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# Types of Some Infinitely Generated Kleinian Groups

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In the previous paper [4] we gave the results between the singular sets of some infinitely generated Kleinian groups and their computing functions and presented three problems; (I) *Is* $G^*(\alpha_0)$ *the* $\mu$*-convergent type, if* $\mathrm{M}_{\frac{\mu}{2}}(E)=0$? (II) *Does it hold* $0<\mathrm{M}_{\frac{\mu_0}{2}}(E)$ *at the Hausdorff dimension* $\mathrm{d}(E)$? (III) *Does it always hold* $\mathrm{d}(E)\geqq\alpha_0$ for $G^*(\alpha_0)$?

The purpose of this paper is to show that (I) and (II) can be solved positively. We shall give in §2 the properties of the limiting computing functions on $G^*(\alpha_0)$ and show in §3 that (I) and (II) are solved by using these properties.

## §1. Preliminaries and Notations.

**1.** Let $\{K_j\}_{j=1}^{p}$ and $\{H_i,H_i'\}_{i=p+1}^{\infty}$ be an infinite number of circles external to one another in the extended complex plane $\widetilde{C}=\{z\ ;|z|\leqq\infty\}$, where $\{H_i,H_i'\}_{i=p+1}^{q}$ tend to only a finite point $Q$ for q→∞. Let $B$ be a domain bounded by these circles. We may assume no loss of generality that these circles are contained in some closed disc $D_0=\{z\ ;|z|\leqq\rho_0\}$.

Let $\{T_j\}_{j=1}^{p}$ be the elliptic transformations with period 2 corresponding to $\{K_j\}_{j=1}^{p}$, each of which transforms the outside of $K_j$ onto the inside of itself. Let $\{T_j\}_{i=p+1}^{\infty}$ be the system of hyperbolic or loxodromic transformations, each $T_i$ of which transforms the outside of $H_i$ onto the inside of $H_i$. Then the system $\mathcal{Y}=\{T_i,T_i^{-1}\}_{i=1}^{\infty}$ $(T_i=T_i^{-1},\ 1\leqq i\leqq p)$ generates an infinitely generated discontinuous gromp denoted by $G$ and we call $\mathcal{Y}$ the generator system of $G$, where $T_i^{-1}$ denotes the inverse of $T_i$.

Take a positive integer q (>p) and consider a subset $\mathcal{Y}_{\mathrm{N}}=\{T_j\}_{j=1}^{p}\cup\{T_i,T_i^{-1}\}_{i=1}^{q}$ (N=2q−p) of $\mathcal{Y}$. Then $\mathcal{Y}_{\mathrm{N}}$ generates a finitely generated subgroup $G_{\mathrm{N}}$ of $G$. If we denote by $B_{\mathrm{N}}$ a domain bounded by $\{K_j\}_{j=1}^{p}\cup\{H_i,H_i'\}_{i=p+1}^{q}$ (N=2q−p), it is well known that $B_N$ coincides with a fundamental domain of $G_N$. We gave some results with respect to the singular set $E_N$ of $G_N$ by using the relations between $E_N$ and the computing functions on $G_N$ ([2]). We shall get $G$ from $G_N$ for $N\to\infty$.

2. Denote by $r(H)$ the radius of a circle $H\in\{H_i,H_i'\}_{i=p+1}^{\infty}$ and assume that there exists some positive constant $K$ independent of H such that it holds

$$\text{(A)} \qquad \frac{r(H)}{l(H)} \leqq K ,$$

where $l(H)=\inf|z-\zeta|$ and the infimum is taken for all points $z\in H$ and for all points $\zeta$ on any circle from $\{H_i,H_i'\}_{i=p+1}^{\infty}-\{H\}$ .

Defining the product $ST$ in $G$ by $ST(z)=S(T(z))$, we can write any element $U$ of $G$ in the form

$$U=T_{i_n}\cdot\ .\ .\ T_{i_2}T_{i_1}\ (T_{i_j}\in\mathcal{Y}\ (1\leqq i\leqq n)\ ;\ T_{i_{j+1}}^{-1} \neq T_{i_j}).$$

We call the positive integer n the grade of U and for simplicity we use the notation $S_{(n)}$ to clarify grade of $U$.

Consider the image $S_{(n)}(B_N)$ by any element $S_{(n)}=T_{i_n}\cdot\ .\ .\ T_{i_2}T_{i_1}$ ($\in G_N$). It is easily seen that $S_{(n)}(B_N)$ is bounded by an outer boundary circle $S_{(n)}(C_{T_{i_1}^{-1}})$ and $(N-1)$ inner boundary circles $S_{(n)}(C_{T_j})$ $(T_j\neq T_{i_1}^{-1}\ ,\ T_j\in\mathcal{Y}_N)$. We shall call such inner boundary circle the circles of grade n. Circles $\{K_j\}_{j=1}^{p}$ and $\{H_i,H_i'\}_{i=p+1}^{q}$ , which bounds $B_N$, are of grade 0. The circles of grade n with respect to $G$ can be defined in the same way.

