Sci. Rep. Kanazawa Univ.,
Vol. 20 No. 1, pp 7-13
June 1975

On Indices of Unit Groups Related to the Genus Number of Galois Extensions

Norikata NAKAGOSHI*

Department of Mathematics, Faculty of Science, Kanazawa University

(Received March 26, 1975)

1. Let K/k be a Galois extension of finite degree, over a finite algebraic number field k and denote by K^* the genus field of K over k. A formula of the genus number $g_{K/k}=(K^* : K)$is given by Y. Furuta [2] :

$$g_{K/k}=\frac{h_k \prod_{\mathfrak{p}} e_{\mathfrak{p}}'}{(K_0 : k)(\varepsilon : \eta)},$$

where (ε) is the unit group of k, (η) is the subgroup of (ε) generated by the units η which are everywhere locally norm, h_k is the class number of k and the other notations are as in [2]. If K/k is cyclic, then $g_{K/k}$ is equal to the ambiguous ideal class number of K over k and we have some results on the relation between class numbers and $(\varepsilon : \eta)$ in [3], [6] and [7].

In the present paper, we treat of the divisors of the index $(\varepsilon : \eta)$ for Abelian extensions and decide the index for Kummer extensions $K=k(\sqrt[l]{\Delta})$, where l is an odd prime, Δ is an l-th power free rational integer and k is the l-th cyclotomic field over the rational number field $\mathbf{Q}$.

2. For each prime divisor $\mathfrak{p}$ of k, we denote by $U_{\mathfrak{p}}$ the unit group of the $\mathfrak{p}$-completion $k_{\mathfrak{p}}$ of k and by $H_{\mathfrak{p}}$ the subgroup of $U_{\mathfrak{p}}$, consisting of all units $\bar{\varepsilon}$ of $U_{\mathfrak{p}}$ such that $\bar{\varepsilon}\equiv 1 \bmod \mathfrak{p}$ (*"Einseinheitengruppe von $k_{\mathfrak{p}}$"*). We put

$$(\eta_{\mathfrak{p}})=(\varepsilon)\cap NU_{\mathfrak{P}},\quad (\eta'_{\mathfrak{p}})=(\varepsilon)\cap H_{\mathfrak{p}},$$

where $U_{\mathfrak{P}}$ is the unit group of $K_{\mathfrak{P}}$ for an extension $\mathfrak{P}$ of $\mathfrak{p}$ in K and N is the norm with respect to $K_{\mathfrak{P}}/k_{\mathfrak{p}}$.

*Present address : Department of Mathematics, College of Liberal Arts, Toyama University

Let $\mathfrak{R}_{\mathfrak{P}}^{\times}$, $\mathfrak{R}_{\mathfrak{p}}^{\times}$ be the cyclic groups, generated by a primitive $(\mathfrak{N}(\mathfrak{P})-1)$-th root of unity and a primitive $(\mathfrak{N}(\mathfrak{p})-1)$-th root of unity, respectively, where $\mathfrak{N}(\mathfrak{p})$ is the absolute norm of $\mathfrak{p}$.

LEMMA. *Let K/k be an Abelian extension of finite degree. Then we have*

(i) $e(\mathfrak{p})\equiv 0$ *mod* $(\varepsilon : \eta_{\mathfrak{p}})$ *for a prime divisor* $\mathfrak{p}$ *of* k,

(ii) *for a tamely ramified prime divisor* $\mathfrak{p}$ *of* k,

$$(\varepsilon : \eta_{\mathfrak{p}})\equiv 0 \ mod \ \frac{(w,\ \mathfrak{N}(\mathfrak{p})-1)}{(w,\ \frac{\mathfrak{N}(\mathfrak{p})-1}{e(\mathfrak{p})})} \text{ and } (\varepsilon : \eta_{\mathfrak{p}}')\equiv 0 \ mod \ (w,\ \mathfrak{N}(\mathfrak{p})-1),$$

where $e(\mathfrak{p})$ is the ramification index of $\mathfrak{p}$ in K/k, w is the number of roots of unity in k and (a, b) is the greatest common divisor for integers a and b.

Proof. If K/k is an Abelian extension of finite degree, then $(U_{\mathfrak{p}} : NU_{\mathfrak{P}})=e(\mathfrak{p})$, hence we have (i).

