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A Problem of Monotone Approximation

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Let $1 \leq k_1 \leq \cdots \leq k_q \leq n$ be the positive integers and let $\varepsilon_i = \pm 1$, $i=1,\ldots,q$. We denote by H_n the set of all algebraic polynomials p of degree n or less, and then we define the sets

$$H_n\binom{k_1,\ldots,k_q}{\varepsilon_1,\ldots,\varepsilon_q} = \{p \in H_n;\ \varepsilon_i p^{(k_i)}(x) \geq 0 \quad \text{for } a \leq x \leq b,\ i=1,\ldots,q\},$$

$$F\binom{k_1,\ldots,k_q}{\varepsilon_1,\ldots,\varepsilon_q} = \{f \in c^{(k_q)}[a,b];\ \varepsilon_i f^{(k_i)}(x) \geq 0 \quad \text{for } a \leq x \leq b,\ i=1,\ldots,q\}.$$

G. G. Lorentz and K. L. Zeller [1] approximated the function f in $F\binom{k_1,\ldots,k_q}{\varepsilon_1,\ldots,\varepsilon_q}$ by the polynomials p in $H_n\binom{k_1,\ldots,k_q}{\varepsilon_1,\ldots,\varepsilon_q}$, and they investigated this problem, such problem is called monotone approximation problem, in detail.

In [1] they proved the following.

Let $f \in F\binom{k_1,\ldots,k_q}{\varepsilon_1,\ldots,\varepsilon_q}$, $p \in H_n\binom{k_1,\ldots,k_q}{\varepsilon_1,\ldots,\varepsilon_q}$ and $f \neq p$, then the set $A(f,p) = \left\{x \in [a,b];\ |f(x)-p(x)| = \|f-p\|\right\}$ is compact and the function $\sigma(x) = \mathrm{SIGN}\,[f(x) - p(x)]$ is continuous on $A(f,p)$. Conversely, if a polynomial $p \in H_n\binom{K_1,\ldots,k_q}{\varepsilon_1,\ldots,\varepsilon_q}$, a compact set $A \subset [a,b]$ and a continuous sign function σ on A are given, then there exists a function $f \in C[a,b]$ for which $A = A(f,p)$ and $\sigma(x) = \mathrm{SIGN}[f(x)-p(x)]$.

However they left whether one can take f here to be continuously differentiable with the properties $\varepsilon_i f^{(k_i)}(x) > 0$, $a \leq x \leq b$, $i=1,\ldots,q$. We can give a solution for this problem.

Theorem. Given a non-empty compact set $A \subset [a,b]$ and a continuous sign function σ, then for any $p \in H_n\binom{k_1,\ldots,k_q}{\varepsilon_1,\ldots,\varepsilon_q}$, $k_q \leq \mathrm{DEG}\ P$, there exists a $f \in C^{(2n-1)}[a,b]$ for which $A = A(f,p)$, $\sigma(x) = \mathrm{SIGN}[f(x)-p(x)]$ on A and $\varepsilon_i f^{(k_i)}(x) \geq 0$ for $a \leq x \leq b$, $i=1,\ldots,q$.

First, we give a lemma. Let $\delta > 0$, $n=1,2,\ldots$, then the function

$$\lambda(x) = \lambda(\delta,x) = \lambda_{[\alpha,\beta]}(\delta,x) = \frac{\delta(x-\alpha)^{2n}}{(x-\alpha)^{2n} + (x-\beta)^{2n}}, \qquad \alpha \leq x \leq \beta$$

has the followlng properties;

(a) $\lambda(\alpha)=0,\ \lambda(\beta)=\delta,\ 0<\lambda(x)<\delta$ for $\alpha<x<\beta$,

(b) $\lambda^{(k)}(\alpha)=\lambda^{(k)}(\beta)=0$ for $1<k<2n-1$,

(c) if $\varepsilon_i p^{(k_i)}(x)>0$ for all $\alpha<x<\beta$, $i=1,\dots,q$, there exist $\delta_1>0$ and $\delta_2>0$ for which $|\lambda^{(k_i)}(\delta_1,x)|<|p^{(k_i)}(x)|$ for all $\alpha\le x\le\beta$, $i=1,\dots,q$, or $|\lambda^{(k_i)}(\delta_2,-x+\alpha+\beta)|\le|p^{(k_i)}(x)|$ for all $\alpha\le x\le\beta$, $i=1,\dots,q$.

