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A Generalization of the Bernstein Polynomials and Limit of Its Iterations

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1. Introduction

The Bernstein polynomial of degree n for a real-valued bounded function F defined on the closed interval $[0, 1]$ is defined by

$$B_n(F)(x) = \sum_{k=0}^{n} F(\tfrac{k}{n})\binom{n}{k}x^k(1-x)^{n-k}.$$

The following theorem is well known (see, for example, G. G. Lorentz [4]).

Theorem A (S. N. Bernstein). If F is continuous on $[0, 1]$, then

$$\operatorname{LIM}_{n\to\infty} B_n(F)(x) = F(x)$$

holds uniformly on this interval.

It goes without saying that this theorem gives a constructive proof of the classical theorem of Weierstrass concerning the uniform approximability of continuous functions by polynomials.

R. P. Kelisky and T. J. Rivlin [3] investigated the convergence behavior of the k-th iteration B_n^k of B_n and obtained the following theorem.

Theorem B. If F is continuous on $[0, 1]$ and n is fixed, then

$$\operatorname{LIM}_{k\to\infty} B_n^k(F)(x) = F(0)+(F(1)-F(0))x$$

holds uniformly on this interval.

So far various generalizations of the Bernstein polynomials have been obtained by many authors. One of them is the following which can be founded in Lorentz [4]:

The m-dimensional Bernstein polynomial of degree n for a real-valued bounded function F defined on the simplex

$$K_m = \left\{(x_1,\ldots,x_m) \in \mathbf{R}^m ;\ x_1 \geq 0,\cdots,x_m \geq 0,\ \sum_{j=1}^{m} x_j \leq 1\right\}$$

is defined by

$$B_{n,m}(F)(x_1,\ldots,x_m) = \sum_{\substack{k_j \geq 0,\ j=1,\cdots m \\ k_1+\cdot\cdot\cdot+k_m \geq n}} F\left(\frac{k_1}{n},\ldots,\frac{k_m}{n}\right)\binom{n}{k_1,\ldots,k_m} \times x_1^{k_1}\ldots x_m^{k_m}(1-x_1-\cdots-x_m)^{n-k_1-\cdots-k_m},$$

where $\binom{n}{k_1,\ldots,k_m} = \frac{n!}{k_1!\cdot\cdot\cdot k_m!(n-k_1-\cdots-k_m)!}$.

Noting K_m to be homeomorphic to a compact subset in $\mathbf{R}^{m+1}$

$$K' = \left\{(x_0,\ x_1,\ldots,x_m) \in \mathbf{R}^{m+1} ;\ x_0 \geq 0,\ x_1 \geq 0,\ldots,x_m \geq 0,\ \sum_{j=0}^{m} x_j = 1\right\},$$

R. Schnable [7] generalized $B_{n,m}$ to space of Radon probability measures on a compact Hausdorff space and investigated the problem of its saturation in his paper [8].

The purpose of this paper is to give a generalization of K_m (we will denote by $K(\Delta)=K$ this) which is different from that generalized by Schnabl [7], to construct the θ-T-operators being a generalization of the m-dimensional Bernstein polynomials and to establish the corresponding results (Theorem 1) to theorems A-B with respect to our θ-T-operators and a result (Theorem 2) concerning the saturation of those. We will notice that in the case of the one-dimensional, the proof of the part (b) in our theorem 1 can be a simple another proof of theorem B, and our θ-T-operators are defined for space of Radon probability measures on a compact Hausdorff space and in the special case θ, T, the classical operators having been constructed in Schnabl [7] are obtained and the proof of our theorem 2 applied to his operators can be a simple another proof of his result in his paper [8] concerning the saturation theorem.

Now, to achieve our purpose, we introduce the following notations.

X : a compact Hausdorff space.

$C(X)$: a Banach algebra of all real-valued continuous functions defined on X with the usual pointwise operations and sup-norm $\|\cdot\|$.

1 : an unit of C(X).

$\Delta(C(X)) = \Delta$: a set of all non-zero homomorphisms of $C(X)$ onto the field of real numbers.

$K(\Delta) = K$: a weak*-closed convex hull of Δ in the dual space $C(X)^*$ of $C(X)$ considered as a Banach space.