It is well-known the direct decompositions

$$U_{\mathfrak{P}}=\mathfrak{R}_{\mathfrak{P}}^{\times}\times H_{\mathfrak{P}},\ U_{\mathfrak{p}}=\mathfrak{R}_{\mathfrak{p}}^{\times}\times H_{\mathfrak{p}},$$

and

$$NU_{\mathfrak{P}}=N\mathfrak{R}_{\mathfrak{P}}^{\times}\times NH_{\mathfrak{P}},\ NH_{\mathfrak{P}}\cap H_{\mathfrak{p}}=NH_{\mathfrak{P}}. \tag{1}$$

On the other hand, $(U_{\mathfrak{p}} : NU_{\mathfrak{P}})=e(\mathfrak{p})$, $(U_{\mathfrak{p}} : H_{\mathfrak{p}})=\mathfrak{N}(\mathfrak{p})-1$ and $(H_{\mathfrak{p}} : H_{\mathfrak{p}}^{n_{\mathfrak{p}}})$ is the power of p, where $n_{\mathfrak{p}}=(K_{\mathfrak{P}} : k_{\mathfrak{p}})$ and p is the prime number divisible by $\mathfrak{p}$ (cf. [4], s. 222-223). Then it follows by the assumption that

$$NH_{\mathfrak{P}}=H_{\mathfrak{p}} \tag{2}$$

and

$$U_{\mathfrak{p}}/NU_{\mathfrak{P}}\cong\mathfrak{R}_{\mathfrak{p}}^{\times}/N\mathfrak{R}_{\mathfrak{P}}^{\times}. \tag{3}$$

Here $N\mathfrak{R}_{\mathfrak{P}}^{\times}$ is generated by a primitive $(\mathfrak{N}(\mathfrak{p})-1)/e(\mathfrak{p})$-th root of unity.

Let ξ_0 be a primitive w-th root of unity in (ε). Then $(\varepsilon : \eta_{\mathfrak{p}})\equiv 0 \mod (\xi_0 : (\xi_0)\cap(\eta_{\mathfrak{p}}))$, and by (1), (2) we have $(\xi_0 : (\xi_0)\cap(\eta_{\mathfrak{p}}))\equiv 0 \mod ((\xi_0)\cap\mathfrak{R}_{\mathfrak{p}}^{\times} : (\xi_0)\cap N\mathfrak{R}_{\mathfrak{P}}^{\times})$.

Finally, $(\varepsilon : \eta'_{\mathfrak{p}})\equiv 0 \mod ((\xi_0)\cap\mathfrak{R}_{\mathfrak{p}}^{\times} : 1)$. Thus (ii) is proved.

PROPOSITION. *Let K/k be an Abelian extension of finite degree and $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ be all the finite prime divisors of k tamely ramified in K. Then*

$$(\varepsilon : \eta)\equiv 0 \ mod \ 2^{\delta}\prod_{i=1}^{t}\frac{(w_{i-1},\ \mathfrak{N}(\mathfrak{p}_i)-1)}{(w_{i-1},\ \frac{\mathfrak{N}(\mathfrak{p}_i)-1}{e(\mathfrak{p}_i)})},$$

where

$$w_0=w,\quad w_i=(w_{i-1},\ (w,\ \frac{\mathfrak{N}(\mathfrak{p}_i)-1}{e(\mathfrak{p}_i)})\ p_i^{\mu_i}\ ),$$

p_i *is the prime number divisible by* $\mathfrak{p}_i$, $p_i^{\mu_i}$ *is the* p_i*-component of* w *for each* i $(1\leqq i\leqq t)$, *and*

$$\delta=\begin{cases}1, & \textit{if at least one of the infinite prime divisors of } k \textit{ is ramified in } K\\ & \textit{and } w_t=2,\\ 0, & \textit{otherwise.}\end{cases}$$

Proof. We put

$$(\eta_{\mathfrak{p}_i})=(\varepsilon)\cap NU_{\mathfrak{P}_i},\quad (\eta_s)=\bigcap_{i=1}^{s}(\eta_{\mathfrak{p}_i}),\quad 1\leqq i\leqq s\leqq t,$$

and let ξ_0 be a primitive w-th root of unity in (ε).

