Sci. Rep. Kanazawa Univ.
Vol. 20, No. 1, pp. 1-6
June 1975

On Embedding Dimensions of Quotient Singularities

Akira IMAI*

Department of Mathematics, Faculty of Science, Kanazawa University
(Received April 14, 1975)

Introduction.

In the case of dimension two the embedding dimensions of quotient singularities were calculated almost completely by E. Brieskorn [1], but in case that the dimension is larger than two they have been scarcely known except for some special examples. The purpose of this paper is to obtain an estimate for embedding dimensions of quotient singularities in general dimensions by calculating their ananalytic local algebras. The author wishes to express his sincere gratitude to Professor E. Sakai for his kind encouragements and valuable discussions.

§ 1. G-invariant analytic local algebras.

In this section we shall enumerate some properties of analytic local algebras and their G-invariant subalgebras according to H. Grauert-R. Remmert [3]. Throughout this paper an algebra means a commutative algebra with a unit element and we assume that a homomorphism between algebras maps a unit element to a unit element.

DEFINITION 1. Let k be a commutative field. A k-algebra A is called a *k-local algebra* if it is a local ring and if the natural monomorphism $k \to A/\mathfrak{m}$ is an isomorphism where $\mathfrak{m}$ is the unique maximal ideal of A.

DEFINITION 2. Let A be a k-local algebra, and $\mathfrak{m}$ be its maximal ideal. The *embedding dimension* of A is the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a k-vector space, and will be denoted by emdim A.

PROPOSITION 1. *Let A be a k-local algebra, $\mathfrak{m}$ be its maximal ideal and $\mathfrak{a}$ be a proper ideal of A. Then we have* $\operatorname{emdim} A - \operatorname{emdim} A/\mathfrak{a} = \dim_k (\mathfrak{a}+\mathfrak{m}^2)/\mathfrak{m}^2$.

The field of complex numbers will be denoted by $\mathbf{C}$, and $\mathbf{C}$-algebra consisting of all convergent power series of n variables $X_1, \ldots, X_n$ will be denoted by $\mathbf{C}\langle X_1, \ldots, X_n\rangle$ or K_n.

---

* Present address: Department of Mathematics, Faculty of Science, Kyushu University

Definition 3. A C-algebra is called an *analytic local algebra* if it is isomorphic to the C-algebra of a form $K_n/\mathfrak{a}$ where $\mathfrak{a}$ is a proper ideal of K_n.

Remark 1. An analytic local algebra is a C-local algebra.

Remark 2. K_n is an analytic local algebra and its embedding dimension is n.

Proposition 2. *Let A be an analytic local algebra. Then the embedding dimension of A is the minimal number of m such that A is isomorphic to the quotient $K_m/\mathfrak{a}$ of K_m by a proper ideal $\mathfrak{a}$ of K_m.*

Definition 4. Let A be an analytic local algebra, and G be a subgroup of Aut A where Aut A is the group of all C-algebra automorphisms of A. Then the subalgebra $A^G := \{f \in A : \phi(f) = f$ for all $\phi \in G\}$ of A is called the *G-invariant subalgebra* of A.

Definition 5. A C-algebra endomorphism γ of $\mathbf{C}\langle X_1, \dots, X_n\rangle$ is said to be linear if there exists a (n, n) C-matrix (a_{ij}) such that

$$\begin{pmatrix} \gamma(X_1) \\ \vdots \\ \gamma(X_n) \end{pmatrix} = \begin{pmatrix} a_{11} \cdots a_{1n} \\ \vdots \quad \vdots \\ a_{n1} \quad a_{nn} \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}.$$

A subgroup of Aut $\mathbf{C}\langle X_1, \dots, X_n\rangle$ is said to be *linear* if all its elements are linear.

