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Note on the Generalized Prouhet-Tarry Problem in an Algebraic Number Field

To Professor Ikuzo Yamamoto in memorial

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1. Introduction. We consider the rational integral solutions of the system of k equations

(1) $\lambda_1{}^i+\lambda_2{}^i+\cdots+\lambda_s{}^i=\mu_1{}^i+\mu_2{}^i+\cdots+\mu_s{}^i \qquad (1\leqq i\leqq k).$

It is plain that these equations are satisfied when the $(\mu_1, \mu_2,\cdots,\mu_s)$ is a permutation of the $(\lambda_1, \lambda_2,\cdots,\lambda_s)$, snch a solution we call a trivial solution. It is easy to prove that there are no other solutions when $s\leqq k$. When $s>k$ there may be non-trivial solutions, and we denote by $N(k)$ the least value of s for which this is true. We have $N(k)\leqq\frac{1}{2}k(k+1)+1$, [1], (numbers in square brackets refer to the Notes at the end of the paper).

If $(\lambda_1, \lambda_2,\cdots,\lambda_s)$ and $(\mu_1, \mu_2,\cdots,\mu_s)$ satisfy

(2) $\lambda_1{}^{k+1}+\lambda_2{}^{k+1}+\cdots+\lambda_s{}^{k+1}\neq\mu_1{}^{k+1}+\mu_2{}^{k+1}+\cdots+\mu_s{}^{k+1},$

then we denote by $M(k)$ the least value of s for which (1) and (2) are true. It is very important and interesting problem to obtaln the estimation of the number of solutions of (1) or (1) and (2) and to know the existence of such solutions. Because these problems are very closely connected with some additive number theoretic problems in particular with the Waring problem. [2].

We consider these problems in an algebraic number field, but for the former we have already some results [3] and so in this note we consider the latter problem in some generalized forms. [2], [4].

Let K be an algebraic number field of degree n over the rational number field P. Let $\mathfrak{o}$ be the integral domain consisting of all integers in K and $K^{(s)}$ $(1\leqq s\leqq r_1)$ be r_1 real conjugate fields and $K^{(t)}$, $K^{(t+r_2)}$ $(r_1+1\leqq t\leqq r_1+r_2)$ be r_2 pairs of complex conjugate fields, so that $n=r_1+2r_2$. We denote by $\gamma^{(i)}$ $(1\leqq i\leqq n)$ the conjugate of $\gamma\in K$. Let $\Omega(T.c)$ $(c>0)$ be the set consisting of all integers $\lambda\in\mathfrak{o}$ satisfying the conditions [5]

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$$\begin{cases} cT \leqq \lambda^{(s)} \leqq 2cT & (1 \leqq s \leqq r_1) \\ cT \leqq |\lambda^{(t)}| \leqq 2cT & (r_1+1 \leqq t \leqq r_1+r_2) \end{cases}$$

where T is a sufficiently large number and $\theta_2-\theta_1>0$ is a sufficiently small real number satisfying the conditions

$$\theta_1 \leqq \text{Arg } \lambda^{(i)} \leqq \theta_2 \text{ and } 0<\theta_1<k_{p+1}\theta_1<\theta_2<\frac{\pi}{2} \qquad (r_i+1 \leqq i \leqq r_1+r_2).$$

We use a letter c (and similarly c_1, c_2) to denote a positive constant depending only on K. k r_1, r_2, θ_1, θ_2, k_1, $\cdots$ k_{p+1}. It is not necessarily the same one each time it occurs.

2. Three theorems and one lemma. we begin with

Lemma 1. *Let* l_i $(1 \leqq i \leqq q)$ *and* $a_{ij}>0$ $(1 \leqq i,j \leqq p$ *be non-negative integers with* $0 \leq l_1<l_2<\cdots<l_p$ *and let* $H>0$. *Let*

$$A = \begin{pmatrix} a_{11}c_1{}^{l_1}, & a_{12}c_2{}^{l_1}, & \cdots, & a_{1p}c_p{}^{l_1} \\ a_{21}c_1{}^{l_2}, & a_{22}c_2{}^{l_2}, & \cdots, & a_{2p}c_p{}^{l_2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p1}c_1{}^{l_p}, & a_{p2}c_2{}^{l_p}, & \cdots, & a_{pp}c_p{}^{l_p} \end{pmatrix}$$

and put

(1) $$d_m = a_{11}a_{22}\cdots a_{mm}c_1{}^{l_1}c_2{}^{l_2}\cdots c_m{}^{l_m}$$

(2) $$\theta_m = \sum_{(i_1\cdots i_m)\neq(1,\cdots,m)} a_{i_1}\cdots a_{i_m}c_{i_1}{}^{l_1}\cdots c_{i_m}{}^{l_m} \quad (1 \leqq m \leqq p)\ .$$