From the direct decomposition of $H_{\mathfrak{p}_i}$ and (2), we have

$$NH_{\mathfrak{P}_i}=H_{\mathfrak{p}_i}=Z_{\mathfrak{p}_i}\times H_{0,\mathfrak{p}_i},$$

where $Z_{\mathfrak{p}_i}$ is the cyclic subgroup of $H_{\mathfrak{p}_i}$ of p_i-th power order and $H_{0,\mathfrak{p}_i}$ is the free part of $H_{\mathfrak{p}_i}$ of rank $(k_{\mathfrak{p}_i}:\mathbf{Q}_{p_i})$.

For $i=1$,

$$(\eta_1)=(\varepsilon)\cap NU_{\mathfrak{P}_1}\supset((\xi_0)\cap N\mathfrak{R}_{\mathfrak{P}_1}^{\times})\times((\xi_0)\cap Z_{\mathfrak{p}_1})$$

and by (3)

$$(((\xi_0)\cap N\mathfrak{R}_{\mathfrak{P}_1}^{\times})\times((\xi_0)\cap Z_{\mathfrak{p}_1}):1)=(w,\ \frac{\mathfrak{N}(\mathfrak{p}_1)-1}{e(\mathfrak{p}_1)})\ p_1^{\mu_1}=w_1.$$

Thus (η_1) contians a primitive w_1-th root of unity.

Moreover we have

$$(\eta_2)=(\eta_{\mathfrak{p}_1})\cap(\eta_{\mathfrak{p}_2})\supset\bigcap_{i=1}^{2}\left\{((\xi_0)\cap N\mathfrak{R}_{\mathfrak{P}_i}^{\times})\times((\xi_0)\cap Z_{\mathfrak{p}_i})\right\},$$

hence (η_2) contains a primitive w_2-th root of unity and similarly as above, (η_s) contains a primitive w_s-th root of unity $(s=1,\ \cdots,\ t)$.

Let ξ_s be a primitive w_s-th root of unity for each s $(1\leqq s\leqq t)$. Then

$$(\eta_{s-1}:\eta_s)\equiv 0\ \mathrm{mod}\ (\xi_{s-1}:(\xi_{s-1})\cap(\eta_s)),$$

and

$$(\xi_{s-1}:(\xi_{s-1})\cap(\eta_s))\equiv 0\ \mathrm{mod}\ ((\xi_{s-1})\cap\mathfrak{R}_{\mathfrak{p}_s}^{\times}:(\xi_{s-1})\cap N\mathfrak{R}_{\mathfrak{P}_s}^{\times}),$$

where $(\eta_0)=(\varepsilon)$.

Suppose that the infinite real prime divisors of k are all unramified in K. Then $(\varepsilon : \eta) \equiv 0 \mod \prod_{i=1}^{t} (\eta_{s-1} : \eta_s)$ and the proposition is proved.

Next, let $\mathfrak{p}_{1,\infty}, \cdots, \mathfrak{p}_{\lambda,\infty}$ be the infinite prime divisors of k ramified in K and put

$$(\eta_{\mathfrak{p}_{j,\infty}}) = (\varepsilon) \cap NU_{\mathfrak{P}_{j,\infty}}, \quad (\eta_{t,\nu}) = (\eta_t) \cap \bigcap_{j=1}^{\nu} (\eta_{\mathfrak{p}_{j,\infty}}), \quad 1 \leqq j \leqq \nu \leqq \lambda.$$

Then

$$(\varepsilon : \eta) \equiv 0 \mod \prod_{s=1}^{t} (\eta_{s-1} : \eta_s) \cdot (\eta_t : \eta_{t,1}) \cdots (\eta_{t,\lambda-1} : \eta_{t,\lambda}), \quad (\eta_{\mathfrak{p}_{j,\infty}}) = \left\{\varepsilon_0 \in (\varepsilon) \mid \varepsilon_0 > 0\right\},$$

$j = 1, \cdots, \lambda$, $w = 2$, $(\eta_t : \eta_{t,1}) = \begin{cases} 1, & \text{if } w_t = 1, \\ 2, & \text{if } w_t = 2, \end{cases}$

and $(\eta_{t,1} : \eta_{t,2}) = \cdots = (\eta_{t,\lambda-1} : \eta_{t,\lambda}) = 1$. Thus we have the proposition.