Propositiop 3. *Let G be a finite linear subgroup of* Aut $\mathbf{C}\langle Y_1, \dots, Y_n\rangle$, *$q_1, \dots, q_m \in \mathbf{C}[Y_1, \dots, Y_n]^G$ (where $\mathbf{C}[Y_1, \dots, Y_n]^G := \mathbf{C}[Y_1, \dots, Y_n] \cap \mathbf{C}\langle Y_1, \dots, Y_n\rangle^G$) be homogeneous polynomials of positive degrees, and assume that $q_1, \dots, q_m$ generate $\mathbf{C}[Y_1, \dots, Y_n]^G$ as a C-algebra. Then the C-algebra homomorphism $\alpha : \mathbf{C}\langle X_1, \dots, X_m\rangle \to \mathbf{C}\langle Y_1, \dots, Y_n\rangle$ which satisfies $\alpha(X_\mu) = q_\mu$ $(\mu = 1, \dots, m)$ is an epimorphism onto $\mathbf{C}\langle Y_1, \dots, Y_n\rangle^G$. Accordingly $\mathbf{C}\langle Y_1, \dots, Y_n\rangle^G \cong \mathbf{C}\langle X_1, \dots, X_m\rangle /$ Ker α, and specially $\mathbf{C}\langle Y_1, \dots, Y_n\rangle^G$ is an analytic local algebra.*

Proposition 4. *Let G be a finite linear and cyclic subgroup of* Aut $\mathbf{C}\langle Y_1, \dots, Y_n\rangle$, *$b$ be its order and γ be its generator. Then there exist a linear automorphism λ and integers $b_1, \dots, b_n$ $(0 \leqq b_\nu < b;\ \nu = 1, \dots, n)$ such that $\mathfrak{Z}(Y_\nu) = \xi^{b_\nu} Y_\nu$ $(\nu = 1, \dots, n;\ \mathfrak{Z} := \lambda\gamma\lambda^{-1})$ where ξ is a primitive b-th root of unity. Hence it follows $\lambda G \lambda^{-1} = \{1, \mathfrak{Z}, \dots, \mathfrak{Z}^{b-1}\}$, $\mathbf{C}\langle Y_1, \dots, Y_n\rangle^G \cong \mathbf{C}\langle Y_1, \dots, Y_n\rangle^{\lambda G \lambda^{-1}}$.*

§ 2. The calculation of K_n^G.

In this section we shall try to express K_n^G in the form $K_m/\mathfrak{a}$ for a finite linear group G. For a positive integer n, GL$(n, \mathbf{C})$ denotes the general linear group over C of dimension n. The following proposition was proved by E. Noether [5].

Proposition 5. *Let*

$$\left\{ \begin{pmatrix} a_{11}^{(\alpha)} & \cdots & a_{1n}^{(\alpha)} \\ \vdots & & \vdots \\ a_{n1}^{(\alpha)} & \cdots & a_{nn}^{(\alpha)} \end{pmatrix} \middle| \ \alpha = 1, \cdots, b \right\}$$

be a subgroup of GL$(n, \mathbf{C})$, *$g^{(\alpha)}$ be the automorphisms of $\mathbf{C}[X_1, \dots, X_n]$ defined by* *relations* $g^{(\alpha)}(X_i)=\sum_{k=1}^{n} a_{ik}^{(\alpha)} X_k$ $(i=1, \ldots, n)$ *and put* $G:=\{g^{(\alpha)} \mid \alpha=1, \ldots, b\}$. *Define polynomials* $Gr_1\cdots r_n(X_i^{(\alpha)})$ *of* nb *variables* $X_i^{(\alpha)}$ $(i=1, \ldots, n,\ \alpha=1, \ldots, b)$ *by*

$$\prod_{\alpha=1}^{b}(u+u_1X_1^{(\alpha)}+\ldots+u_nX_n^{(\alpha)})=\sum ur u_1^{r_1}\ldots u_n^{r_n}Gr_1\ldots r_n(X_i^{(\alpha)})\ (r+r_1+\cdots+r_n=b)$$

and let $Fr_1\ldots r_n(X_1, \ldots, X_n)$ *be the polynomials of* $X_1, \ldots, X_n$ *which are got by substituting* $X_i^{(\alpha)}$ *of* $Gr_1\cdots r_n(X_i^{(\alpha)})$ *by* $\sum_{k=1}^{n} a_{ik}^{(\alpha)} X_k$. *Then* $\mathbf{C}[X_1, \ldots, X_n]^G$ *is generated as a* $\mathbf{C}$*-algebra by all of* $Fr_1\cdots r_n(X_1, \ldots, X_n)$ $((r_1, \ldots, r_n)\neq(0, \ldots, 0))$.