Also, let $\gamma>0$, then the function

$$\mu(x)=\mu(\gamma,x)=\mu_{[\alpha,\beta]}(\gamma,x)=\begin{cases}\lambda_{[\alpha,\frac{\alpha+\beta}{2}]}(\gamma,x), & \alpha\le x\le\frac{\alpha+\beta}{2}\\ \lambda_{[\frac{\alpha+\beta}{2},\beta]}(\gamma,-x+\frac{\alpha+\beta}{2}+\beta), & \frac{\alpha+\beta}{2}<x\le\beta,\end{cases}$$

has the following properties;

(a′) $\mu(\alpha)=\mu(\beta)=0,\ 0<\mu(x)<\gamma$ for $\alpha<x<\beta$,

(b′) $\mu^{(k)}(\alpha)=\mu^{(k)}(\beta)\ (=\mu^{(k)}\frac{\alpha+\beta}{2}))=0$ for $1\le k\le 2n-1$,

(c′) if $\varepsilon_i p^{(k_i)}(x)>0$ for all $\alpha<x<\beta$, $i=1,\dots,q$, there exists $\gamma>0$ for which $|\mu^{(k_i)}(\gamma,x)|\le|p^{(k_i)}(x)|$ for all $\alpha\le x\le\beta$, $i=1,\dots,q$.

Thus we get the following lemma.

Lemma. Let $\varepsilon_i p^{(k_i)}(x)>0$ for all $\alpha<x<\beta$, $i=1,\dots,q$, and let $\lambda_{[\alpha,\beta]}(\delta,x)$ and $u_{[\alpha,\beta]}(\gamma,x)$ be the functions defined as above. Then we have the following functions.

(1) Let $f_1(x)=p(x)+\frac{\delta}{2}-\lambda(\delta,x)$, then f_1 has the following properties;

(a) $f_1(\alpha)=p(\alpha)+\frac{\delta}{2},\ f_1(\beta)=p(\beta)-\frac{\delta}{2},\ |f_1(x)-p(x)|<\frac{\delta}{2}$ for $\alpha<x<\beta$,

(b) $\varepsilon_i f_1^{(k_i)}(x)\ge 0$ for $\alpha\le x\le\beta$, $i=1,\dots,q$,

(c) $f_1^{(k_i)}(\alpha)=p^{(k_i)}(\alpha),\ f_1^{(k_i)}(\beta)=p^{(k_i)}(\beta)$ for $i=1,\dots,q$.

(2) Let $f_2(x)=p(x)-\frac{\delta}{2}+\lambda(\delta,x)$, then f_2 has the following properties;

(a) $f_2(\alpha)=p(\alpha),\ f_2(\beta)=p(\beta)+\frac{\delta}{2},\ |f_2(x)-p(x)|<\frac{\delta}{2}$ for $\alpha<x<\beta$ and f_2 satisfies the above conditions (1),(b) and (c).

(3) Let $f_3(x)=p(x)+\frac{\delta}{2}-\mu(\gamma.x),\ 0<\gamma<\delta$, then f_3 has the following properties;

(a) $f_3(\alpha)=p(\alpha)+\frac{\delta}{2},\ f_3(\beta)=p(\beta)+\frac{\delta}{2},\ |f_3(x)-p(x)|<\frac{\delta}{2}$ for $\alpha<x<\beta$, and f_3 satisfies the above conditions (1) (b) and (c).

(4) Let $f_4(x)=p(x)-\frac{\delta}{2}+\mu(\gamma,x),\ 0<\gamma<\delta$, then f_4 has the following properties;

(a) $f_4(\alpha)=p(\alpha)-\frac{\delta}{2},\ f_4(\beta)=p(\beta)-\frac{\delta}{2},\ |f_4(x)-p(x)|<\frac{\delta}{2}$ for $\alpha<x<\beta$, and f_4 satisfies the above conditions (1),(b) and (c).