3. We note that if K/k is a tamely ramified Abelian extension of finite degree, then $(\varepsilon : \eta)$ can be expressed as follows ; let ρ_i be a primitive $(\mathfrak{N}(\mathfrak{p}_i) - 1)$-th root of unity and $(\eta_s \mod \mathfrak{p}_i)$ be the subgroup of $\mathfrak{R}_{\mathfrak{p}_i}^{\times}$, generated by the representatives of $\eta_s \mod \mathfrak{p}_i$, $1 \leqq i \leqq s \leqq t$. Then

$$(\eta_{s-1})/(\eta_s) \cong (\eta_{s-1} \mod \mathfrak{p}_s)/(\eta_{s-1} \mod \mathfrak{p}_s) \cap N\mathfrak{R}_{\mathfrak{P}_s}^{\times},$$

where $(\eta_0) = (\varepsilon)$. We take also the rational integers $a(\mathfrak{p}_s)$ and $b(\mathfrak{p}_s)$ such that

$$(\eta_{s-1} \mod \mathfrak{p}_s) = \langle \rho_s^{a(\mathfrak{p}_s)} \rangle, \quad (\eta_{s-1} \mod \mathfrak{p}_s) \cap N\mathfrak{R}_{\mathfrak{P}}^{\times} = \langle \rho_s^{b(\mathfrak{p}_s)} \rangle.$$

Then $b(\mathfrak{p}_s)$ is the least common multiple of $a(\mathfrak{p}_s)$ and $e(\mathfrak{p}_s)$ and we have

$$(\eta_{s-1} : \eta_s) = \frac{d_0(\mathfrak{p}_s)}{d(\mathfrak{p}_s)},$$

where $d(\mathfrak{p}_s) = (a(\mathfrak{p}_s), \mathfrak{N}(\mathfrak{p}_s) - 1)$ and $d_0(\mathfrak{p}_s) = (b(\mathfrak{p}_s), \mathfrak{N}(\mathfrak{p}_s) - 1)$. Therefore, it follows

$$(\varepsilon : \eta) = 2^{\delta} \prod_{i=1}^{t} \frac{d_0(\mathfrak{p}_i)}{d(\mathfrak{p}_i)},$$

where δ is defined in the proposition.

4. Examples.

(a). Put $k = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{\Delta})$, where Δ is a square free positive rational integer, having prime factors $p \neq 2$. For each odd prime factor p of Δ we have

$$\frac{(2, p-1)}{(2, \frac{p-1}{2})} = \begin{cases} 1, & \text{if } p \equiv 1 \mod 4, \\ 2, & \text{if } p \not\equiv 1 \mod 4. \end{cases}$$

Then we have by (i) of the lemma that $(\varepsilon : \eta_p) = 2$, if $p \not\equiv 1 \mod 4$.

(b). Let $K=k(\sqrt[l]{\Delta})$, where l is an odd prime number, Δ is an l-th power free rational integer having prime factors $p\neq l$ and k is the l-th cyclotomic field over $\mathbf{Q}$.

For a prime factor p of Δ not equal to l, we have

$$\frac{(2l,\ \mathfrak{N}(\mathfrak{p})-1)}{(2l,\ \frac{\mathfrak{N}(\mathfrak{p})-1}{l})}=\begin{cases}1, & \text{if } \mathfrak{N}(\mathfrak{p})\equiv 1 \bmod l^2,\\ l, & \text{if } \mathfrak{N}(\mathfrak{p})\not\equiv 1 \bmod l^2.\end{cases}$$

Hence we have $(\varepsilon:\eta_{\mathfrak{p}})=l$, if $\mathfrak{N}(\mathfrak{p})\not\equiv 1 \bmod l^2$.

(c). Let $k=\mathbf{Q}(\zeta)$, $\zeta=\exp(2\pi i/3\cdot5\cdot7\cdot11)$ and $K=k(\sqrt[3\cdot5]{7\cdot11})$.

Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be the prime divisors of k, lying above 7, 11, respectively. Then

$$\frac{(2\cdot3\cdot5\cdot7\cdot11,\ \mathfrak{N}(\mathfrak{p}_i)-1)}{(2\cdot3\cdot5\cdot7\cdot11,\ \frac{\mathfrak{N}(\mathfrak{p}_i)-1}{3\cdot5})}=\begin{cases}3, & \text{for } i=1,\\ 5, & \text{for } i=2,\end{cases}$$

and $(\varepsilon:\eta)\equiv 0 \bmod 3\cdot5$.