Now let G be a linear subgroup of Aut $\mathbf{C}\langle X_1, \ldots, X_n\rangle$ of order $b<\infty$ and G' be the group of automorphisms of $\mathbf{C}[X_1, \ldots, X_n]$ induced by G. Applying proposition 5 to G', we see that $Fr_1\cdots r_n(X_1, \ldots, X_n)$ $((r_1, \ldots, r_n)\neq(0, \ldots, 0))$ generate $\mathbf{C}[X_1, \ldots, X_n]^{G'}$, that they are homogeneous polynomials of positive degree or zeros, and that their number is $(b+1)\cdots(b+n)/n!-1$. Let $\{F_1, \ldots, F_m\}$ be the set of all non-zero elements among $Fr_1\cdots r_n(X_1, \ldots, X_n)$ $((r_1, \ldots, r_n)\neq(0, \ldots, 0))$. Then the $\mathbf{C}$-algebra homomorphism $\alpha:\mathbf{C}\langle Y_1, \ldots, Y_m\rangle\to\mathbf{C}\langle X_1, \ldots, X_n\rangle$ satisfying $\alpha(Y_1)=F_1, \ldots, \alpha(Y_m)=F_m$ is an epimorphism of $\mathbf{C}\langle Y_1, \ldots, Y_m\rangle$ onto $\mathbf{C}\langle X_1, \ldots, X_n\rangle^G$ and satisfies $\mathbf{C}\langle X_1, \ldots, X_n\rangle^G\cong \mathbf{C}\langle Y_1, \ldots, Y_n\rangle/\mathrm{Ker}\ \alpha$ according to Proposition 3. Furthermore we assume that G is a cyclic group and that $b\neq 1$, then there exists $\lambda\in$ Aut $\mathbf{C}\langle X_1, \ldots, X_n\rangle$ such that $\lambda G\lambda^{-1}=\{1, \mathfrak{J}, \ldots, \mathfrak{J}^{b-1}\}$, $\mathfrak{J}(X_\nu)=\xi^{b_\nu}X_\nu$ $(\nu=1, \ldots, n)$ and $\mathbf{C}\langle X_1, \ldots, X_n\rangle^G\cong\mathbf{C}\langle X_1, \ldots, X_n\rangle^{\lambda G\lambda^{-1}}$ where ξ is a primitive b-th root of unity. Put $\widetilde{G}=\lambda G\lambda^{-1}$ and let $\widetilde{G}'$ be the group of automorphisms of $\mathbf{C}[X_1, \ldots, X_n]$ induced by $\widetilde{G}$. Now we apply Proposition 5 to $\widetilde{G}'$, then $F_{10\cdots0}, \ldots, F_{0\cdots01}$ are of the form $c_1X_1, \ldots, c_nX_n$ $(c_1, \ldots, c_n\in\mathbf{C})$ and $F_{m0\cdots0}, \ldots, F_{0\cdots0m}$ $(m=2, \ldots, b)$ are of the form $c_{m1}X_1^m, \ldots, c_{mn}X_n^m$ $(c_{m1}, \ldots, c_{mn}\in\mathbf{C})$, so if $c_j\neq 0$, $F_{0\cdots\underset{j}{m}\cdots0}$ $(m=2, \ldots, b)$ can be written as a polynomial of $F_{0\cdots\underset{j}{1}\cdots0}$ Hence $\mathbf{C}[X_1, \ldots, X_n]^{\widetilde{G}'}$ is generated by polynomials of positive degree, the number of which is not larger than $(b+1)\cdots(b+n)/n!-1-n$. By the preceding argument and proposition 2, we have the following theorem.