Proof of Theorem. Given A and σ, then we can find the finite sets of points; $\{a_j\}_{j=1}^m,\ \{b_j\}_{j=1}^m\subset A$ such that $a\le a_1\le b_1<a_2\le b_2<\cdots<a_m\le b_m\le b$, $A=\cup_{j=1}^m \mathbf{A}_j$,

$A_j=[a_j,b_j]\cap A,\ j=1,\dots,m,$ and σ is constant on each A_j, but σ changes its sign on the successive A_j and A_{j+1}. Here we will assume that $a\in A$ and $b\notin A$, though we can consider four cases such that (1) $a\in A$ and $b\in A$, (2) $a\in A$ and $b\notin A$, (3) $a\notin A$ and $b\in A$, (4) $a\notin A$ and $b\notin A$. Then

$$[a,b]\setminus A=\cup_{j=1}^{m-1}\ (b_j,a_{j+1})\cup(b_m,b)\cup\cup_{i=1}^{\infty}\ (c_i,d_i).$$

We define the sets

$$\mathcal{A}=\ \{[b_j,a_{j+1}];j=1,\dots,m-1\}\ ,\ \mathcal{B}=\ \{[c_j,d_i];i=1,2,\dots\}\ ,$$
$$Z=\ \{x\in[a,b]\setminus A;P^{(k_i)}(x)=0 \text{ for some } i=1,\dots,q\}\ ,$$
$$\mathcal{A}_1=\ \{I\in\mathcal{A};I\cap Z\neq\phi\}\ ,\mathcal{A}_1=\ \{I\in\mathcal{A};I\cap Z=\phi\}\ ,$$
$$\mathcal{B}_1=\ \{I\in\mathcal{B};I\cap Z\neq\phi\}\ ,\mathcal{B}_1=\ \{I\in\mathcal{B};I\cap Z=\phi\}$$

Here Z is finite, because $k_q\leq\text{DEG p}$. If $I=[b_j,a_{j+1}]\in\mathcal{A}_1$, we have

$$b_j<z_j=\text{MIN}(I\cap Z)\leq z_j'=\text{MAX}(I\cap Z)<a_{j+1}.$$

Then we define the sets

$$\mathcal{A}_2=\ \{[b_j,z_j],[z_j',a_{j+1}];[b_j,a_{j+1}]\in\mathcal{A}_1\}\ ,$$
$$\mathcal{A}_2=\ \{[z_j,z_j'];[b_j,a_{j+1}]\in\mathcal{A}_1\}\ .$$

Similarly we define as following;

$$c_i<z_i''=\text{MIN}(I\cap Z)<z_i'''=\text{MAX}(I\cap Z)<d_i \text{ for } I=[c_i,d_i]\in\mathcal{B}_1,$$
$$\mathcal{B}_2=\ \{[c_i,z_i''],[z_i''',d_i];[c_i,d_i]\in\mathcal{B}_1\}\ ,$$
$$\mathcal{B}_2=\ \{[z_i'',z_i'''];[c_i,d_i]\in\mathcal{B}_1\}\ .$$

We also define as following;

$$z_o=\begin{cases}\text{MIN}([b_m,b]\cap z) & \text{if } (b_m,b]\cap z\neq\phi,\\ b & \text{if } [b_m,b]\cap z=\phi\end{cases}$$
$$\mathcal{C}=\{[b_m,z_o]\}\ ,\ \bar{c}=\ \{[z_o,b]\}$$

To decide the required function f we will give f on each interval defined as above.

Since the total number of the intervals in $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_2$ and $\mathcal{C}$ is finite, we can take the function f such as (1) or (2) in Lemma with a constant $\delta^*>0$. On A we take f such that $f(x)=p(x)+\sigma(x)\delta^*$. On each interval in $\mathcal{A}_1$ we take f such that it is continuously connected with f defined on $\mathcal{A}$ and it is given by (1) or (2) in Lemma with δ^*. On each interval in $\mathcal{B}_1$ we take f such that it is continuously connected with f defined on A and it is given by (3) or (4) in Lemma with δ^* and $0<\gamma<\delta^*$. On each interval in $\mathcal{A}_2, \mathcal{B}_2$, and $\mathcal{C}$ we take f such that $f(x)=p(x)$. Lastly, on each interval in $\mathcal{A}_2$, $\mathcal{B}_2$ and $\mathcal{C}$ we take f such that it is continuously connected with f defined as above and it equals to one of the following functions;

$$f(x)=p(x)+\frac{\delta^*}{2}-\lambda(\frac{\delta^*}{2},x), \qquad f(x)=p(x)-\frac{\delta^*}{2}+\lambda(\frac{\delta^*}{2},x).$$

It is clear for this function f to satisfy the conditions required in theorem.

(q.e.d.)

Reference

〔1〕 G.G.Lorentz and K.L.Zeller, Monotone Approximation by Algebraic Polynomials, Trans. Amer. Math. Soc., 149〔1970〕 1-18.