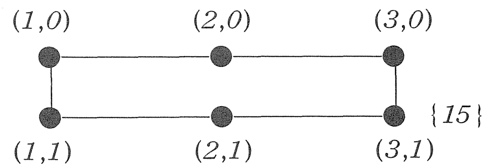
1) $O(n)$ diagram is made from $O(n-1)$ diagram,
2) diagrams for group $O(2l)$ (or D_l) and $O(2l+1)$ (or B_l), both with rank l, are l-dimensional arraies of points (IRs),
3) spinor and vector IRs are grouped into separated diagrams which have the similar structure for the same group. Vector (spinor) diagrams of group $O(n)$ are made from vector (spinor) diagrams of group $O(n-1)$. Procedures of preparation are identical for both vector and spinor diagrams.

The IRs of group $O(n)$ are labelled by $(\lambda_1(n), \lambda_2(n), \ldots, \lambda_l(n))$ with $\lambda_1(n)\geqq\lambda_2(n) > \cdots \geqq |\lambda_l(n)|$. The rank of group is l. $\lambda_i(n)$, $i=1, 2, \ldots, l$, are simultaneously integral for vector IRs and simultaneously half-integral for spinor IRs. For odd n $\lambda_i(n)$ are positive and for even $n(O(n)=O(2l)=D_l)$ $\lambda_1(2l), \lambda_2(2l), \ldots, \lambda_{l-1}(2l)$ are positive and $\lambda_l(2l)$ are positive or negative. For even n there exist complex IRs $(\lambda_1(2l), \ldots, \lambda_{l-1}(2l), \lambda_l(2l))$ and $(\lambda_1(2l), \ldots, \lambda_{l-1}(2l), -\lambda_l(2l))$. The fourth rule is:

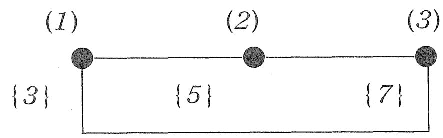
4) for even n and $\lambda_l(n(=2l))\neq 0$ there exist conjugate IRs and points on diagrams are doubly degenerated.

The fifth rule is explained by actual examples.

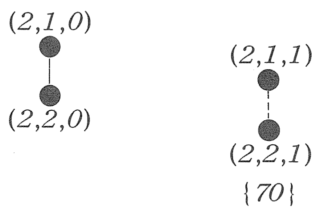
5) IR {*15*} with label (*3,1*) for group $O(4)$ occupies the point on diagram as



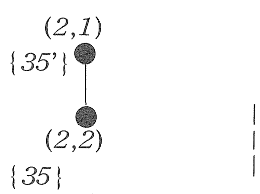
The corresponding part on the diagrams for group $O(3)$ is



Therefore IR {*15*} of group $O(4)$ is composed of IRs {*3*}, {*5*}, and {*7*} of group $O(3)$. The IR {*70*} with label (*2,2,1*) for group $O(6)$ occupies the point on diagram as



The left half of this figure is on the first floor and the right on the second floor. The step, which is not drawn in the diagrams, connects any IR on each floor only to immediately upper and definite IR on the next floor. The corresponding part on the diagram for group $O(5)$ is



Therefore IR {*70*} of group $O(6)$ is composed of IRs {*35'*} and {*35*} of group $O(5)$.

The fifth rule is identical with the recipe for preparing of similar genetic diagrams for special unitary group $SU(n)$ with arbitrary dimension which has been stated in reference[3].

The differences between genetic diagrams for group $O(n)$ and those for $SU(n)$ are: (1) diagrams for group $SU(n)$ always consist of infinite repeats of the identical subdiagram, while diagrams for group $O(n)$ have no repeat at all, (2) group $O(n)$ has two separated and independent diagrams-one for vector IRs and the other for spinor IRs-, while group $SU(n)$ has only a single diagram, and (3) points on diagram for group $SU(n)$ are in one-to-one correspondence with definite Young diagrams, while those for $O(n)$ have no concern at all with Young diagrams.

The inequalities[1),2)] such as

$$\lambda(3) \geqq \lambda(2) \geqq -\lambda(3), \quad \lambda_1(4) \geqq \lambda(3) \geqq |\lambda_2(4)|,$$
$$\lambda_1(5) \geqq \lambda_1(4) \geqq \lambda_2(5) \geqq \lambda_2(4) \geqq -\lambda_2(5),$$
$$\lambda_1(6) \geqq \lambda_1(5) \geqq \lambda_2(6) \geqq \lambda_2(5) \geqq |\lambda_3(6)|,$$
$$\lambda_1(7) \geqq \lambda_1(6) \geqq \lambda_2(7) \geqq \lambda_2(6) \geqq \lambda_3(7) \geqq \lambda_3(6) \geqq -\lambda_3(7), \ldots,$$

are automatically built-in within our five rules for preparation of diagrams for group $O(n)$. Diagrams for vector and spinor IRs of group $O(4)$ are shown in **Fig.** 1(a) and (b), respectively, together with diagrams for vector and spinor IRs of group $O(3)$ shown in **Fig.** 2(a) and (b), respectively.