Now let us impose a restriction with respect to the accumulation of circles for $G$. Consider the circle $C_{T_i}:|z-\alpha(T_i)|=r_{T_i}$ of radius $r_{T_i}$ with center $\alpha(T_i)$ for any $T_i(\in\mathcal{Y})$. Take some boundary circle $C_{T_j}(T_j\neq T_i)$ of $B$ and denote the distance from $\alpha(T_i)$ to $C_{T_j}$ by $\rho_j(T_i)$, that is,

$$\text{(1.1)} \qquad \rho_j(T_i)=\inf_{z\in C_{T_j}}|z-\alpha(T_i)|\ .$$

It is obvious that $\rho_j(T_i)>r(T_i)$ and from the property (A)

$$\text{(1.2)} \qquad \frac{r(T_i)}{\rho_j(T_i)} \leqq \frac{K}{K+1}\ .$$

We assume that there exists a positive constant $K_1(\alpha)$ depending only on some positive number $\alpha$ $(0<\alpha<2)$ satisfying

$$\text{(B)} \qquad W(T_j,\alpha)= \sum_{T_i \in \mathcal{Y}}' \left(\frac{r(T_i)}{\rho_j(T_i)^2}\right)^{\alpha} \leqq K_1(\alpha),$$

where $\sum'_{T_i \in \mathcal{Y}_1}$ denotes the sum with respect to all $T_i$ ($\neq T_j$). Then we can determine the unique number $\alpha_0$ ($\geqq 0$) such that

$$\text{(1.3)} \qquad \alpha_0 = \inf \{\alpha\,;\, K_1(\alpha) < +\infty\} \ .$$

We note that $\alpha_0$ is always equal to 0 for $G_{\rm N}$.

From now we shall call such discontinuous group with these properties (A) and (B) the Kleinian group with properties (A) and (B) and denote it by $G^*(\alpha_0)$ and the generator system by $\mathcal{Y}^*(\alpha_0)$.

3. Let $S_{(n)} = T_{i_n} \cdot\ .\ .\ T_{i_2} T_{i_1}$ ($T_{i_j} \in \mathcal{Y}_{\rm N}$) be any element of $G_{\rm N}$ and assume that $T_{i_1}^{-1} \neq T$ for a fixed element $T$ ($\in \mathcal{Y}_{\rm N}$) and take any point $z \in D_T$. If we denote by $R_{S_{(n)}}$ the radius of the isometric circle of $S_{(n)}$, we obtain easily

$$\text{(1.4)} \qquad \left| \frac{dS_{(n)}(z)}{dz} \right|^{\frac{\mu}{2}} = \left(\frac{R_{S_{(n)}}}{|z - S_{(n)}^{-1}(\infty)|}\right)^{\mu} \quad (0<\mu<4),$$

where $S_{(n)}^{-1}$ denotes the inverse $(S_{(n)})^{-1} = T_{i_1}^{-1}\ .\ .\ .\ T_{i_n}^{-1}$ of $S_{(n)}$. Here we note that $z \in D_T$ and $S_{(n)}^{-1}(\infty) \in D_{T_{i_1}^{-1}} \neq D_T$.

Forming the sum of $({\rm N}-1)^n$ terms with respect to all $S_{(n)}$ ($\in G_{\rm N}$) such that $T_{i_1}^{-1} \neq T$ and $T_{i_j} \neq T_{i_{j+1}}^{-1}$ ($1 \leqq j \leqq n-1$), we had the function

$$\text{(1.5)} \qquad \chi_{n,{\rm N}}^{(\mu;T)}(z) = \sum_{S_{(n)} \in G_{\rm N}} \left(\frac{R_{S_{(n)}}}{|z - S_{(n)}^{-1}(\infty)|}\right)^{\mu},$$

and called $\chi_{n,{\rm N}}^{(\mu;T)}(z)$ the $\mu$-dimensional computing function of order n on $T$. The domain of definition of $\chi_{n,{\rm N}}^{(\mu;T)}(z)$ is $D_T$ ([2]).

Since each term in the sum $\chi_{n,{\rm N}}^{(\mu;T)}(z)$ is positive, $\chi_{n,{\rm N}}^{(\mu;T)}(z)$ has necessarily the unique limit containing the infinity for any $z \in D_T$, if N tends to the infinity. Thus we can define the function

$$\text{(1.6)} \qquad \lim_{{\rm N}\to\infty} \chi_{n,{\rm N}}^{(\mu;T)}(z) = \lim_{{\rm N}\to\infty} \sum \left(\frac{R_{S_{(n)}}}{|z - S_{(n)}^{-1}(\infty)|}\right)^{\mu} = \sum_{S_{(n)} \in G} \left(\frac{R_{S_{(n)}}}{|z - S_{(n)}^{-1}(\infty)|}\right)^{\mu},$$

and we shall call it the $\mu$-dimensional limiting computing function of order n on $T$ and denote it by $\chi_{n,\infty}^{(\mu;T)}(z)$.

Now let us give the following definition ([4]).

Definition. Let $\{\chi_{n,\infty}^{(\mu;T)}(z)\}$ $(n=1,2,\cdots)$ be the sequence of the $\mu$-dimensional limiting computing function on $T\in\mathcal{Y}$. If it holds

$$\lim_{n\to\infty}\chi_{n,\infty}^{(\mu;T)}(z)=0 \qquad (\text{or } \infty) \tag{1.7}$$

for some element $T$ of $\mathcal{Y}$ and some point $z\in D_T$, we call $G$ the $\mu$-convergent (or divergent) type. If it holds

$$0<\varliminf_{n\to\infty}\chi_{n,\infty}^{(\mu;T)}(z)\leqq\varlimsup_{n\to\infty}\chi_{n,\infty}^{(\mu;T)}(z)<\infty \tag{1.8}$$

for some $T$ $(\in\mathcal{Y})$ and some point $z\in D_T$, we call $G$ the $\mu$-finite type.