Then there exst p natural numbers c_i $(1 \leqq i \leqq p)$, *satisfying* (1), (2) *and*

$$\varphi_m(c_1,\cdots,c_m) = d_m - H\theta_m > 0 \qquad (1 \leqq m \leqq p),$$

where c_i $(1 \leqq i \leqq p)$ *depending only on* H, d_i, l_i $(1 \leqq i \leqq p)$ *and* a_{ij} $(1 \leqq i,\ j \leqq p)$.

Proof. We shall prove by induction on m. If $m=1$, then we obtain a natural number c_1 satisfying $\varphi(c_1)=a_1c_1{}^{l_1}>0$. Suppose our result holds for $\varphi_j(c_1\cdots c_j)$ $(1 \leqq j \leqq m-1)$. It follows at once that

$$\varphi_m(c_1\cdots c_m) = a_{mm}c_m{}^{l_m}\varphi_{m-1}(c_1\cdots c_{m-1}) - H\psi(c_1\cdots c_m),$$

where $\psi(c_1\cdots c_m)=\psi_1(c_m)$ is a polynomial in c_m of degree l_m-1. We have, by the induction assumptions, $\varphi_{m-1}(c_1\cdots c_{m-1})>0$, where c_i $(1 \leqq i \leqq m-1)$ are defined and natural numbers. Thus we can choose a natural number c_m so that $\varphi_m(c_1\cdots c_m)>0$. This proves the result for m.

We say for simplicity "We define $(c_1,\cdots,c_p)$ by the matrix A and $H>0$."

Theorem 1. *Let* a_{ijv} $(1 \leqq i,\ j \leqq p+1,\ 1 \leqq v \leqq l)$ *and* k_i $(1 \leqq i \leqq p+1)$ be natural numbers and suppose that $k_1<k_2<\cdots<k_{p+1}$ *and let* $H_v>0$ $(1 \leqq v \leqq l)$. *We put*

$$A_v = \begin{pmatrix} a_{1,1,v}\, c_{1,v}^{k_1-1}, & \cdots\cdots, & a_{1,p+1,v}\, c_{p+1,v}^{k_1-1} \\ a_{2,1,v}\, c_{1,v}^{k_2-1}, & \cdots\cdots, & a_{2,p+1,v}\, c_{p+1,v}^{k_2-1} \\ \cdots & \cdots & \cdots \\ a_{p+1,1,v}\, c_{1,v}^{k_{p+1}-1}, & \cdots, & a_{p+1,p+1,v}\, c_{p+1,v}^{k_{p+1}-1} \end{pmatrix} \quad (1 \leqq v \leqq l).$$

We define $(c_{1v}, \cdots, c_{p+1,v})$ $(1 \leqq v \leqq l)$ *by the matrices* A_v *and* H_v $(1 \leqq v \leqq l)$. *Let* $r_K(\nu_1, \cdots, \nu_p)$ *be te number of solutions* λ_{uv} *of the system of Diophantine equations*

$$(3) \qquad \sum_{u=1}^{p+1} (a_{j,u,1}\, \lambda_{u,l}^{k_i} + \cdots + a_{i,u,l}\, \lambda_{u,l}^{k_i}) = \nu_i \qquad (1 \leqq i \leqq p)$$

where

$$\beta = 1 - \frac{1}{k_{p+1}} \text{ and } \lambda_{uv} \in \Omega\ (T^{\beta^{v-1}},\ c_{uv}) \quad (1 \leqq u \leqq p+1,\ 1 \leqq v \leqq l).$$

Then there are $\nu_i \in \mathfrak{o}$ $(1 \leqq i \leqq p)$ *satisfying*

$$r_K(\nu_1 \cdots \nu_p) \geqq c T^{n(k_{p+1}(p+1)(1-\beta^l) - (k_1 + \cdots + k_p))}$$

where c *is a constant depending only on* K, r_1, r_2, θ_1, θ_2, c_{uv}, k_i, a_{ij} *and* β.