THEOREM 1. *Let* G *be a linear subgroup of* Aut K_n, *of order* $b<\infty$. *Then we have*

$$\text{(1)}\qquad \operatorname{emdim} K_n^G \leqq \frac{(b+1)\cdots(b+n)}{n!}-1.$$

Furthermore if G *is cyclic and* $b\neq 1$, *then we have*

$$\text{(2)}\qquad \operatorname{emdim} K_n^G \leqq \frac{(b+1)\cdots(b+n)}{n!}-1-n.$$

REMARK 3. Let g be a automorphism of $K_n=\mathbf{C}\langle X_1, \ldots, X_n\rangle$ which satisfies $g(X_i)=-X_j$ $(j=1, \ldots, n)$. We take a cyclic group of order 2 generated by g as G in Theorem 1. Then an equality holds for (2) of Theorem 1.

PROOF. If $b=2$, $(b+1)\cdots(b+n)/n!-1-n=n(n+1)/2$, so it sufficies to prove emdim $K_n^G=n(n+1)/2$. Let G' be the group of automorphisms of $\mathbf{C}[X_1, \ldots, X_n]$ induced by G, then $\mathbf{C}[X_1, \ldots, X_n]^{G'}$ is generated by X_kX_l $(k,l=1, \ldots, n)$ according to Proposition 5. The number of X_kX_l $(k, l=1, \ldots, n)$ is $n(n+1)/2$. Put $m:=n(n+1)/2$. The homomorphism α of $K_m=\mathbf{C}\langle Y_1, \ldots, Y_m\rangle$ to $K_n=\mathbf{C}\langle X_1, \ldots, X_n\rangle$ which satisfies $\alpha(Y_1)=X_1^2$, $\alpha(Y_2)=X_1X_2, \ldots, \alpha(Y_m)=X_n^2$ is an epimorphism to $\mathbf{C}\langle X_1, \ldots, X_n\rangle^G$ according to proposition 3 and we have $\mathbf{C}\langle X_1, \ldots, X_n\rangle^G \cong \mathbf{C}\langle Y_1, \ldots, Y_m\rangle/\mathrm{Ker}\,\alpha$. If $\alpha(\sum a_{\nu_1\cdots\nu_m}\cdot Y_1^{\nu_1}\ldots Y_m^{\nu_m})=0$, $\sum a_{\nu_1\cdots\nu_m}Y_1^{\nu_1}\ldots Y_m^{\nu_m}\in\mathbf{C}\langle Y_1, \ldots, Y_m\rangle$, then $\sum a_{\nu_1\cdots\nu_m}(X_1^2)^{\nu_1}(X_1X_2)^{\nu_2}\ldots(X_n^2)^{\nu_m}=0$, and we have $a_{0\cdots0}=0$, $a_{10\cdots0}=\ldots=a_{0\cdots01}=0$ by comparing the constant terms and the coefficients of X_kX_l of both sides of this equality. This shows that $\mathrm{Ker}\,\alpha\subset\mathfrak{m}^2$ and that $\dim_{\mathbf{C}}(\mathrm{Ker}\,\alpha+\mathfrak{m}^2)/\mathfrak{m}^2=0$ where $\mathfrak{m}$ is the maximal ideal of K_m. Hence we have $0=\dim_{\mathbf{C}}(\mathrm{Ker}\,\alpha+\mathfrak{m}^2)/\mathfrak{m}^2=\mathrm{emdim}\,K_m-\mathrm{emdim}\,K_m/\mathrm{Ker}\,\alpha=m-\mathrm{emdim}\,K_m/\mathrm{Ker}\,\alpha$ according to Proposition 1 and Remark 2, i. e. emdim $K_n^G=\mathrm{emdim}\,K_m/\mathrm{Ker}\,\alpha=m=n(n+1)/2$.

§ 3. Quotient singularities.

To simplify the notation, we write X for an analytic space $(X, \mathfrak{A})$ with a structure sheaf $\mathfrak{A}$.

DEFINITION 6. A *singularity* is a pair (X, x) of an analytic space X and a point x of X.

DEFINITION 7. Singularities (X, x) and (Y, y) are said to be *isomorphic* if there exists a biholomorphic map of a neighbourhood of x in X onto a neighbourhood of y in Y which maps x to y.

Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions over $\mathbf{C}^n$, G be a group of analytic automorphisms acting properly and discontinuously on $\mathbf{C}^n$, and p be the natural projection of $\mathbf{C}^n$ onto the quotient space $\mathbf{C}^n/G$. To any open set $U\subset\mathbf{C}^n/G$, we associate the ring $F_U:=\{\phi\mid\phi:U\to\mathbf{C}$ continuous, $\phi\circ p\in\Gamma(p^{-1}(U), \mathcal{O})\}$ and to any pair of open sets $V\subset U$ we associate the restriction map $f_V^U: F_U\to F_V$. Then the system $\{F_U, f_V^U\}$ forms a presheaf of rings over $\mathbf{C}^n/G$. Let $\mathcal{O}_{\mathbf{C}^n/G}$ be the sheaf of rings associated to this presheaf. Hereafter, we shall use these notation without explanations.

PROPOSITION 6. (H. Cartan [2]) *The ringed space* $(\mathbf{C}^n/G, \mathcal{O}_{\mathbf{C}^n/G})$ *is an analytic space and p is an analytic map. Let* 0 *be the origin of* $\mathbf{C}^n$, *and* $p_0^*: (\mathcal{O}_{\mathbf{C}^n/G})_{p(0)}\to\mathcal{O}_0$ *be the* $\mathbf{C}$-*algebra homomorphism defined by* $(f)_{p(0)}\mapsto(f\circ p)_0$. *Here we use the subscripts* 0 *and* $p(0)$ *in order to associate the sheaves or mappings with their stalks or germs at the point* 0 *and* $p(0)$. *Furthermore let* G_0 *be the isotropy subgroup of* G *at* 0, *and let* G_0^* *be the group of all* $\mathbf{C}$-*algebra automorphisms* $g^*: \mathcal{O}_0\to\mathcal{O}_0\,;\,h_0\mapsto(h\circ g)_0$, $g\in G$. *Then* $(\mathbf{C}^n/G, p(0))$ *and* $(\mathbf{C}^n/G_0, p'(0))$ *are isomorphic where p' is the natural projection of* $\mathbf{C}^n$ *onto* $\mathbf{C}^n/G$, *and* p_0^* *is an isomorphism of* $(\mathcal{O}_{\mathbf{C}^n/G})_{p(0)}$ *onto* $\mathcal{O}_0^{G_0^*}$

DEFINITION 8. A singularity (X, x) is called a *quotient singularity* if (X, x) is isomorphic to $(\mathbf{C}^n/G, p(0))$. We write $(\mathbf{C}^n/G, 0)$ for $(\mathbf{C}^n/G, p(0))$.

The following remark is due to H. Cartan [2] or E. Brieskorn [1].

REMARK 4. Let $X_1, \dots, X_n$ be a coordinate system of $\mathbf{C}^n$, put $g(X)=(g_1(X), \dots, g_n(X))$ for $g\in G_0$ and let

$$g_j(X_1, \dots, X_n) = \sum_{\nu_1\cdots\nu_n} a^{(j)}_{\nu_1\cdots\nu_n} X_1^{\nu_1} \dots X_n^{\nu_n}$$

be the Taylor expansion of g_j at 0. We identify the stalk $\mathcal{O}_0$ with $\mathbf{C}\langle X_1, \dots, X_n\rangle$, then for $g\in G_0$ g^* is given by

$$\mathbf{C}\langle X_1, \dots, X_n\rangle \to \mathbf{C}\langle X_1, \dots, X_n\rangle \; ; X_j \mapsto \sum_{\nu_1\cdots\nu_n} a^{(j)}_{\nu_1\cdots\nu_n} X_1^{\nu_1} \dots X_n^{\nu_n}.$$

Put

$$g'(X)=(a^{(1)}_{10\dots0} X_1+\dots+a^{(1)}_{0\dots01}X_n, \dots, a^{(n)}_{10\dots0}X_1+\dots+a^{(n)}_{0\dots01}X_n)$$

and

$$\sigma(x) := \sum_{g\in G_0} g'^{-1}\circ g(X)/b \quad (\text{where } b := \# G_0),$$

then σ is a biholomorphic map of a neighbourhood of 0 onto a neighbourhood of 0 such that $\sigma(0)=0$, and $G_0' := \sigma G_0 \sigma^{-1}$ is a group of linear automorphisms of $\mathbf{C}^n$ such that $(\mathbf{C}^n/G_0, 0)$ is isomorphic to $(\mathbf{C}^n/G_0', 0)$. Here, of course, it follows $\# G_0=\# G_0'$. Furthermore, let G_0^*, $G_0'^*$ be the group of $\mathbf{C}$-algebra automorphisms of $\mathbf{C}\langle X_1, \dots, X_n\rangle$ induced respectively by G_0, G_0', then G_0^* and $G_0'^*$ are conjugate each other. And $G_0'^*$ is linear according to the linearity of G'_0.