4. In the previous paper [4] we obtained the following results with respect to $E$.

PROPOSITION 1. (i) *Let $G^*(\alpha_0)$ be a Kleinian group with properties* (A) *and* (B). *Then $G^*(\alpha_0)$ is the $\mu$-divergent type, if and only if $M_{\frac{\mu}{2}}(E)=\infty$.* (ii) *If $G^*(\alpha_0)$ is the $\mu$-convergent type, then it holds $M_{\frac{\mu}{2}}(E)=0$.*

The Hausdorff dimension $d(E)$ of the singular set $E$ of $G$ is defined in the following:

$$d(E)=\sup\left\{\frac{\mu}{2}\,;\, M_{\frac{\mu}{2}}(E)=\infty\right\}=\inf\left\{\frac{\mu}{2}\,;\, M_{\frac{\mu}{2}}(E)=0\right\}.$$

We had the following proposition ([4]).

PROPOSITION 2. *Let $d(E)=\frac{\mu_0}{2}$ be the Hausdorff dimension of the singular set $E$ of $G^*(\alpha_0)$. If $\frac{\mu_0}{2}>\alpha_0$, then it holds that $M_{\frac{\mu_0}{2}}(E)<+\infty$.*

We presented in [4] the following three problems. (I) *Is $G^*(\alpha_0)$ the $\mu$-convergent type, if $M_{\frac{\mu}{2}}(E)=0$ ?* This is the converse of (ii) in PROPOSITION 1. (II) *Does it hold $0<M_{\frac{\mu_0}{2}}(E)$ at the Hausdorff dimension $d(E)=\frac{\mu_0}{2}$ ? If so, is $G^*(\alpha_0)$ the $\mu_0$-finite type ?* (III) *Does it always hold $\frac{\mu_0}{2}\geqq\alpha_0$ for $G^*(\alpha_0)$ ?*

In the case of $G_N$, the above problems (I) and (II) were solved positively in [2] and it is not necessary to consider the problem (III), since $\alpha_0=0$.

REMARK. We obtained in [2] the result that $0<M_{\frac{\mu_N}{2}}(E_N)<+\infty$ is equivalent to that

$0<\varliminf_{n\to\infty}\chi_{n,\infty}^{(\mu_N;T)}(z)\leqq\varlimsup_{n\to\infty}\chi_{n,N}^{(\mu_N;T)}(z)<+\infty$ for any $T$ and any $z\in D_T$. But in the process of proving LEMMA 5 of this theorem there is a mistake. But this is easily corrected. The main part of this correction is similar to the method using inequalities (3.6) and (3.7) in the process of proving THEOREM 3 (§3) of this paper.

## §2. Properties of the limiting computing function $\chi_{n,\infty}^{(\mu;T)}(z)$.

5. The purpose of this paper is to solve the problems (I) and (II). But now the problem (III) is still open.

At first we shall give some properties which will be necessary to solve these problems.

Let $G^*(\alpha_0)$ be the $\mu$-convergent type. Then it holds for some element $T\in\mathcal{Y}^*(\alpha_0)$ and some point $z\in D_T$

$$\lim_{n\to\infty}\chi_{n,\infty}^{(\mu;T)}(z)=\lim_{n\to\infty}(\lim_{N\to\infty}\chi_{n,N}^{(\mu;T)}(z))=0.$$

Since it holds for any N

$$\chi_{n,N}^{(\mu;T)}(z)\leqq\chi_{n,\infty}^{(\mu;T)}(z),\tag{2.1}$$

we have $\lim_{n\to\infty}\chi_{n,N}^{(\mu;T)}(z)=0$

and hence

$$\lim_{N\to\infty}(\lim_{n\to\infty}\chi_{n,N}^{(\mu;T)}(z))=0.\tag{2.2}$$

Therefore we obtain

$$\lim_{N\to\infty}(\lim_{n\to\infty}\chi_{n,N}^{(\mu;T)}(z))=\lim_{n\to\infty}(\lim_{N\to\infty}\chi_{n,N}^{(\mu;T)}(z))=0.\tag{2.3}$$

Next we suppose that $G^*(\alpha_0)$ is the $\mu$-divergent type. Let $G_N$ and $\frac{\mu_N}{2}$ for any integer N ($>0$) be the subgroup of $G^*(\alpha_0)$ and its Hausdorff dimension of the singular set $E_N$. We have already found in [3] that $\frac{\mu_N}{2}$ increases strictly according as the increment of the boundary circles of $B_N$ and $\lim_{N\to\infty}\frac{\mu_N}{2}=\frac{\mu_0}{2}$ $(=d(E))$. Further we have from PROPOSITION 1 that $\varlimsup_{n\to\infty}\chi_{n,\infty}^{(\mu_0;T)}(z)<+\infty$ for any element $T\in\mathcal{Y}^*(\alpha_0)$ and any point $z\in D_T$ is equivalent to that $M_{\frac{\mu_0}{2}}(E)<+\infty$. Then from the definition of the Hausdorff dimension we have

