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On the Number of Prime Factors of Integers

to Professor IKUZO YAMAMOTO on his 70th Birthday

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1. Introduction. Throughout the paper, the letters p, p_1, p_2, ... will be reserved for prime numbers. Let $\omega(n)$ be the number of distinct prime factors of a positive integer n, and let x be a positive real number. Let $f_i(\xi)$ $(1\leqq i\leqq k)$ be polynomials in ξ, satisfying the following conditions :

(c_1) Each $f_i(\xi)$, $(1\leq i\leq k)$ has integral coefficients ;
(c_2) Each $f_i(\xi)$, $(1\leq i\leq k)$ is of positive degree ;
(c_3) Each $f_i(\xi)$, $(1\leq i\leq k)$ is positive for $\xi\geq 1$;
(c_4) $f_1(\xi)$, ..., $f_k(\xi)$ are relatively prime in pairs.

Let r_i $(1\leq i\leq k)$ be the number of the primitive and irreducible factors of $f_i(\xi)$. Let c_1, c_2,..., be positive absolute constants. We put $g(x)=(\mathrm{lnln}x)^{1/4k}(\mathrm{lnlnln}x)^{1/2k}$. Let A{…} denote the number of positive integers with some conditions...... We put, for integers $n\geq 3$ and for $1\leq i\leq k$,

$$\frac{\omega\{f_i(n)\}-r_i\ \mathrm{lnln}\ n}{\sqrt{r_i\ \mathrm{lnln}\ n}}=u_i(n)$$

To each integer $n\geq 3$, there corresponds a point $(u_1(n),...,u_k(n))$ in a k-dimensional space R^k. Let E be a Jordan-measurable set, bounded or unbounded, in R. Let A(x;E) denote the number of integers n $(3\leq n\leq x)$, for which the points $(u_1(n),...,u_k(n))$ belong to the set E. Tanaka obtained the following Theorem A :

Theorem A. $$\lim_{x\to\infty}\frac{A(x;E)}{x}=(2\pi)^{-k/2}\int_E \exp\Big(-\frac{1}{2}\sum_{i=1}^{k}u_i^2\Big)du_1...du_k.$$
The integral is the sence of Riemann.[3].

Similarly, by using the sieve method of A. Selberg [2], [1] and Tanaka's method [3], we shall prove the following main theorem :

Main Theorem. *Let α_i, β_i $(1\leq i\leq k)$ be any real numbers with $\alpha_i<\beta_i$ $(1\leq i\leq k)$. We put*

$$A(x)=A\Big\{3\leq n\leq x\ ;\ r_i\ \mathrm{lnln}\ n+\alpha_i\sqrt{r_i\ \mathrm{lnln}\ n}<\omega\{f_i(n)\}<r_i\ \mathrm{lnln}\ n+\beta_i\sqrt{r_i\ \mathrm{lnln}\ n}\Big\},\quad (1\leq i\leq k).$$

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Then we have

$$\frac{A(x)}{x}=(2\pi)^{-k/2}\prod_{i=1}^{k}\int_{\alpha_i}^{\beta_i}e^{-\frac{1}{2}u_i^2}du_i+O\left(\frac{\mu^{2(k+1)}(\ln\ln\ln x)^{1-1/2k}}{(\ln\ln x)^{1/4k}}\right),$$

where $$\mu=\max_{1\leq i\leq k}(1,\ |\alpha_i|,\ |\beta_i|)\ .$$

The O-term is uniform with respect to sufficiently large x.

2. Selberg's sieve method.

Lemma 1. *Let n be a positive integer.* Let $z\geq 2$, $\ln z\geq c_1\ln n$, *where* c_1 *is a sufficiently small constant. Let Q be an arbitrary set of primes, none of which exceeds z. Let* D *be the set of all positive square-free integers which are divisible only by primes of* Q ; *assume that* $1\in$D. *Further, let* $a(m)$ $(1\leq m\leq n)$ *be integers, such that the number of all* $a(m)$ $(1\leq m\leq n)$ *which are divisible by an integer d of* D *is equal to* $n\vartheta(d)+R(d)$, *where* $\vartheta(d)$ *is a multiplicative function, defined on* D,