I : an identity operator on $C(X)$.

HOM $(C(X))_1$: a set of all homomorphisms T of $C(X)$ into itself with $T(1)=1$.

$\theta = <\theta_n(j)>_{n,j\geq 1}$: an infinite matrix whose entries satisfy the following

(i) $\theta_n(j)>0,\ (1\leq j\leq n),\ \theta_n(j)=0,\ (j>n)$;

(ii) $\sum_{j=1}^{n} \theta_n(j)=1$, for each $n\geq 1$;

(iii) $\operatorname{Lim}_{n\to\infty} \sum_{j=1}^{n} \theta_n^2(j)=0$, where $\theta_n^2(j)=(\theta_n(j))^2$.

$T=<T_{n,j}>_{n,j\geq 1}$: an infinite matrix whose entries satisfy the following

(i) $T_{n,j} \in \operatorname{Hom}(C(X))_1$, for all $n, j\geq 1$;

(ii) $\operatorname{Sup}_{n\geq 1} \sum_{j=1}^{n} \| (T_{n,j}-I)(f) \|^2<+\infty$, for each $f \in C(X)$.

We call θ and T θ-facter and T-facter, respectively. The simple examples of these are the following:

(1) $$\theta_n(j)=\frac{1}{n},\ (1\leq j\leq n),\ \theta_n(j)=0,\ (j>n);\ T_{n,j}=I.$$

We write simply $\theta=<\frac{1}{n}>$ and $T=<I>$ in stead of the θ-facter and T-facter defined by (1), respectively.

For each $x \in X$ and $T \in \operatorname{Hom}(C(X))_1$, we define $\varphi_{x,T}$ by $\varphi_{x,T}(f)=T(f)(x)$ to be an element of Δ, and for θ-facter, T-facter and $n=1, 2,\ldots$, define the mapping $\Phi_n(\theta, T;\bullet)$: $X^n \to K$ by $\Phi_n(\theta, T; (x_1,\ldots,x_n)) = \sum_{j=1}^{n} \theta_n(j)\ \varphi_{x_j,T_{n,j}}$.

2. A generalization of the m-dimensional Bernstein polynomials — definition of the θ-T-operators —, and lemmas

Using $C(X)$ to be isometrically isomorphic with $C(\Delta)$ and Urysohn's lemma, we have the following lemma:

lemma 1. If $T \in \operatorname{Hom}(C(X))_1$ is fixed, then the mapping φ_T: $X\to\Delta$ defined by $\varphi_T(X)=\varphi_{x,T}$ is continuous. Furthermore, if T is an isomorphism of $C(X)$ onto itself, then φ_T is a homeomorphism of X onto Δ.

From lemma 1 it is immediate that $\Phi_n(\theta, T;\bullet)$ is continuous. Furthermore, it is readily seen that $\mu\geq 0$ and its linear functional norm $\|\mu\|=1$ hold for each $\mu \in K$. So, in view of Riesz's representation theorem for the linear functionals, we identify $\mu \in K$ with a Radon probability measure on X and write $\hat{f}(\mu)=\mu(f)=\int_X f(x)d\mu(x)$ for each $f \in C(X)$.

Definition. Let F be a real-valued bounded function defined on K, so that $F\circ\Phi_n(\theta, T;\bullet) \in C(X^n)$, and $\mu \in K$. Then we define

$$B_n(\theta, T; F)(\mu)=\int_X \cdots \int_X F(\Phi_n(\theta, T; (x_1,\ldots,x_n)))\, d\mu(x_1)\cdots d\mu(x_n).$$

We call $\{B_n(\theta, T;\bullet)\}_{n=1}^{\infty}$ θ-T-operators.

Remark. R. Schnabl [7] has introduced $\{\mu \in C(X)^*;\ \mu\geq 0,\ \mu(1)=1\}=P$ being identified with the space of Radon probability measures on X as a generalization of K_m. Since $\Phi_n(\theta,T; (x_1,\ldots,x_n)) \in P$, we can define $B_n(\theta, T;\bullet)$ with respect to P, too and putting $\theta=<\frac{1}{n}>$ and $T=<I>$, then it is easily seen that the classical operators having been constructed in his paper [7] are obtained.