(2.4) $$\frac{\mu}{2} < \frac{\mu_0}{2} \ (=\mathrm{d}(E)).$$

So there exists a positive integer N such that $\frac{\mu}{2} < \frac{\mu_N}{2}$ Hence we have from THEOREM 2 in [2] that

(2.5) $$\lim_{n\to\infty} \chi_{n,N}^{(\mu;T)}(z)=\chi_{\infty,N}^{(\mu;T)}(z)=\infty.$$

Since $\chi_{n,N}^{(\mu;T)}(z)$ is a monotone increasing function of N, we have

(2.6) $$\lim_{N\to\infty} \chi_{\infty,N}^{(\mu;T)}(z)=\infty.$$

Thus from the assumption on $G^*(\alpha_0)$ and (2.6) we have

(2.7) $$\lim_{N\to\infty} (\lim_{n\to\infty} \chi_{n,N}^{(\mu;T)}(z))=\lim_{n\to\infty}(\lim_{N\to\infty} \chi_{n,N}^{(\mu;T)}(z))=\infty.$$

Arranging the above result we have the following theorem.

THEOREM 1. *Let $G^*(\alpha_0)$ be a Kleinian group with properties* (A) *and* (B). *If $G^*(\alpha_0)$ is the $\mu$-convergent* (*or divergent*) type, then it holds

(2.8) $$\lim_{N\to\infty} (\lim_{n\to\infty} \chi_{n,N}^{(\mu;T)}(z))=\lim_{n\to\infty}(\lim_{N\to\infty} \chi_{n,N}^{(\mu;T)}(z))=0 \quad (\text{or } \infty).$$

**6.** We defined the types of $G^*(\alpha_0)$ and divided them into three types. But it is natural to arise the following problem: Does there exist the type other than the above ones? In other words, there may exist the case that

$$\varliminf_{n\to\infty} \chi_{n,\infty}^{(\mu;T)}(z) < \varlimsup_{n\to\infty} \chi_{n,\infty}^{(\mu;T)}(z)=\infty$$

or

$$0=\varliminf_{n\to\infty} \chi_{n,\infty}^{(\mu;T)}(z) < \varlimsup_{n\to\infty} \chi_{n,\infty}^{(\mu;T)}(z)$$

for some element $T$ and some point $z\in D_T$ and some number $\mu$.

We shall prove that for any fixed number $\mu$ there are only three types of $G^*(\alpha_0)$, that is, the $\mu$-convergent, divergent and finite types. For this purpose we shall give the following Lemma.

LEMMA 1 ([4]). *Let $G^*(\alpha_0)$ be a Kleinian group defined in the above.* (i) *Then it holds*

$$(2.5)\qquad (4K+1)^{-\mu}\chi_{n,\infty}^{(\mu;T)}(z_0) \leqq \chi_{n,\infty}^{(\mu;T)}(z) \leqq (4K+1)^{\mu}\chi_{n,\infty}^{(\mu;T)}(z_0)$$

*for any element $T \in \mathfrak{Y}^*(\alpha_0)$ and any two points $z$ and $z_0 \in D_T$, where $K$ is a positive constant in the property* (A). (ii) *Let $T$, $T^*$ and $T^{**}$ ($\neq T^{-1}, \neq T^{*-1}$) be arbitrary elements of $\mathfrak{Y}^*(\alpha_0)$. Then there exists a positive constant $K(G^*(\alpha_0), T^*, T^{**}, \mu)$ depending only on $G^*(\alpha_0)$, $T^*$, $T^{**}$ and $\mu$ ($> 2\alpha_0$) such that it holds for any two points $z$ ($\in D_T$) and $z^* = T^*T^{**}(z)$ ($\in D_{T^*}$)*

$$(2.10)\qquad \chi_{n+2,\infty}^{(\mu;T)}(z) \geqq K(G^*(\alpha_0), T^*, T^{**}, \mu)\chi_{n,\infty}^{(\mu;T^*)}(z^*).$$

Proof. Since the proof of (i) is given in [4], we shall give only the proof of (ii) here. Take any element $S_{(n+2)} = S_{(n)}T^*T^{**} = T_{i_n}\cdots T_{i_1}T^*T^{**}$ of grade n+2 such that $T^* \neq T_{i_1}^{-1}$.
Since

$$\left|\frac{dS_{(n+2)}(z)}{dz}\right|^{\frac{\mu}{2}} = \left(\left|\frac{dS_{(n)}(z^*)}{dz^*}\right|\left|\frac{dT^*(T^{**}(z))}{dT^{**}(z)}\right|\left|\frac{dT^{**}(z)}{dz}\right|\right)^{\frac{\mu}{2}},$$

we have from the definition of the computing function

$$(2.11)\qquad \begin{aligned}&\sum_{S_{(n+2)}\in G^*(\alpha_0)}\left(\frac{R_{S_{(n+2)}}}{|z-S^{-1}_{(n+2)}(\infty)|}\right)^{\mu}\\ &= \sum_{S_{(n+2)}\in G^*(\alpha_0)}\left(\frac{R_{S_{(n)}}}{|z^*-S_{(n)}^{-1}(\infty)|}\right)^{\mu}\left(\frac{R_{T^*}}{|T^{**}(z)-T^{*-1}(\infty)|}\right)^{\mu}\left(\frac{R_{T^{**}}}{|z-T^{**-1}(\infty)|}\right)^{\mu}.\end{aligned}$$

Since all circles $\{K_j\}_{j=1}^{p}$ and $\{H_i, H_i'\}_{i=p+1}^{\infty}$ are contained in a closed disc $D_0$, it holds $|z-T^{**-1}(\infty)| \leqq 2\rho_0$ and $|T^{**}(z)-T^{*-1}(\infty)| \leqq 2\rho_0$.