$$0\leq\vartheta(d)<1\ \ \textit{for}\ d>1\ ,\ |R(d)|\leq c_2 d\vartheta(d),$$

$$\vartheta(p)\leq\frac{c_3}{c_3+p}\ \ \textit{for}\ p\in Q.$$

Then the number of $a(m)$ $(1\leq m\leq n)$ *which are not divisible by any prime of* Q *is*

$$n\prod_{p\in Q}(1-\vartheta(p))\left\{1+O\left(e^{-c_4\frac{\ln n}{\ln z}}\right)\right\}.$$

Proof. Kubilius [1], lemma 1–4.

We shall denote by l_i $(1\leq i\leq k)$ the degree of the polynomial $f_i(\xi)$, and for any prime p, we denote by $\nu_i(p)$ the number of incongruent solutions of the congruence $f_i(\xi)\equiv 0$ (mod p). We put

$$l\ =\ l_1+\cdots\cdots+l_k.$$
$$\nu(p)\ =\ \nu_1(p)+\cdots\cdots+\nu_k(p).$$

It is plain that $\nu_i(p)\leq l_i$ for $1\leq i\leq k$ and $\nu(p)\leq l$.

In virtue of the condition (c_4) we can take a positive number r_1 such that, for any prime $p>r_1$, no two of the congruences $f_i(\xi)\equiv 0\pmod{p}$ $(1\leq i\leq k)$ have common solution and therefore the congruence

$$f_1(\xi)\ldots f_k(\xi)\equiv 0\pmod{p}\qquad (p>r_1)$$

has $\nu(p)$ incongruent solutions (see lemma 2-1 of [3]).

Lemma 2. *Let a be an integer, and let* $d<x$ *be a positive integer. Let* $z\geqq 2$, $z\leq c_5\ln x$, *where* c_5 *is a sufficiently small constant. Let* p_j $(1\leq j\leq h)$ *be prime numbers such that* $p_j\nmid d$ $(1\leq j\leq h)$ *and* $\max(r_1,\ l)<p_j\leq z$ $(1\leq j\leq h)$. *We put*

$$F(x;\ a,\ d\ ;\ p_1,\ p_2,\ \ldots,\ p_h)$$
$$=A\{n\leq x\ ;\ n\equiv a \bmod d\ ;\ f_1(n)\ \ldots\ f_k(n)\not\equiv 0\pmod{p_j}\ (1\leq j\leq h)\}.$$

Then, we have uniformly with respect to sufficiently large x,

$$F(x\ ;\ a,\ d;\ p_1,\dots,\ p_h)=\frac{x}{d}\prod_{j=1}^{h}(1-\frac{\nu(p_j)}{p_j})\left\{1+O(e^{-c_6\frac{\ln x}{\ln z}})\right\}.$$

Proof. Clearly, we can assume that $-d<a\leq 0$. For any numbers $n\leq x$ which satisfy $n\equiv a$ mod d, we put $n=a+td$, $(1\leq t\leq n_1)$, $n_1=\left[\frac{x-a}{d}\right]$. Thus $F(x;a,\ d;p_1,\dots,\ p_h)$ is equal to the number of t $(1\leq t\leqq n_1)$ satisfying the conditions $f_1(a+td)\dots f_k(a+td)\not\equiv 0$ mnd $p_j(1\leq j\leq h)$. Let H be the set of all positive square-free integers which are divisible only by primes p_j $(1\leq j\leq h)$. We denote by $\nu(g)$ the number of incongruent solutions of the congruence $f_1(\xi)\cdots\cdots f_k(\xi)\equiv 0$ mod g, where g is an element of the set H. Then $\nu(g)$ is a multiplicative function defined on H. Since $(d,g)=1$, the number of incongruent solutions of the congruence $f_1(a+\xi d)\dots f_k(a+\xi d)\equiv 0$ mod g is equal to $\nu(g)$. If we denote by S_0 the number of t $(1\leq t\leq n_1)$ satisfying the condition $f_1(a+td)\dots f_k(a+td)\equiv 0$ mod g and put $\vartheta(g)=\frac{\nu(g)}{g}$, then we have

$$S_0=n_1\vartheta(g)+R(g)\quad \text{for } g>1,\ |R(g)|<g\vartheta(g),$$

$$\vartheta(p_j)=\frac{\nu(p_j)}{p_j}\leq\frac{l}{p_j}\leq\frac{c_7}{p_j+c_7}\qquad (1\leq j\leq h).$$