5. Notations being as above, we have the following

THEOREM. *Let l be an odd prime and ζ be a primitive l-th root of unity. We put $k=Q(\zeta)$ and $K=k(\sqrt[l]{\Delta})$, where Δ is an l-th power free rational integer. Let $\mathfrak{p}$ be a prime divisor of k, lying above a rational prime p and t be a number of ramified prime divisors of k in K.*

(i) *If $\mathfrak{N}(\mathfrak{p})\equiv 1 \bmod l^2$ for each prime factor p of Δ and l is a regular prime, then*

$$(\varepsilon:\eta)=1 \text{ and } g_{K/k}=h_k\cdot l^{t-1}.$$

(ii) *If $\mathfrak{N}(\mathfrak{p})\not\equiv 1 \bmod l^2$ for some prime factor p of Δ and $\Delta^{l-1}\equiv 1 \bmod l^2$, then*

$$(\varepsilon:\eta)=l \text{ and } g_{K/k}=h_k\cdot l^{t-2}.$$

Proof. (A) Let ρ be a primitive $(\mathfrak{N}(\mathfrak{p})-1)$-th root of unity for the prime divisor $\mathfrak{p}$ of Δ $(p\neq l)$.

(a_1) In the case where $\mathfrak{N}(\mathfrak{p})\not\equiv 1 \bmod l^2$.

If $(\varepsilon:\eta_{\mathfrak{p}})=1$, i.e. $(\varepsilon)=(\eta_{\mathfrak{p}})=(\varepsilon)\cap(N\mathfrak{R}_{\mathfrak{P}}^{\times}\times H_{\mathfrak{p}})$ by (1) and (2), then $\zeta\in N\mathfrak{R}_{\mathfrak{P}}^{\times}$. Since $N\mathfrak{R}_{\mathfrak{P}}^{\times}=\langle\rho^l\rangle$ by (3) and $e(\mathfrak{p})=l$, $(\mathfrak{N}(\mathfrak{p})-1)/l\equiv 0 \bmod l$. Therefore, if $\mathfrak{N}(\mathfrak{p})\not\equiv 1 \bmod l^2$, then $(\varepsilon:\eta_{\mathfrak{p}})\neq 1$ and $\zeta\notin N\mathfrak{R}_{\mathfrak{P}}^{\times}$. Hence we have $(\varepsilon:\eta_{\mathfrak{p}})=l$ by (i) of the lemma.

Moreover, (ε) contains a primitive $2l$-th root of unity and the roots of unity in $(\eta_{\mathfrak{p}})$ are ± 1, a system of the fundamental units of (ε) can be chosen the same as that of $(\eta_{\mathfrak{p}})$.

(a_2) In the case where $\mathfrak{N}(\mathfrak{p})\equiv 1 \bmod l^2$ and l is a regular prime.

Let

$$\mathfrak{N}(\mathfrak{p})\equiv 1 \bmod l^{\mu},\quad \not\equiv 1 \bmod l^{\mu+1},\ \mu\geqq 2.$$

We denote by ζ_j a primitive l^j-th root of unity and by (ε_j) the unit group of $k_j=Q(\zeta_j)$ for each j $(1\leqq j\leqq\mu,\ \zeta_1=\zeta,\ k_1=k,\ (\varepsilon_1)=(\varepsilon))$.

Let $\mathfrak{p}=\mathfrak{p}_1=\mathfrak{p}_1^{(1)}\cdots\mathfrak{p}_1^{(l^{\mu-1})}$ be the prime ideal decomposition of $\mathfrak{p}=\mathfrak{p}_1$ in k_μ. Since $\mathfrak{N}(\mathfrak{p}_1)=\mathfrak{N}(\mathfrak{p}_1^{(s)})$, $s=1,\ \cdots,\ l^{-1}$, each ε_μ of (ε_μ) can be written as

$$\varepsilon_\mu=\rho^x\cdot\eta^{(s)},\quad \rho^x\in\mathfrak{R}_{\mathfrak{p}}^{\times},\quad \eta^{(s)}\in H_{\mathfrak{p}_1^{(s)}},\quad s=1,\ \cdots,\ l^{\mu-1},$$

where $H_{\mathfrak{p}_1^{(s)}}$ are the "Einseinheitengruppen" of the $\mathfrak{p}_1^{(s)}$-completions of k_μ. Then

$$N_{k_\mu/k_1}\varepsilon_\mu=(\rho^l)^{x\cdot l^{\mu-2}}\cdot\prod_{s=1}^{l^{\mu-1}}\eta^{(s)}.$$