Remark. We put $k_i = i$ $(1 \leqq i \leqq k+1)$ and all $a_{ijv} = 1$ and let

$$s \geqq \left[\frac{\log 2 - \log\ (1 + \frac{1}{k+1})}{-\log\ (1 - \frac{1}{k+1})} + 1 \right] (k+1),$$

then we have the solutions (non trivial) of Tarry-type equations

$$\sum_{u=1}^{p+1} \sum_{v=1}^{l} \lambda_{uv}^{i} = \sum_{u=1}^{p+1} \sum_{v=1}^{l} \mu_{uv}^{i} \qquad (1 \leqq i \leqq k+1).$$

Proof of Theorem 1. Let E denote the number of set $(\lambda_{1v}, \cdots, \lambda_{p,v})$ $(1 \leqq v \leqq l)$ of solutions of the equations (3), Then we have

$$E \geqq c \prod_{u=1}^{p+1} \prod_{v=1}^{l} (c_{uv} T^{\beta^{v-1}})^n = c_1\ T^{n k_{p+1}(p+1)\ (1-\beta^l)}.$$

Let F denote the number of sets $(\nu_1, \cdots, \nu_p)$. Then we have

$$F \leqq c_2\ T^{n(k_1 + \cdots + k_p)}$$

Thus we have

$$r_K(\nu_1 \cdots \nu_p) \geqq E/F \geqq c\ T^{n(k_{p+1}(p+1)(1-\beta^l) - (k_1 + \cdots + k_p))}.$$

Theorem 2. *Let* $a_{i,j}$ $(1 \leqq i,\ j \leqq p+1)$ *and* k_i $(1 \leqq i \leqq p+1)$ *be natural numbers and suppose that* $k_1 < k_2 < \cdots k_{p+1}$ *and put*

$$(3)\quad \begin{cases} \sum_{j=1}^{s} a_{ij}\ \lambda_j^{ki} = \sum_{j=1}^{s} a_{ij}\ \mu_j^{ki} & (1 \leqq i \leqq p) \\ \sum_{j=1}^{s} a_{p+1,j}\ \lambda_j^{k_{p+1}} \neq \sum_{j=1}^{s} a_{p+1,j}\ \mu_j^{k_{p+1}} & \end{cases}$$

where λ_i *and* μ_i *belong to some* $\Omega\ (T_i,\ c_i)$ $(1 \leqq i \leqq s)$ *with sufficiently large* T_i $(1 \leqq i \leqq s)$. *We denote by* $W_K(k_1, \cdots, k_{p+1})$ *the least value of* s *for which* (3) *has a solution. Then*

$$W_K\,(k_1,\cdots,k_{p+1}) \leqq \left[\frac{\log(1+\dfrac{k_1+\cdots+k_p}{k_{p+1}})}{-\log(1-\dfrac{1}{k_{p+1}})} + 1 \right] (p+1).$$

Theorem 3. *Let* a_{ij} $(1 \leqq i \leqq p+1,\ 1 \leqq j \leqq s)$ *and* k_i $(1 \leqq i \leqq p+1)$ *be natural numbers and suppose that* $k_1 < k_2 < \cdots k_{p+1}$ *and puts (two sets of the system of Diopfantine equations)*

$$(4)\quad \begin{matrix} R_m : \\ (m=1,2) \end{matrix} \begin{cases} \sum_{j=1}^{s} a_{ij}\ \lambda_j^{ki} = \nu_i & (1 \leqq i \leqq p) \\ \sum_{j=1}^{s} a_{ij}\ \lambda_j^{ki} = \mu_m & (\mu_1 \neq \mu_2) \end{cases}$$

where λ_j $(1 \leqq k \leqq s)$ *belongs to some domain* $\Omega(T_j,\ c_j)$ *with sufficiently large* T_j. *If*

$$s \geqq \left[\frac{\log(1+\dfrac{k_1+\cdots+k_p}{k_{p+1}})}{-\log(1-\dfrac{1}{k_{p+1}})} + 1 \right] (p+1)$$

then there are ν_i $(1 \leqq i \leqq p) \in \mathfrak{o}$ *and* $\mu_1,\ \mu_2 \in \mathfrak{o}$ $(\mu_1 \neq \mu_2)$ *for which* (4) *has solutions.*