Hence we have

$$(2.12)\qquad \begin{aligned}&\sum_{S_{(n+2)}\in G^*(\alpha_0)}\left(\frac{R_{S_{(n+2)}}}{|z-S^{-1}_{(n+2)}(\infty)|}\right)^{\mu}\\ &\geqq \left(\frac{R_{T^*}}{2\rho_0}\right)^{\mu}\left(\frac{R_{T^{**}}}{2\rho_0}\right)^{\mu}\sum_{S_{(n)}\in G^*(\alpha_0)}\left(\frac{R_{S_{(n)}}}{|z^*-S_{(n)}^{-1}(\infty)|}\right)^{\mu}.\end{aligned}$$

Thus we have from (2.12)

$$\chi_{n+2,\infty}^{(\mu;T)}(z) \geqq K(G^*(\alpha_0), T^*, T^{**}, \mu)\,\chi_{n,\infty}^{(\mu;T^*)}(z^*).$$

q.e.d.

7. By using the above LEMMA 1 we have the following result.

LEMMA 2. *Assume that it holds* $\overline{\lim_{n\to\infty}}\ \chi^{(\mu;T^*)}_{n,\infty}(z_0)=\infty$ *for some element* $T^*$ *and some point* $z_0\in D_{T^*}$. *Then* $G^*(\alpha_0)$ *is the* $\mu$*-divergent type.*

Proof. From the assumption there exists some subsequence $\{\chi^{(\mu;T^*)}_{n_i,\infty}(z_0)\}$ $(i=1,2,\cdots)$ for some $T^*$ and some point $z_0\in D_{T^*}$ such that

(2.13) $$\lim_{i\to\infty}\chi^{(\mu;T^*)}_{n_i,\infty}(z_0)=\infty.$$

Take another element $T^{**}$ $(\neq T^{*^{-1}})\in\mathcal{Y}^*(\alpha_0)$ and let it be fixed. Then from (i) of LEMMA 1 there exists a positive integer $N_0$ depending on large number $M$ $(>1)$, $T^*$, $T^{**}$, $G^*(\alpha_0)$ and $\mu$ such that it holds for any integer $n_{i_0}\geqq N_0$ $(>0)$ and any point $z^*\in D_{T^*}$

(2.14) $$\chi^{(\mu;T^*)}_{n_{i_0},\infty}(z^*) > \frac{(4K+1)^{\mu}M}{K(G^*(\alpha_0),\ T^*,T^{**},\mu)},$$

where $K$ is a constant in the property (A) and $K(G^*(\alpha_0),T^*,T^{**},\mu)$ is a constant in (ii) of LEMMA 1. Then we have from (ii) of LEMMA 1

(2.15) $$\chi^{(\mu;T)}_{n_{i_0}+2,\infty}(z) > M$$

for any element $T$ $(\neq T^{**^{-1}})$ uniformly on $D_T$. If we take another element $T_1^{**}$ $(\neq T^{*^{-1}})$ different from $T^{**}$ and similar steps as in the above, then there exists a positive integer $N_0'$ depending on $M$ $(>1)$, $T^*$, $T_1^{**}$, $G^*$ $(\alpha_0)$ and $\mu$ such that for any $n_{i_0}'\geqq N_0'$

(2.16) $$\chi^{(\mu;T^*)}_{n'_{i_0},\infty}(z^*) > \frac{(4K+1)^{\mu}M}{K(G^*(\alpha_0),T^*,T_1^{**},\mu)}$$

and hence it holds for any $T$ $(\neq T_1^{**^{-1}})$

(2.17) $$\chi^{(\mu;T)}_{n'_{i_0}+2,\infty}(z) > M$$

uniformly on $D_T$. If we take $N_0^*=\max(N_0,N_0')$, then it holds for any element $T\in\mathcal{Y}^*(\alpha_0)$ and any $n_i^*\geqq N_0^*$

(2.18) $$\chi^{(\mu;T)}_{n_i^*+2,\infty}(z) > M$$

uniformly on $D_T$. Since $\lim_{N\to\infty}\chi^{(\mu;T)}_{n_i^*+2,N}(z)=\chi^{(\mu;T)}_{n_i^*+2,\infty}(z)$, we can take a large positive integer

N depending only on small $\varepsilon$ such that it holds for any element $T \in \mathcal{Y}_N$

(2.19) $$\chi^{(\mu;T)}_{n_i^*+2,N}(z) \geqq M-\varepsilon > 1.$$

We know that this inequality is a sufficient condition for $M_{\frac{\mu}{2}}(E_N)$ to be infinity ([2]). Hence we obtain that $M_{\frac{\mu}{2}}(E)=+\infty$. Hence from (i) of PROPOSITION 1 we can conclude that $G^*(\alpha_0)$ is the $\mu$-divergent type.

q.e.d.