Let $X=\{0, 1,\dots,m\}$. Then we have that C(X) is isometrically isomorphic with $\mathbf{R}^{m+1}$ being a Banach algebra with norm $\|(x_0, x_1,\dots,x_m)\| = \mathrm{MAX}_{0\leq j\leq m} |x_j|$, and therefore $C(X)^*$ is a Banach space $\mathbf{R}^{m+1}$ with norm $\|(x_0, x_1,\dots,x_m)\| \sum_{j=0}^{m} |x_j|$. Using lemma 1 as $T=I$, we are able to see that Δ is identified with $\{e_0, e_1,\dots,e_m\}$, where $e_j=(0,\dots,0, \overset{j}{1}, 0,\dots,0)$ for $j=0, 1,\dots,m$, and therefore K agrees with both K' being a (weak*-) closed convex hull of $\{e_0, e_1,\dots,e_m\}$ and P. Furthermore, using lemma 1, it is seen that our θ-T-operators are polynomial operators if every entry of T-facter is an isomorphism of $C(X)$ onto itself, and they are the m-dimensional Bernstein polynomials $B_{n,m}$, $n=1,2,\dots$ if $\theta=<\frac{1}{n}>$ and $T=<I>$, by the arguments similar to Schnabl [7].

By the immediate calculations, we have the following lemma:

Lemma 2. (a) For each $n=1, 2,\dots$, $B_n(\theta, T;\circ)$ is a positive linear operator from $C(K)$ into itself with its operator norm $\|B_n(\theta, T;\circ)\|=1$.

(b) For each $f\in C(X)$ and $\nu\in K$, the following equations hold:

$$B_n(\theta, T;\hat{f})=\sum_{j=1}^{n}\theta_n(j)\widehat{T_{n,j}(f)},$$

$$B_n(\theta, T;\hat{f}^2)=\Big(\sum_{j=1}^{n}\theta_n(i)\widehat{T_{n,j}(f)}\Big)^2+\sum_{j=1}^{n}\theta^2{}_n(j)(\widehat{T_{n,j}(f^2)}-\widehat{T_{n,j}(f)}^2),$$

and

$$B_n(\theta, T;(\hat{f}(\mu)-\hat{f}(\nu))^2, \nu)=(\sum_{j=1}^{n}\theta_n(j)(\widehat{T_{n,j}-I)(f)}(\nu))^2 + \sum_{j=1}^{n}\theta^2{}_n(j)(\widehat{T_{n,j}(f^2)}-\widehat{T_{n,j}(f)}^2)(\nu).$$

3. The main theorems

We denote by $B_n^k(\theta, T;\circ)$ the k-th iteration $B_n(\theta, T;\circ)$, that is,

$$B_n^1(\theta, T;\circ)=B_n(\theta, T;\circ), \text{ and } B_n^k(\theta, T;\circ)=B_n(\theta, T;B_n^{k-1}(\theta, T;\circ)), \quad k\geqq 2.$$

Theorem 1. Let $F\in C(K)$. Then the following statements hold.

(a) If k is fixed, then $\mathrm{LIM}_{n\to\infty}\|B_n^k(\theta, T;F)-F\|=0$

(b) In the case of $T=<I>$, if n is fixed, then

$$\mathrm{LIM}_{n\to\infty}\|B_n^k(\theta, T;F)-\widehat{F\circ\Phi_1(\theta, T;\circ)}\|=0$$

Proof. (a) for brevity, we write simply $B_n(\theta, T;\circ)=B_n$ and $\Phi_n(\theta, T;\circ)=\Phi_n$. Since $B_n^k(F)-F=\sum_{j=0}^{k-1}B_n^j(B_n(F)-F)$ and $\|B_n\|=1$, it will suffice to prove (a) in the case of $k=1$. Let $\varepsilon>0$ be given. By compactness of K and continuity of F there exists a finite subset $\mathscr{F}$ in $C(X)$ such that

$$(2) \qquad |F(\mu)-F(\nu)| \leq \varepsilon+2\|F\|\sum_{f\in\mathscr{F}}(\hat{f}(\mu)-\hat{f}(\nu))^2$$

for all $\mu, \nu \in K$. Since B_n is a positive linear operator with $B_n(1)=1$, we can operate on the variable μ in (2) and obtain