3. Proofs of Theorem 2 and Theorem 3.

Lemma 2. Let a_{ij} $(1 \leqq i,\ j \leqq p)$ and k_i $(1 \leqq i \leqq p)$ be natural numbers and suppose that $k_1 < k_2 < \cdots k_p$. Let $a_i \geqq 0$ $(1 \leqq i \leqq p)$ be real numbers and let $H > 0$. We put

$$A = \begin{pmatrix} a_{11}\, c_1^{k_1-1}, \cdots, a_{1p}\, c_p^{k_1-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{p1}\, c_1^{k_p-1}, \cdots, a_{pp}\, c_p^{k_p-1} \end{pmatrix}$$

We define $(c_1, \cdots, c_p)$ *by the matrix* A *and* $A\ H > 0$. *We put*

$$(1)\qquad \alpha_i = a_{i1}\lambda_1^{k_i} + \cdots + a_{ip}\lambda_p^{k_i} \qquad (1 \leqq i \leqq p),$$

where

$$\lambda_i \in \Omega(T, c_j) \qquad (1 \leqq i \leqq p).$$

Let E *denote the number of sets* $(\lambda_1, \cdots, \lambda_p)$ *for which* $\alpha_i^{(s)}$ $(1 \leqq s \leqq r_1)$ *and* Re $\alpha_i^{(t)}$, Im $\alpha_i^{(t)}$ $(r_1+1 \leqq t \leqq r_1+r_2)$ *fall into any given intervals whose length are respectively* $O(T^{a_i+k_i-1})$ $(1 \leqq i \leqq p)$. *Then*

$$E \ll T^{n(a_1+\cdots+a_p)}.$$

Proof. Instead of the given intervals of lengths $O(T^{a_i+k_i-1})$ $(1 \leqq i \leqq p)$ it is sufficient to consider intervals whose length is respectively $O(T^{k_i-1})$ $(a_i=0)$ $(1 \leqq i \leqq p)$. Because Lemma 2 follows from the fact that an intervals of length $O(T^{a_i+k_i-1})$ can be covered by $O(T^{a_i})$ intervals of length $O(T^{k_i-1})$. Let $(\lambda_1, \cdots, \lambda_p)$ and $(\lambda_1', \cdots, \lambda_p')$ be two sets of values for which we have (1) with λ_i, $\lambda_i' \in \Omega(T, c_j)$. We put $\alpha_i - \alpha_i' = \beta_i$, $\max\limits_{1 \leqq j \leqq n} \beta_i^{(j)} = \| \beta_i \|$ and $\lambda_i - \lambda_i' = \mu_i$ $(1 \leqq i \leqq p)$. Then we have from (1)

$$(2)\qquad \begin{cases} B_{11}\mu_1 + \cdots + B_{1p}\mu_p = \beta_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ B_{p1}\mu_1 + \cdots + B_{pp}\mu_p = \beta_p \end{cases}$$

where

$$B_{ij} = a_{ij}\,(\lambda_j^{k_i-1} + \lambda_j^{k_i-2}\lambda_j' + \cdots + \lambda_j'^{k_i-1}) \quad (1 \leqq i,\ j \leqq p)$$

and

$$\| \beta_i \| \ll T^{k_i-1}.$$

By the definition of $c_1, c_2, \cdots, c_p$ $(H = c2^{k^{1+\cdots+k_p-p}})$, we have

$$B_{ii} \in \Omega(T^{k_i-1}, k_i a_{ij}(2(j)^{k_i-1}) \qquad (1 \leqq i,\ j \leqq p)$$

and further

$$(3)\qquad \min_{1 \leqq q \leqq n} |B_{11}^{(q)} \cdots B_{pp}^{(q)}| \geqq c_1 k_1 k_2 \cdots k_p\, a_{11} \cdots a_{pp}\, c_1^{k_1-1} \cdots c_p^{k_p-1}\, T^{k_1+\cdots+k_p-p}$$

$$(4)\qquad \sum_{(i_1 \cdots i_p) \neq (1 \cdots p)} |B_{1i_1}^{(q)} \cdots B_{pi_p}^{(q)}| < c_1 k_1 \cdots k_p (2T)^{k_1+\cdots+k_p-p}$$

$$\times \sum_{(i_1 \cdots i_p) \neq (1 \cdots p)} a_{1i_1} \cdots a_{pi_p}\, c_{i_1}^{k_i-1} \cdots c_{i_p}^{k_p-1}.$$