8. Next we shall prove the following result.

LEMMA 3. *Assume that it holds* $\underline{\lim}_{n\to\infty} \chi^{(\mu;T^*)}_{n,\infty}(z_0)=0$ *for some element* $T^* \in \mathcal{Y}^*(\alpha_0)$ *and some point* $z_0 \in D_{T^*}$. *Then* $G^*(\alpha_0)$ *is the* $\mu$*-convergent type, that is,* $\lim_{n\to\infty} \chi^{(\mu;T^*)}_{n,\infty}(z_0)=0$.

Proof. Assume that $\overline{\lim}_{n\to\infty} \chi^{(\mu;T^*)}_{n,\infty}(z_0)=k\ (>0)$ for some $T^*$ and some $z_0 \in D_{T^*}$. Then there exists a subsequence $\{\chi^{(\mu;T^*)}_{n_i,\infty}(z_0)\}$ $(i=1,2,\cdots)$ of the $\mu$-dimensional computing functions on $T$ such that

(2.20) $$\lim_{i\to\infty} \chi^{(\mu;T^*)}_{n_i,\infty}(z_0)=k.$$

Hence there is a positive integer $N_0=N_0(\varepsilon,n_i)$ depending only on any $\varepsilon$ and $n_i$ such that it holds for $N \geqq N_0$

(2.21) $$\chi^{(\mu;T^*)}_{n_i,N}(z_0) > k-\varepsilon > 0.$$

On the other hand from the assumption $\underline{\lim}_{n\to\infty} \chi^{(\mu;T^*)}_{n,\infty}(z_0)=0$ there exists a subsequence $\{\chi^{(\mu;T^*)}_{n_j,\infty}(z_0)\}$ $(j=1,2,\cdots)$ such that

(2.22) $$\lim_{j\to\infty} \chi^{(\mu;T^*)}_{n_j,\infty}(z_0)=0.$$

Since $\chi^{(\mu;T^*)}_{n_j,N}(z_0) \leqq \chi^{(\mu;T^*)}_{n_j,\infty}(z_0)$ for any integer N $(>0)$, it holds from (2.22)

$$\lim_{j\to\infty} \chi^{(\mu;T^*)}_{n_j,N}(z_0)=0.$$

Hence from the property (LEMMA 2 in [2]) of the computing function, we have

(2.23) $$\lim_{n\to\infty} \chi^{(\mu;T^*)}_{n,N}(z_0)=0.$$

Therefore we can determine the order $n_i$ depending only on $\delta=k-\varepsilon$ in $\{\chi_{n_i,N}^{(\mu;T^*)}(z_0)\}$ $(i=1,2,\cdots)$ so that it may hold

$$\chi_{n_i,N}^{(\mu;T^*)}(z_0) < \delta, \tag{2.24}$$

which contradicts (2.21). Thus it must hold that $\overline{\lim}_{n\to\infty} \chi_{n,\infty}^{(\mu;T^*)}(z_0)=0$ and hence it completes the proof of this Lemma.

q.e.d.

Then we have from LEMMAS 2 and 3 the following theorem.

THEOREM 2. *There are only three types of $G^*(\alpha_0)$ for any fixed number $\mu$, that is, the $\mu$-con vergent, divergent and finite type.*

## § 3; Relations between the Hausdorff measure of the singular sets of $G^*(\alpha_0)$ and their limiting computing functions.

**9.** Now let us solve the problem (I). For this purpose we shall show that $\underline{\lim}_{n\to\infty} \chi_{n,\infty}^{(\mu;T)}(z)>0$ implies $M_{\frac{\mu}{2}}(E)>0$. If we suppose $\overline{\lim}_{n\to\infty} \chi_{n,\infty}^{(\mu;T)}(z)=+\infty$, then we have from PROPOSITION 1 and LEMMA 2 that $G^*(\alpha_0)$ is the $\mu$-divergent type and hence $M_{\frac{\mu}{2}}(E)=+\infty$. So we may consider only the case that $G^*(\alpha_0)$ is the $\mu$-finite type, that is,

$$0<\underline{\lim}_{n\to\infty} \chi_{n,\infty}^{(\mu;T)}(z) \leqq \overline{\lim}_{n\to\infty} \chi_{n,\infty}^{(\mu;T)}(z) < +\infty \tag{3.1}$$

for some element $T\in\mathcal{Y}^*(\alpha_0)$ and some point $z\in D_T$. Then (3.1) holds for any element $T$ and any point $z\in D_T$ from LEMMAS 1 and 2.

Take a large positive integer N and let it be fixed. Then there is a some positive number $K$ depending only on N such that it holds

$$\chi_{n,N}^{(\mu;T)}(z) < K \tag{3.2}$$

for any element $T\in\mathcal{Y}_N$ and any point $z\in D_T$. Further take a small number $\delta$ and consider a covering consisting of closed discs with radii $r_{R_{(n_j)}}$ $(<\delta)$ bounded by circles of grade $n_j$ by $S_{(n_j)}$ $(\in G_N)$ to the singular set $E_N$. Extract a subcovering of $E_N\frown D_{T^*}$ from this covering for some element $T^*$ $(\in\mathcal{Y}_N)$.