(3) $$|B_n(F)(\nu)-F(\nu)| \leq \varepsilon+2\|F\| \sum_{f\in\mathscr{F}} B_n((\hat{f}(\mu)-\hat{f}(\nu))^2, \nu)$$

for all $\nu \in K$. Using lemma 2 and the conditions of θ-facter and T-facter, we see that

(4) $$\operatorname*{LIM}_{n\to\infty} B_n((\hat{g}(\mu)-g(\nu))^2, \nu)=0$$

uniformly for $\nu \in K$, whenever $g \in C(X)$. Therefore, by (3) and (4), we have that $\operatorname*{LIM}_{n\to\infty}\|B_n(F)-F\|=0$.

(b) Putting $n=1$ in (3), and using lemma 2, $\theta_1(1)=1$ and $B_1(F)=F\circ\Phi_1$, we obtain

$$|\widehat{F\circ\Phi_1}(\nu)-F(\nu)| \leq \varepsilon+2\|F\| \sum_{f\in\mathscr{F}} (\hat{f^2}-\hat{f}^2)(\nu)$$

for all $\nu \in K$. Furthermore, since B_n^k is a positive linear operator with $B_n^k(1)=1$, we obtain

(5) $$|B_n^k(\widehat{F\circ\Phi_1})(\nu)-B_n^k(F)(\nu)| \leq \varepsilon+2\|F\| \sum_{f\in\mathscr{F}} (B_n^k(\hat{f^2})-B_n^k(\hat{f}^2))(\nu)$$

for all $\nu \in K$. Using lemma 2 ang induction on k, we see that

(6) $$B_n^k(\hat{g})=\hat{g},\ B_n^k(\hat{g}^2)=\hat{g}^2+(1-\sum_{j=1}^{n}\theta_n^2(j))^k(\hat{g}^2-\hat{g^2})$$

for all $g \in C(X)$. Therefore, by (5) and (6),

$$|(\widehat{F\circ\Phi_1})(\nu)-B_n^k(F)(\nu)| \leq \varepsilon+2\|F\|(1-\sum_{j=1}^{n}\theta_n^2(j))^k \sum_{f\in\mathscr{F}}(\hat{f^2}-\hat{f}^2)(\nu)$$

for all $\nu \in K$. Here, taking norm, we obtain

(7) $$\|\widehat{F\circ\Phi_1}-B_n^k(F)\| \leq \varepsilon+2\|F\|(1-\sum_{j=1}^{n}\theta_n^2(j))^k \sum_{f\in\mathscr{F}}\|\hat{f^2}-\hat{f}^2\|.$$

Since $\operatorname*{LIM}_{k\to\infty}(1-\sum_{j=1}^{n}\theta_n^2(j))^k=0$ for each fixed n, and $\varepsilon>0$ is arbitrary, (7) completes the proof of (b). q. e. d.

Remark 1. In the case of $X=\{0, 1\}$, if we put $\theta=<\frac{1}{n}>$ in (b) of theorem 1, then before-mentioned Kelisky and Rivlin's result (Theorem B) is obtained. However, our proof is different and simple.

Remark 2. From the proof of theorem 2, we see that if $T=<I>$, and $\{K_n\}$ is a sequence of positive integers, then

(8) $$\operatorname{LIM}_{n\to\infty} K_n \sum_{j=1}^{n} \theta_n^2(j)=0 \text{ if and only if } \operatorname{LIM}_{n\to\infty} \| B_n^{k_n}(\theta, T; F)-F\| =0$$

for all $F \in C(K)$, and

(9) $$\operatorname{LIM}_{n\to\infty} K_n \sum_{j=1}^{n} \theta_n^2(j)=+\infty \text{ if and only if } \operatorname{LIM}_{n\to\infty} \| B_n^{k_n}(\theta, T; F)-F\circ\Phi_1(\theta, T; \bullet)\| =0$$

for all $F \in C(K)$.

It follows from the result (a) of theorem 1 that $\{B_n(\theta, T; \bullet)\}_{n=1}^{\infty}$ is a storong approximation process on $C(K)$. With the help (6) and (9) we can now prove the following theorem concerning the saturation of $\{B_n(\theta, T; \bullet)\}_{n=1}^{\infty}$ for $T=<I>$.