Hence we have from (3) and (4)

$$(5)\qquad \det \| B_{ij}^{(q)} \| \geqq c_1 T^{k_1+\cdots k_p-p}$$

and

$$(6)\qquad \begin{vmatrix} \beta_1^{(q)} B_{12}^{(q)} \cdots B_{1p}^{(q)} \\ \cdots\cdots\cdots\cdots \\ \beta_p^{(q)} B_{p2}^{(q)} \cdots B_{pp}^{(q)} \end{vmatrix} = O(T^{k_1+\cdots+k_p-p}) \qquad (1 \leqq d \leqq n).$$

It follows from (2), (5) and (6), by the Cramer's method. that

$$\| \mu_i \| \ll 1 \qquad (1 \leqq i \leqq p).$$

This proves the required result.

Lemma 3. *We use the same notations and assumptions as Theorem* 1. *Let E be the number of solutions* λ_{uv} $(1 \leqq u \leqq p+1,\ 1 \leqq v \leqq l)$ *of the system of the Diophantine equations*

$$(7)\qquad \sum_{u=1}^{p+1} (a_{iu1} \lambda_{u1}^{ki} + \cdots + a_{iul} \lambda_{ul}^{ki}) = \nu_i \qquad (1 \leqq i \leqq p+1),$$

where $\nu_i \in \mathfrak{o}$ $(1 \leqq i \leqq p+1)$ *are given. Then for any given* ν_i

$$(8)\qquad E \ll c\, T^{n(k_{p+1}p-k_1\cdots-k_p)(1-\beta l)}.$$

proof. We can write the equations (7) as

$$(9)\qquad \sum_{u=1}^{p+1} a_{iu1} \lambda_{u1}^{ki} = \alpha_{i1} = \nu_i - \Big(\sum_{j=2}^{l} \sum_{u=1}^{p+1} a_{iuj} \lambda_{uj}^{ki}\Big)$$

$$= \nu_i - \Big(\sum_{j=2}^{l} \alpha_{ij}\Big),$$

where

$$\alpha_{ij} = \sum_{u=1}^{p+1} a_{iuj} \lambda_{uj} \qquad (1 \leqq j \leqq p+1,\ 1 \leqq j \leqq l).$$

From the assumptions $\lambda_{uj} \in \Omega(T^{\beta j-1}, c_{jv})$ the sum in bracket is always

$$\ll T^{\beta k_i} \qquad (1 \leqq i \leqq p+1)$$

Thus, for given values of $(\nu_1, \cdots \nu_{p+1})$, $\alpha_i^{(s)}$ $(1 \leqq s \leqq r_1)$, Re $\alpha_i^{(t)}$, Im $\alpha_j^{(t)}$ $(r_1+1 \leqq t \leqq r_1+r_2)$ fall into the intervals where length is respectively $O(T^{\beta k_i})$ $(1 \leqq i \leqq p+1)$. It follows from Lemma 2 $\left(a_i = k_i\, \beta - (k^i - 1) = 1 - \frac{k_i}{k_{p+1}} > 0\right)$ that the number of possible choices for $(\lambda_{11}, \cdots, \lambda_{p+1,1})$ is $\ll T^{n(p+1-\frac{k_1+\cdots+k_{p+1}}{k_{p+1}})}$. Having chosen this set $(\lambda_{11}, \cdots {}_{p+1,1})$ we can write the equation (9) as

$$\alpha_{i2} = \nu_j - \alpha_{i1} - \Big(\sum_{j=2}^{l} \alpha_{ij}\Big).$$

The sum in brecket is $\ll T^{\beta 2k_j}$. Hence for given $(\nu_1, \cdots, \nu_{p+1})$ and given $(\lambda_{11}, \cdots, \lambda_{p+1,1})$, the number of possible choices for the set $(\lambda_{12},\cdots,\lambda_{p+1,2})$

$$\ll T^{n\beta\left(p+1-\frac{(k_1+\cdots+k_{p+1})}{k_{p+1}}\right)}$$

continuing in this way, we find that the number of possible selections of l sets $(\lambda_{1u},\cdots,\lambda_{p+1,u})$ $(1\leqq u\leqq l)$ satisfying (7) is

$$\leqq c\, T^{n(k_{p+1}p-k_1\cdots-k_p)(1-\beta^l)}.$$

This proves (8).
Theorem 2 is a corollary of Theorem 3.