DEFINITION 9. For a singularity (X, x), the *embedding dimension* of (X, x) is the minimal number of m such that some neighbourhood of x is biholomorphic to the analytic subset of a domain in $\mathbf{C}^m$. The embedding dimension of (X, x) is denotod by emdim (X, x).

PROPOSITION 7. (R. D. Gunning [4]) *For a singularity* (X, x), emdim (X, x) *is equal to the embedding dimension of the stalk of the structure sheaf of* X *at* x *as a* $\mathbf{C}$-*local algebra.*

According to Theorem 1, Remark 4, Proposition 6 and Proposition 7, we obtain the following theorem.

THEOREM 2. *Let* G *be a group of analytic automorphisms acting properly and discontinuously on* $\mathbf{C}^n$, G_0 *be the isotropy subgroup of* G *at the origin* 0 *of* $\mathbf{C}^n$ *and put* $b := \# G_0$. *Then we have*

$$\text{(1)} \qquad \operatorname{emdim}(\mathbf{C}^n/G, 0) \leq \frac{(b+1)\cdots(b+n)}{n!} - 1.$$

Furthermore if G_0 *is cyclic and* $b\neq 1$, *then we have*

$$\text{(2)} \qquad \operatorname{emdim}(\mathbf{C}^n/G, 0) \leq \frac{(b+1)\cdots(b+n)}{n!} - 1 - n.$$

Remark 5. We take the cyclic group generated by an automorphism $g(X_1, \ldots, X_n) = (-X_1, \ldots, -X_n)$ of $\mathbf{C}^n$ as G in (2) of Theorem 2. Then an equality holds for (2) of Theorem 2 according to Remark 3.

Now we consider the problem whether there exists an example such that an equality holds for (1) of theorem 2 in case of $b \neq 1$. For this purpose we state a definition and a proposition.

Definition 10. A subgroup of $\mathrm{GL}(n, \mathbf{C})$, G is said to be *small* if no $g \in G$ has 1 as an eigen-value of multiplicity $n-1$.

Proposition 9. (D. Prill [6]) *Let G be a finite subgroup of* $\mathrm{GL}(n, \mathbf{C})$, *and H be the subgroup of G generated by all of elements which have* 1 *as eigen-values of multiplicity $n-1$. Then H is a normal subgroup of G and there exists a subgroup K of* $\mathrm{GL}(n, \mathbf{C})$ *which is isomorphic to G/H such that* $(\mathbf{C}^n/G, 0)$ *is isomorphic to* $(\mathbf{C}^n/K, 0)$.

According to Remark 4, Proposition 8 and Theorem 2, if there exists an example for which an equality holds for (1) of Theorem 2 in case of $b \neq 1$, there exists such an example in case that G is an finite small subgroup of $\mathrm{GL}(n, \mathbf{C})$ and not cyclic. But it may be difficult to know whether such an example exists or not.

References

[1] E. Brieskorn, Rational Singularitäten komplexer Flächen, Inv. Math., **4** (1968), 336–358.

[2] H. Cartan, Quotient d'un espace analytique par un groupe d'automorphismes, Algebraic geometry and topology, 90–102, Princeton, 1957.

[3] H. Grauert und R. Remmert, Analytische Stellenalgebren, Springer, 1971.

[4] R. C. Gunning, Lectures on complex analytic varieties (The local parametrization theorem), Princeton 1970.

[5] E. Noether, Der Endlichkeitssatz der Invarianten endlicher Gruppen, Math. Ann., **77** (1916), 89–92.

[6] D. Prill, Local classification of quotients of complex manifolds by discontinuous groups, Duke. Math. J., **34** (1967), 375–386.