Theorem 2. Let $T=<I>$ and $F \in C(K)$. Then the following are equivalent:

(i) $\| B_n(\theta, T; F)-F\| =o(\sum_{j=1}^{n} \theta_n^2(j))$;

(ii) $F=\widehat{F\circ\Phi_1(\theta, T; \bullet)}$, therefore $B_n(\theta, T; F)=F$ for all n.

Proof. By (6), it is obvious that (ii) implies (i). Suppose now that (i) is satisfied. Put $\varepsilon_n=(\sum_{j=1}^{n} \theta_n^2(j))^{-1} \| B_n(\theta, T; F)-F\|$. Then $\operatorname{LIM}_{n\to\infty} \varepsilon_n=0$. and

(10) $$\| B_n^k(\theta, T; F)-F\| < k\varepsilon_n \sum_{j=1}^{n} \theta_n^2(j).$$

We now choose a sequence $\{k_n\}$ of positive integers such that

$$\operatorname{LIM}_{n\to\infty} k_n \sum_{j=1}^{n} \theta_n^2(j)=+\infty \quad \text{and} \quad \operatorname{LIM}_{n\to\infty} k_n \varepsilon_n \sum_{j=1}^{n} \theta_n^2(j)=0.$$

Putting $k=k_n$ in (10), and letting n tend to ∞, we obtain

(11) $$\operatorname{LIM}_{n\to\infty} \| B_n^{k_n}(\theta, T; F)-F\| =0.$$

Thus, it follows from (9) and (11) that $F=\widehat{F\circ\Phi_1(\theta, T; \bullet)}$, and so $B_n(\theta, T; F)=F$ for all n, by (6). q. e. d

In the case of $T=<I>$, it is seen from (6) that there exists a noninvariant element $F_0 \in C(K)$ of $\{B_n(\theta, T; \bullet)\}_{n=1}^{\infty}$ such that

$$\| B_n(\theta, T; F_0)-F_0\| =O(\sum_{j=1}^{n} \theta_n^2(j)).$$

This fact and theorem 2 state that in case $T=<I>$, $\{B_n(\theta, T; \bullet)\}_{n=1}^{\infty}$ is saturated with order $\sum_{j=1}^{n} \theta_n^2(j)$ on $C(K)$ and its trivial class agrees with a subset $\widehat{C(X)}$ in $C(K)$, consisting of all $\hat{f}$ for which $f \in C(X)$. For the saturation problems concerning the classical Bernstein polynomial operators $\{B_n\}_{n=1}^{\infty}$ see [1], [2], [5], [6] and [9].

Remark 3. R. Schnabl [8] has used "semi-group methods" to prove that a sequence of his operators (see §1, Remark) is saturated with order $\frac{1}{n}$ on *C(P)*. Our proof of theorem 2 is quite true with respect to his operators. Therefore, we have given a simple another proof of his result.

References

[1] B. Bajšanski and R. Bojanic, A note on approximation by Bernstein polynomials, Bull. Amer. Math. Soc. **70**, (1964), 675–677.

[2] K. De Leeuw, On the degree of approximation by Bernstein polynomials, J. d'Anal. Math., **7** (1959), 89–104.

[3] R. P. Kelisky and T. J. Rivlin, Iterates of Bernstein polynomials, Pacific J. Math., **21** (1967), 511–520.

[4] G. G. Lorentz, Bernstein polynomials, Toronto, 1953.

[5] G. G.Lorentz, Approxxmation of Functione, Holt, New York, 1966.

[6] C. Micchelli, The Saturation Class and Iterates of the Bernstein Polynomials, J. Approx. Th., 8 (1973), 1–18.

[7] R. Schnabl, Eine Verallgemeinerung der Bernsteinpolynome, Math. Ann., **179** (1968), 74–82.

[8] R. Schnabl, Zum Saturationsproblem der Verallgemeinerten Bernsteinoperatoren, Abstract spaces and Approximation, Proc. Oberwolfach (Basel, 1969), 281–289.

[9] R. Schnal, Zum globalen Saturationsproblem der Folge der Bernsteinoperatoren, Acta. Sci. Math., (Szeged) **31** (1970), 351–358.