

On Divergence of Fourier Series

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1. Introduction

Y. M. Chen [2] has proved that there exists a function of the class $L(\log^+\log^+L)^{1-\varepsilon}$ with the almost everywhere divergent Fourier series by construction similar to the well-known classical example.

The purpose of this paper is to show that there exists an everywhere divergent Fourier series of $L(\log^+\log^+L)^{1-\varepsilon}$, by Y. M. Chen's result [1] with respect to the best possibility of the order of the partial sum of Fourier series of $L(0, 2\pi)$.

2. The main theorem

We shall give the following theorem.

THEOREM. Let $\lambda(t)$, $\phi(t)$ be defined for $t \geq 0$ and satisfy the following conditions ;

- (i) $\lambda(t)$ is positive, decreasing to 0, convex, and $-t\lambda'(t)$ is slowly varying.
- (ii) There exists a function $g(x) \in L(0, 2\pi)$ such that for every point $x \in [0, 2\pi)$,

$$s_n(g)(x) > \frac{1}{\lambda(n)} \text{ for infinitely many } n.$$

- (iii) $\phi(t)$ is non-negative and $t\phi(t)$ is increasing and convex.

- (iv) $-\left(\frac{1}{x}\right)^2 \lambda'\left(\frac{1}{x}\right) \phi\left(\left(\frac{1}{x}\right)^{1+\delta}\right) \in L(0, 2\pi)$ for some $\delta > 0$.

Then, there exists a function $f(x) \in L\phi(L)(0, 2\pi)$ such that the Fourier series diverges for all $x \in [0, 2\pi)$.

3. Proof of the theorem

We put

$$\varphi(x) = \sum_{j=0}^{\infty} \lambda(j) \cos jx$$

where this series converges for all $x \in (0, 2\pi)$ to a function $\varphi(x) \in L(0, 2\pi)$ and is the Fourier series of $\varphi(x)$, because $\lambda(j)$ decreases to 0 and is convex.

Now we consider the function

$$f(x) = (\varphi * g)(x) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(x-t)g(t)dt.$$

We shall show that this function $f(x)$ is satisfactory.

(1) The Fourier series of $f(x)$ diverges for all $x \in [0, 2\pi)$.

Otherwise, there should be a $x_0 \in [0, 2\pi)$ such that $s_n(f)(x_0)$ converges and let

$$s_n = s_n(f)(x_0) = \sum_{j=-n}^n \hat{\varphi}(j) \hat{g}(j) e^{ijx_0} \rightarrow s \quad (n \rightarrow \infty)$$

where
$$\hat{\varphi}(j) = \begin{cases} \frac{1}{2} \lambda(|j|) & \text{for } j \neq 0 \\ \lambda(0) & \text{for } j = 0 \end{cases}.$$

Then by Abel's transformation,

$$\hat{\varphi}(n) s_n(g)(x_0) = \hat{\varphi}(n) \sum_{j=1}^n \frac{1}{\hat{\varphi}(j)} (s_j - s_{j-1}) + o(1) = s_n - \sum_{j=1}^{\infty} c_{n,j} s_j + o(1)$$

where
$$c_{n,j} = \begin{cases} \hat{\varphi}(n) \left\{ \frac{1}{\hat{\varphi}(j+1)} - \frac{1}{\hat{\varphi}(j)} \right\} & \text{for } 1 \leq j \leq n-1 \\ 0 & \text{for } j \geq n \end{cases} \geq 0.$$

Since the matrix $(c_{n,j})$ satisfies the Toeplitz condition,

$$\hat{\varphi}(n) s_n(g)(x_0) \rightarrow s - s = 0 \quad (n \rightarrow \infty).$$

On the other hand,

$$\hat{\varphi}(n) s_n(g)(x_0) > \hat{\varphi}(n) \frac{1}{\lambda(n)} = \frac{1}{2} \lambda(n) \frac{1}{\lambda(n)} = \frac{1}{2}$$

for infinitely many n . So we have the contradiction.

(2) $f(x) \in L\mathcal{O}(L)$.