Denote the sum of radii of this subcovering by $\sum_{j=1}^{p}(r_{S_{(n_j)}})^{\frac{\mu}{2}}$. Then from the relation between the radii of the image circles and the isometric circles of these elements $S_{(n_j)}$

([1]), we have

$$\sum_{j=1}^{p}(r_{S_{(n_j)}})^{\frac{\mu}{2}} \geqq K(G_N,\mu) \sum_{j=1}^{p}(R_{S_{(n_j)}})^{\mu}, \qquad S_{(n_j)}=T^*S_{(n_j-1)}. \tag{3.3}$$

Since $\sum_{j=1}^{p}(R_{S_{(n_j)}})^{\mu}= \sum_{j=1}^{p}(R_{S^{-1}_{(n_j)}})^{\mu}$, we have from (3.3)

$$\sum_{j=1}^{p}(r_{S_{(n_j)}})^{\frac{\mu}{2}} \geqq K(G_N,\mu) \sum_{j=1}^{p}(R_{S^{-1}_{(n_j)}})^{\mu} = K(G_N,\mu) \sum_{j=1}(R_{S^{-1}_{(n_j-1)}T^{*-1}})^{\mu}. \tag{3.4}$$

On the other hand we have from the property of $\chi^{(\mu;T)}_{n,N}(z)$ ([2])

$$\chi^{(\mu;T)}_{n,N}(S_{(m)}(\infty))= \sum_{S_{(n)}} (\frac{R_{S_{(n)}S_{(m)}}}{R_{S_{(m)}}})^{\mu}, \qquad S_{(m)}=TS_{(m-1)}. \tag{3.5}$$

Denote $\max_{1\leqq j\leqq p}(n_j)=n^*$. Consider each element $R_{S^{-1}_{(n_j)}}$ of the sum (3.4).

We put $S^{-1}_{(n_j)} =TS^{-1}_{(n_j-1)} =TS^{-1}_{(n_j-2)}\ T^{*-1}$ and denote $n^*-n_j$ by $n^*(j)$. Then we have from (3.5)

$$K > \sum_{S^{-1}_{(n^*(j))}} (\frac{R_{S^{-1}{}_{(n^*(j))}S^{-1}{}_{(n_j)}}}{R_{S^{-1}_{(n_j)}}})^{\mu}. \tag{3.6}$$

Hence from (3.4) and (3.6)

$$\begin{aligned}\sum_{j=1}^{p} (R_{S^{-1}_{(n_j)}})^{\mu} &> K \sum_{S_{(n^*-1)}} (R_{S_{(n^*-1)}T^{*-1}})^{\mu}\\ &=K(R_{T^{*-1}})^{\mu} \sum_{S_{(n^*-1)}} (R_{S_{(n^*-1)}T^{*-1}})^{\mu}/(R_{T^{*-1}})^{\mu}\\ &=K(R_{T^{*-1}})^{\mu}\ \chi^{(\mu;T^{*-1})}_{n^*-1,N}(T^{*^{-1}}(\infty)).\end{aligned} \tag{3.7}$$

Then we have from (3.4) and (3.7)

$$\sum_{j=1}^{p} (r_{S_{(n_j)}})^{\frac{\mu}{2}} \geqq K(G_N,\mu)\ K\ (R_{T^{*-1}})^{\mu}\ \chi^{(\mu;T^{*-1})}_{n^*-1,N}(T^{*^{-1}}(\infty)). \tag{3.8}$$

From the well known property about the Hausdorff measure (Proposition 3 in [1]),

$$\mathrm{M}_{\frac{\mu}{2}}(E_N\cap D_{T^*}) \geqq \varkappa^{-1}(\frac{k_0(\mathrm{N})}{2})^{\frac{\mu}{2}} K(G_N,\mu)K(R_{T^{*-1}})^{\mu}\ \chi^{(\mu;T^{*-1})}_{n^*-1,N}(T^{*^{-1}}(\infty)), \tag{3.9}$$

where $\varkappa$ is an absolute constant and $k_0$(N) is a constant depending only on N. Since $E\supset E_N$, we have

$$\mathrm{M}_{\frac{\mu}{2}}(E\cap D_{T^*}) \geqq K^*(G_N,\mathrm{N},T^*,\mu)\ \chi^{(\mu;T^{*-1})}_{n^*-1,N}(T^{*^{-1}}(\infty)), \tag{3.10}$$

where $K^*(G_N,\mathrm{N},T^*,\mu)=\varkappa^{-1}(\frac{k_0(\mathrm{N})}{2})^{\frac{\mu}{2}}\ K(G_N,\mu)K(R_{T^{*-1}})^{\mu}$ is a constant depending only on $G_N$, N, $T^*$ and $\mu$.

It holds that

$$\lim_{n\to\infty}\chi_{n,\infty}^{(\mu;T)}(z_0)=k\ >\ 0 \tag{3.11}$$

for some element $T\in\mathcal{Y}^*(\alpha_0)$ and any point $z\in D_T$. Hence there is an integer $n_0(\varepsilon)$ ($>0$) depending on any number $\varepsilon$ satisfying $k>2\varepsilon$ such that it holds for any integer $n\geqq n_0(\varepsilon)$

$$|\chi_{n,\infty}^{(\mu;T)}(z)\ \geqq\ k-\varepsilon. \tag{3.12}$$

Then it is easily seen that there is a positive integer $N_0$ depending on n and $\varepsilon$ such that it holds

$$\chi_{n,N}^{(\mu;T)}(z)\ \geqq\ k-2\varepsilon\ >\ 0 \tag{3.13}$$

for any integer $N\geqq N_0$ and any point $z\in D_T$.

