Sci. Rep. Kanazawa Univ.,
Vol. 14, No. 2, pp. 59~60
December 1969

A Note on a Definition of (G)-convergence

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(Recieved 23 October 1969)

In a Lecture note given by G. Stolzenberg[1], the definition of (G)-convergence is given as follows :
let X be a metric space and $\{S_i\}$ be a sequence of closed subsets of X, then it is said that the sequence $\{S_i\}$ converges to a closed subset S if for any compact set $K\subset X$, $\{S_i\cap K\}$ is a convergent sequence in Comp (K) and $S=\bigcup_K \lim_{i\to\infty} (S_i\cap K)$.

Moreover it is denoted that if X is σ-compact, then a family of closed subsets of X is normal in the above sense. In our former papers[2], we used the above property for a family of analytic sets in a domain of C^n. However, recently, M. Kita[3] pointed out that no convergent sequence of points is normal in the above definition. Therefore we amend the definition of (G)-convergence as follows.

Definition. Let (X, ρ) be a metric space and $\{S_i\}$ be a sequence of closed subsets of X. We say that $\{S_i\}$ converges geometrically to a closed subset S of X if

(i) for any point $p\in S$, there is a sequence $\{p_i\}$ of points such that $p_i\in S_i$ and $p_i\to p$.

(ii) for any compact set K and positive number ε, there is a positive integer $\nu_0=\nu_0$ (K, ε) such that $S_\nu\cap K\subset S^{(\varepsilon)}\cap K$ for $\nu\geqq\nu_0$, where $S^{(\varepsilon)}=\bigcup_{q\in S}\{q'\in X;\ \rho(q,q')<\varepsilon\}$.

Note that from this definition the following properties are obtained immediately.

1. If $S\cap K=\phi$, then $S_\nu\cap K=\phi$ for sufficiently large ν.
2. If a sequence $\{p_i\}$ of points $p_i\in S_i$ converges to $p\in X$, then $p\in S$.
3. If $\{S_i\}$ converges geometrically to S and T, then $S=T$.

LEMMA 1. *If* X *is σ-compact, then a family* $\mathfrak{S}$ *of closed subsets is normal.*

1) Volumes, limits and extensions of analytic varieties, Lecture note in Math., No. 19, Springer Verlag (1966).
2) On normarity of a family of analytic sets, Sci. Rep. Kanazawa Univ., **12** (1967), 209-213.
On a family of pure-dimensional analytic sets, ibid, **13** (1968), 73-82.
A remark on the theorem of Bishop, Proc. Japan Acad. **45** (1969), 243-246.
3) Department of Mathematics, Faculty of Science, Tokyo University.

Proof. Let $\{K_\nu\}$ be a sequence of compact subsets of X such that $K_1 \subset K_2 \subset K_3 \ldots$, and $\bigcup_{\nu=1}^{\infty} K_\nu = X$. Take any sequence $\{S_i\}$ of $\mathfrak{S}$. Since Comp (K_1) is compact metric space, there is a subsequence $\{S_i^{(1)}\}$ of $\{S_i\}$ such that $S_i^{(1)} \cap K_1$ converges to T_1 in Comp (K_1). Also, since Comp (K_2) is compact, there is a subsequence $\{S_i^{(2)}\}$ of $\{S_i^{(1)}\}$ such that $S_i^{(2)} \cap K_2$ converges to T_2 in Comp (K_2). We continue this process. A diagonal sequence $\{S_i^{(i)}\}$ converges to T_j in Comp (K_j) for any j. Let $T = \cup T_\nu$. From the property of the space Comp (K), we have only to prove that T is closed. Evidently we may assume $T \neq \phi$. Let $\{p_j\}$ be a sequence of points in T such that $p_j \to p \in X$. Put $K = \{p, p_j \ {}_{j=1,2,\ldots}\}$, then for some positive integer j_0, $K \Subset \mathring{K}_{j_0}$. We shall show that $p_j \in T_{j_0}$ for any j. Let $p_j \notin T_{j0}$. Since $p_j \in T_{j'}$ for some j', there is a sequence of points $q_\nu^{(j)} \in S_\nu^{(\nu)}$ such that $q_\nu^{(j)} \to p_j$ $(\nu \to \infty)$, and since $p_j \in \mathring{K}_{j_0}$, we may assume that $q_\nu^{(j)} \in K_{j_0}$.

Let $d = \rho(p_j, T_{j_0})$. Since $S_\nu^{(\nu)} \cap K_{j_0}$ converges to T_{j_0}, there is a positive integer ν_0 such that $S_\nu^{(\nu)} \cap K_{j_0} \subset T_{j_0}^{(d/2)} \cap K_{j0}$ for $\nu \geqq \nu_0$. Also we may assume that $\rho(q_\nu^{(j)}, p_j) < \dfrac{d}{2}$ for $\nu \geqq \nu_0$. Since $q_\nu^{(j)} \in S_\nu^{(\nu)} \cap K_{j_0}$, $q_\nu^{(j)} \in T_{j_0}^{(d/2)}$ for $\nu \geqq \nu_0$. On the other hand, $\rho(q_\nu^{(j)}, T_{j_0}) = \min_{t \in T_{j_0}} \rho(q_\nu^{(j)}, t) = \rho(q_\nu^{(j)}, t_0) \geqq \rho(p_j, t_0) - \rho(q_\nu^{(j)}, p_j) > d - \dfrac{d}{2} = \dfrac{d}{2}$. This is a contradiction and $p_j \in T_{j_0}$ for any j. Since T_{j_0} is closed $p \in T_{j_0} \subset T$. Q.E.D.

LEMA 2. *Let $\{S_i\}$ be a sequence of purely k-dimensional analytic sets in a domain D of C^n. If $\{S_i\}$ converges analytically to a purely k-dimensional analytic set S in D, then $\{S_i\}$ converges geometrically to S.*

Proof. This is a direct conclusion of an analytic convergence.

LEMMA 3. *Let X be a metric space and $\{S_i\}$ be a sequence of closed subsets of X which converges geometrically to S. If there are positive constant N, M such that for any point $p \in S_i$, $H_d(S_i \cap B(p:r)) \geq Nr^d$ where H_d is a d-dimensional Hausdorff measure and $B(p:r)$ is a relatively compact open ball of radius r with center p in X, and that $H_d(S_i) < M$ for all i, then for any compact set K, $H_d(S \cap K) \leqq M4^d/N$. Moreover if X is σ-compact, then $H_d(S) \leqq M4^d/N$.*

The proof of this Lemma is the same as that given in the Lecture note of G. Stolzenberg. From Lemma 3, the following Theorem of Bishop[4] holds.

THEOREM OF BISHOP. *Let $\{S_i\}$ be a sequence of purely k-dimensional analytic sets in a domain D of C^n which converges geometrically to a non-empty closed set S. If the volumes of S_i are uniformly bounded, then S is also an analytic set in D.*

4) Conditions for the analyticity of certain sets. Mich. Math. J., 11 (1964), 289-304.