In order to prove this, as $t\mathcal{O}(t)$ is increasing and convex, by Jensen's inequality we need only to show that

$$\varphi(x) \in L\mathcal{O}(L).$$

Since $\lambda(t)$ decreases to 0 and $-t\lambda'(t)$ is slowly varying, then

$$\varphi(x) \approx -\frac{\pi}{2} \left(-\frac{1}{x}\right)^2 \lambda'\left(-\frac{1}{x}\right) \quad \text{as } x \rightarrow +0$$

(A. Zygmund [3], p. 189).

Therefore

$$\begin{aligned} & \int_0^{2\pi} |\varphi(x)| \mathcal{O}(|\varphi|)(x) dx \\ & \leq \text{const.} \int_0^{2\pi} \left(-\frac{\pi}{2}\right) \left(\frac{1}{x}\right)^2 \lambda'\left(\frac{1}{x}\right) \mathcal{O}\left(\left(-\frac{\pi}{2}\right) \left(\frac{1}{x}\right)^2 \lambda'\left(\frac{1}{x}\right)\right) dx \\ & \leq \text{const.} \int_0^{2\pi} \left(-\frac{\pi}{2}\right) \left(\frac{1}{x}\right)^2 \lambda'\left(\frac{1}{x}\right) \mathcal{O}\left(\text{const.} \left(\frac{1}{x}\right)^{1+\delta}\right) dx \\ & \leq \text{const.} \int_0^{2\pi} (-1) \left(\frac{1}{x}\right)^2 \lambda'\left(\frac{1}{x}\right) \mathcal{O}\left(\left(\frac{1}{x}\right)^{1+\delta}\right) dx < \infty. \end{aligned}$$

Q.E.D.

4. Corollaries

COROLLARY 1. Given any number ε ($0 < \varepsilon \leq 1$), there exists a function of the class $L(\log^+ \log^+ L)^{1-\varepsilon}$ such that the Fourier series diverges everywhere.

PROOF. We choose η such that $0 < \eta < \varepsilon$. We put $\lambda(t) = (\log \log t)^{-(1-\eta)}$ for $t \geq 3$ and extend it to be convex for $t \geq 0$, and put $\mathcal{O}(t) = (\log^+ \log^+ t)^{1-\varepsilon}$. Then the conditions (i) and (iii) are satisfied. The condition (ii) is Y. M. Chen's remarkable result [1]. Since

$$-\left(\frac{1}{x}\right)^2 \lambda'\left(\frac{1}{x}\right) \mathcal{O}\left(\left(\frac{1}{x}\right)^{1+\delta}\right) \leq \text{const.} \frac{1}{x} \left(\log \frac{1}{x}\right)^{-1} \left(\log \log \frac{1}{x}\right)^{-\{1+(\varepsilon-\eta)\}}$$

as $x \rightarrow +0$, the condition (iv) is satisfied.

COROLLARY 2. If for any given positive sequence $\mu_n = o(\log n)$ there exists a function $g(x) \in L$ such that for very point $x \in [0, 2\pi)$

$$s_n(g)(x) > \mu_n \quad \text{for infinitely many } n,$$

then for any given number ε ($0 < \varepsilon \leq 1$), there exists a function of the class $L(\log^+ L)^{1-\varepsilon}$ such that the Fourier series diverges everywhere.

The condition of this corollary is known as Zygmund's conjecture.

PROOF. As before, we choose η such that $0 < \eta < \varepsilon$. We put $\lambda(t) = (\log t)^{-(1-\eta)}$ for $t \geq 2$ and extend it to be convex for $t \geq 0$, and put $\mathcal{O}(t) = (\log^+ t)^{1-\varepsilon}$. Since

$$-\left(\frac{1}{x}\right)^2 \lambda'\left(\frac{1}{x}\right) \mathcal{O}\left(\left(\frac{1}{x}\right)^{1+\delta}\right) \leq \text{const.} \frac{1}{x} \left(\log \frac{1}{x}\right)^{-\{1+(\varepsilon-\eta)\}}$$

as $x \rightarrow +0$, therefore the all conditions of the theorem are satisfied.

REFERENCES

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