Proof of Theorem 3. From Theorem 1 there is a set $(\nu_1,\cdots,\nu_p)$ $(\nu_i\in\mathfrak{o})$ for which we have

$$r_K(\nu_1\cdots\nu_p)\geqq c\, T^{n(k_{p+1}(p+1)(1-\beta^l)-(k_1+\cdots+k_p))}.$$

We put

$$\sum_{u=1}^{p+1}(a_{iu1}\lambda_{u1}^{k_i}+\cdots+a_{iul}\lambda_{ul}^{k_i})=\nu_i \qquad (1\leqq i\leqq p). \tag{10}$$

If for all solution λ_{uj} $(1\leqq u\leqq p+1,\ 1\leqq j\leqq l)$ of (10) we have the same one value μ as

$$\mu=\sum_{u=1}^{p+1}(a_{p+1,u,1}\lambda_{u1}^{k_{p+1}}+\cdots+a_{p+1,u,l}\lambda_{ul}^{k_{p+1}}), \tag{11}$$

then we must obtain a contradiction.

We denote by E the number of solutions of the system of Diophantine equations (10) and (11). Then

$$E\ll T^{n(k_{p+1}p-k_1\cdots-k_p)(1-\beta^l)}. \tag{12}$$

But we have plainly

$$E\geqq r_K(\nu_1,\cdots,\nu_p). \tag{13}$$

If we take

$$l\geqq\left[\frac{\log\left(1+\frac{k_1+\cdots+k_p}{k_{p+1}}\right)}{-\log\left(1-\frac{1}{k_{p+1}}\right)}\right]+1,$$

then we have from Theorem 1

$$\begin{aligned} r_K(\nu_1\cdots\nu_p) &\ll T^{n(k_{p+1}p-k_1\cdots-k_p)(1-\beta^l)} \\ &< T^{n(k_{p+1}(p+1)(1-\beta^l)-(k_1+\cdots+k_p))}\leqq r_K(\nu_1\cdots\nu_p). \end{aligned} \tag{14}$$

From (12), (13) and (14) yields a contradiction. Thus it follows that we have two different values μ_1, μ_2 $(\mu_1\neq\mu_2)$ for which (4) holds. This proves Theorem 3.

Notes

[1] (1) G. H. Hardy and E. M. Wright: An introduction to the theory of numbers, Chap. XXI. 21. 9 3rd ed. Oxford, 1954.
(2) L. K. Hua: An introduction to the theory of numbers (chinese) (数論導引) Chap. 18, § 7. Peking, 1957.

[2] (1) L. K. Hua: Additive Primzahltheorie, Teubner, Leipzig, 1959.
(2) L. K. Hua: Abschätzungen von Exponentialsummen und ihre Anwendung in der Zahlentheorie § 29. Enzykl. der Math. 1, 2, Heft 13, Teil, 1959.
(3) И. М. Виноградов: К вопросv о верхней граниче для G(n), ИАН СССР, Серзя Матем. 23 (1959), 637–642.

[3] (1) B. J. Birch: Waring problem in algebraic number fields. Proc. Cambridge Philos. Soc. 57 (1961), 449–459.
(2) O. Körner: Über Mittelwerte trigonometrischer Summen und ihre Anwendung in algebraischen Zahlkörpern, Math. Ann. 147 (1962), 205–239.
(3) Y. Eda: On the mean-value theorem in an algebraic number field, Jap. J. Math. 36 (1967), 5–21. We must correct (3) in the Main Theorem $s \geqq \frac{n}{2(n-1)} k(k+1)+\tau k-1$.

[4] (1) We hope that Hua's method [2]-[1]-Chap. X is treated in an algebraic number field. [2]-(3).

[5] we can replce a constant c by c_s for $1 \leqq s \leqq r_1$ *and* c_t for $r_1+1 \leqq t \leqq r_1+r_2$.