Since $N\mathfrak{R}_{\mathfrak{P}}^{\times}=\langle\rho^l\rangle$ by (3) and $e(\mathfrak{p})=l$, $N_{k_\mu/k_1}\varepsilon_\mu\in(\eta_{\mathfrak{p}_1})=(\eta_{\mathfrak{p}})$ and $N_{k_\mu/k_1}(\varepsilon_\mu)\subset(\eta_{\mathfrak{p}})$.

For the regular prime l, we see in [6] that

$$N_{k_{j+1}/k_j}(\varepsilon_{j+1})=(\varepsilon_j),\quad j=1,\ \cdots,\ \mu-1,\ \cdots$$

Hence $(\varepsilon:\eta_{\mathfrak{p}})=1$.

(B) We know by (a_1) that if $(\varepsilon:\eta_{\mathfrak{p}})=(\varepsilon:\eta_{\mathfrak{p}'})=l$ for two tamely ramified prime divisors $\mathfrak{p}$ and $\mathfrak{p}'$ of k, then $(\eta_{\mathfrak{p}})=(\eta_{\mathfrak{p}'})$.

Now, it is proved in [1] that $\mathfrak{l}=(1-\zeta)$ is unramified in K if and only if $\Delta^{l-1}\equiv1 \bmod l^2$. Thus the theorem is proved.

6. Example.

Let l be an odd prime and ζ be a primitive l-th root of unity. We put $k=Q(\zeta)$and $K=k(\sqrt[l]{\Delta})$, where Δ is an l-th power free rational integer such that

$$\Delta=\prod_{i=1}^{t}\Delta_i,\quad \Delta_i\neq1,\quad (\Delta_i,\ \Delta_j)=1 \text{ for } 1\leq i<j\leq t$$

and

$$\Delta_i^{l-1}\equiv 1 \bmod l^2\ (i=1,\ \ldots,\ t).$$

Then we have

$$d^{(l)}C_K\geq t-1+d^{(l)}C_k,$$

where $d^{(l)}C_k$ is the l-rank of the ideal class group C_k of k.

In particular, if l is a regular prime and $\Delta_i=p_i$ are prime numbers satisfying the above conditions and their orders modulo l are all $l-1$, then

$$K^*=\overline{k}(\sqrt[l]{p_1},\ \ldots\ldots,\ \sqrt[l]{p_t}),$$

where $\bar{k}$ is the absolute class field of k.

Proof. $K(\sqrt[l]{\Delta})/K$ are unramified extensions by [1] ($i=1, \ldots, t$) and then $\prod_{i=1}^{t} K(\sqrt[l]{\Delta_i}) = k(\sqrt[l]{\Delta_1}, \ldots, \sqrt[l]{\Delta_t})$ is an unramified Abelian extension over K, of degree l^{t-1}.

In particular, if l is a regular prime and $\Delta_i=p_i$ are primes mentioned as above, then $(\varepsilon : \eta) = 1$ and p_i are all primes in k. Hence $g_{K/k}=h_k \cdot l^{t-1}$ and $K^*=\bar{k}(\sqrt[l]{p_1}, \ldots, \sqrt[l]{p_t})$.

References

[1] Akiyama, S., On the class numbers of certain Kummer extensions (in Japanese), Sugaku (Tokyo), **21** (1967), 216-217.

[2] Furuta, Y., The genus field and genus number in algebraic number fields, Nagoya Math. J., **29** (1967), 281-285.

[3] Furuta, Y., Über das Geschlecht und die Klassenzahl eines relativ-Galoisschen Zahlkörpers vom Primzahlgrad, Nagoya Math. J., **37** (1970), 197-200.

[4] Hasse, H., Zahlentheorie, Akademie-Verlag, Berlin, 1963,

[5] Hasse, H., "Bericht", Physica-Verlag, 1965.

[6] Kuroda, S-N., Über die Klassenzahl eines relativ-zyklischen Zahlkörpers vom Primzahlgarad, Proc. Japan Akad., XL(1964), 623-626.

[7] Yokoi, H., On the class number of a relatively cyclic number field, Nagoya Math. J., **29** (1967), 31-44.