The reader will be able to prepare any diagrams for group $O(n)$ with arbitrarily larger n with the help of five rules which have been explained in this note. For the

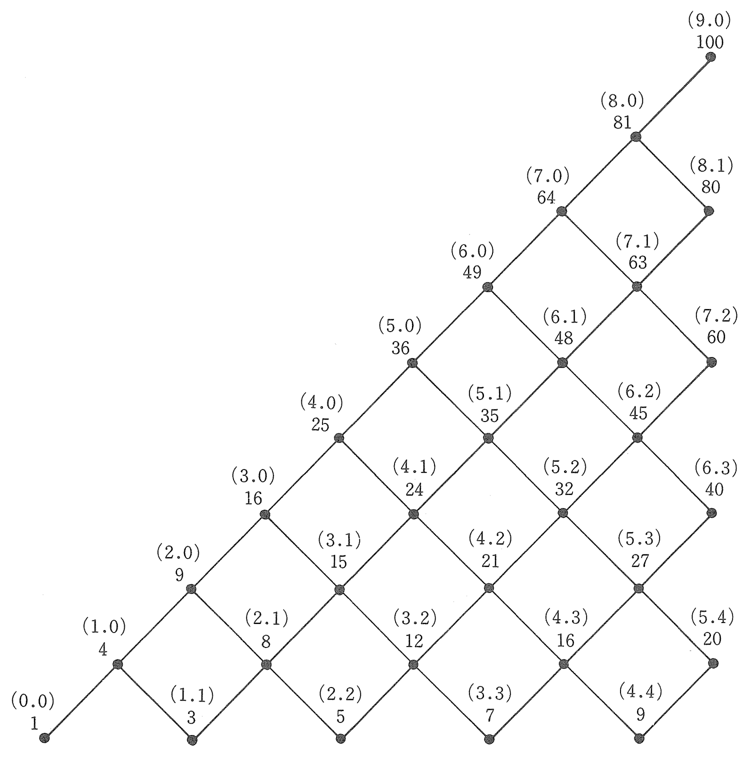


Fig. 1(a) Diagram for $O(4)$ vector IRs
Points denote IRs with label $(\lambda_1(4), \lambda_2(4))$. Numbers at points denote the dimension of IR. For $\lambda_2(4) \neq 0$ this diagram denotes the half of complex IRs.

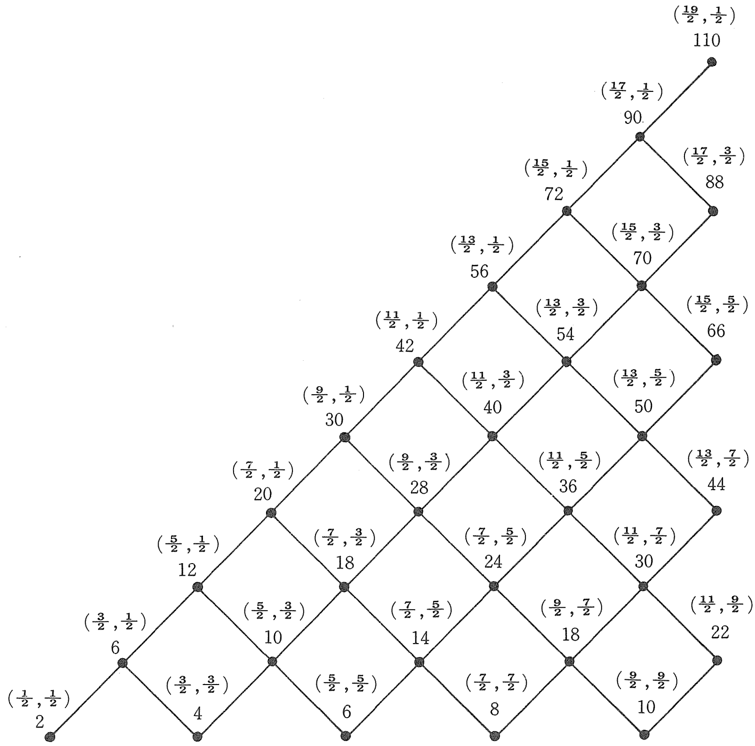


Fig. 1(b) Diagram for $O(4)$ spinor IRs
All IRs are complex. This diagram denotes the half of complex IRs. So every points are doubly degenerated.

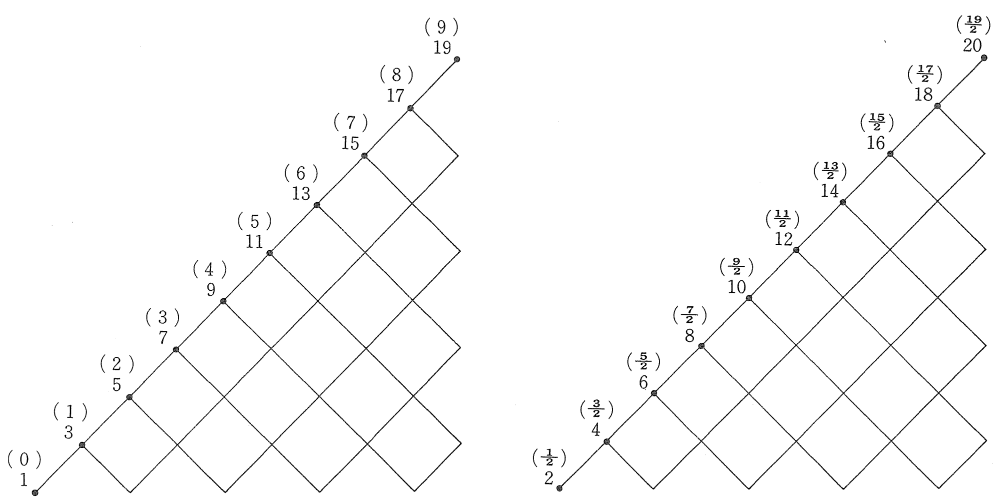


Fig. 2 Diagram for $O(3)$ vector (a) and spinor (b) IRs
Label of IR is $(\lambda(3))$. These diagrams are used to prepare $O(4)$ diagram.

sake of saving of space preparations of diagrams for $O(n)$, $n>4$ are left for reader's exercises.

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