At first we gave a small number $\delta$. Then we can take a grade number n so that the radii of the image circles by $S_{(n)}\in G_N$ for any integer N may be less than $\delta$. Hence from (3.10) and (3.13) we obtain for $T=T^{*^{-1}}$

$$\mathrm{M}_{\frac{\mu}{2}}(E\frown D_{T^*})\ \geqq\ K^*(G_N,\mathrm{N},T^*,\mu)(k-2\varepsilon)\ >\ 0. \tag{3.14}$$

Since (3.14) holds for any integer N ($>N_0$), we can conclude that

$$\mathrm{M}_{\frac{\mu}{2}}(E\frown D_{T^*})\ >\ 0$$

and therefore

$$\mathrm{M}_{\frac{\mu}{2}}(E)\ >\ 0.$$

Then from the contraposition of the above fact we can claim that $\mathrm{M}_{\frac{\mu}{2}}(E)=0$ implies that $G^*(\alpha_0)$ is the $\mu$-convergent type. Thus we have from (ii) of PROPOSITION 1 the following theorem, which solves the Problem (I).

THEOREM 3. *Let $G^*(\alpha_0)$ be a Kleinian group with properties* (A) *and* (B). *Then $G^*(\alpha_0)$ is the $\mu$-convergent type if and only if $\mathrm{M}_{\frac{\mu}{2}}(E)=0$.*

**10.** Now let us consider the Problem (II). We know that $M_{\frac{\mu_0}{2}}(E) < +\infty$ at the Hausdorff dimension $d(E)=\frac{\mu_0}{2}$ from PROPOSITION 2 and that there are only three types of $G^*(\alpha_0)$ from THEOREM 2. So it is easily conjectured from THEOREM 3 that $G^*(\alpha_0)$ is the $\mu_0$-finite type. We shall lead the contradiction under the assumption of the $\mu_0$-convergent type at the Hausdorff dimension.

Since $\lim_{n\to\infty} \chi_{n,\infty}^{(\mu;T)}(z)=+\infty$ is equivalent to $M_{\frac{\mu}{2}}(E)=+\infty$, $G^*(\alpha_0)$ is the $(\mu_0-\delta)$-divergent type for any number $\delta$ $(<\mu_0)$ from the definition of $d(E)$. Hence however small $\delta$ is, it holds for some element $T\in\mathcal{Y}^*(\alpha_0)$ and some point $z_0\in D_T$

(3.15) $$\lim_{n\to\infty} \chi_{n,\infty}^{(\mu;T)}(z_0)=\infty, \quad (\mu=\mu_0-\delta).$$

Take a large integer N and let it be fixed. Then $\chi_{n,N}^{(\mu;T)}(z_0)$ is a function of n and $\mu$. Take a integer $n_1$ $(>0)$. Since $\chi_{n_1,N}^{(\mu;T)}(z_0)$ is a continuous and monotone decreasing function of $\mu$ for some fixed integer $n_1$ $(>0)$, we have

(3.16) $$\chi_{n_1,N}^{(\mu_0;T)}(z_0)=\lim_{\mu\to\mu_0} \chi_{n_1,N}^{(\mu;T)}(z_0)=k(n_1,\mu_0).$$

Since $\chi_{n,N}^{(\mu;T)}(z_0)$ is a monotone increasing function of N, we have

(3.17) $$\chi_{n_1,\infty}^{(\mu_0;T)}(z_0) \geqq k(n_1,\mu_0).$$

As $G^*(\alpha_0)$ is the $\mu$-divergent type $(\mu<\mu_0)$, there is a large integer $n_2$ $(>n_1)$ such that

(3.18) $$\chi_{n_2,N}^{(\mu;T)}(z_0) > \chi_{n_1,N}^{(\mu;T)}(z_0).$$

Hence from (3.17) and (3.18) we have

(3.19) $$\chi_{n_2,\infty}^{(\mu_0;T)}(z_0) \geqq k(n_1,\mu_0).$$

Continuing these procedures infinitely many times, we have the subsequence of computing functions $\{\chi_{n_i,\infty}^{(\mu_0;T)}(z_0)\}$ $(i=1,2,\cdots)$ so that it may holds for any $n_i$

(3.20) $$\chi_{n_i,\infty}^{(\mu_0;T)}(z_0) \geqq k(n_1,\mu_0).$$

If we suppose that $G^*(\alpha_0)$ is the $\mu_0$-convergent type, it holds

(3.21) $$\lim_{n\to\infty} \chi_{n,\infty}^{(\mu_0;T)}(z_0)=0.$$

This fact contradicts (3.20). Thus $G^*(\alpha_0)$ is not the $\mu_0$-convergent type. Therefore we can claim that $G^*(\alpha_0)$ is the $\mu_0$-finite type and hence $M_{\frac{\mu_0}{2}}(E)>0$, which solves the Problem (II). Hence we have from PROPOSITION 2 the following theorem.

THEOREM 4. *Let $G^*(\alpha_0)$ be a Kleinian group with properties* (A) *and* (B). *Assume that* $\frac{\mu_0}{2}>\alpha_0$. *Then $G^*(\alpha_0)$ is the $\mu_0$-finite type if and only if* $0<M_{\frac{\mu_0}{2}}(E)<+\infty$.

## References


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