By lemma 1 we have

$$F(x\ ;\ a,\ d\ ;\ p_1,\dots,p_h)=n_1\prod_{j=1}^{h}(1-\vartheta(p_j))\left\{1+O(e^{-c_8\frac{\ln n_1}{\ln z}})\right\}$$

$$=\frac{x}{d}(1+O(\frac{1}{x}))\prod_{j=1}^{h}(1-\vartheta(p_j))\left\{1+O(e^{-c_9\frac{\ln x}{\ln z}})\right\}$$

$$=\frac{x}{d}\prod_{j=1}^{h}(1-\frac{\nu(p_j)}{p_j})\left\{1+O(e^{-c_6\frac{\ln x}{\ln z}})\right\}.$$

3. The proof of the main theorem. We denote by $\pi=\pi(x)$ the set of all prime numbers p which lie in the interval

$$e^{(\ln\ln x)^2}<p<x^{1/(8r\ln\ln x)}.$$

Let $\omega'(n)$ be the number of distinct prime factors p $(p\in\pi)$ of n. The following three lemmas are obtained by Tanaka [3].

Lemma 3.

$$A\ \{n\leq x\ ;\ \exists i,\ \omega\{f_i(n)\}-\omega\{f_i(n)\}>g(x)\}=O\left(\frac{x\ \ln\ln x}{g(x)}\right),$$

where $g(x)=(\ln\ln x)^{1/4k}(\ln\ln\ln x)^{1/2k}$.

Proof. Tanaka [1] lemma 3-1.

Lemma 4. *For x so large that any prime which belongs to to the set π is greater than l, we put*

$$y_i = y_i(x) = \prod_{p \in \pi} \frac{(p-1)\nu_i(p)}{p(p-\nu(p))} \qquad (1 \leq i \leq k),$$

then we have

$$y_i = r_i \ \ln\ln x + O(\ln\ln\ln x) \ (1 \leqslant i \leqslant k).$$

Proof. Tanaka [3], lemma 3-2.

We shall denote by $\mathfrak{M}(t)$, where t is a positive integer, the set of positive numbers n subject to the following conditions : (i) n is composed only of primes which belong to the set π ; (ii) n is square-free ; (iii) n has t prime factors.

Lemma 5. *Let $t_1,\dots, t_k$ be positive integers such that $t_i < 2r_i$ lnln x $(1 \leq i \leq k)$, then we have*

$$\sum_{m_i \in \mathfrak{M}(t_i)}{}' \frac{\nu_1(m_1)\cdots\nu_k(m_k)}{m_1'\cdots m_k'} = \frac{y_1^{t_1}\cdots y_1^{t_k}}{t_1!\cdots t_k!} + O\left(\frac{1}{\ln x}\right),$$

where the summation on the left-hand side is extended over the systems of positive integers $(m_1,\dots,m_k)$ subject to the conditions that $m_i \in \mathfrak{M}(t_i)$ $(1 \leq i \leq k)$, and furthermore that $m_1,\dots,m_k$ are relatively prime in pairs, the latter condition being signified by the dash attached to $\sum$. The meaning of $\nu_1(m_1),\dots, \nu_k(m_k)$ and *$m_1',\dots, m_k'$ are as follow :*

$$\nu_i(m_i) = \prod_{p \mid m_i} \nu_i(p) \qquad (1 \leq i \leq k),$$

$$m_i' = \prod_{p \mid m_i} \frac{p(p-\nu(p))}{p-1} \qquad (1 \leq i \leq k).$$

The O-term is uniform with respect to the numbers $t_1,\dots,t_k$ such that $t_i < 2r_i$ lnln *x $(1 \leq i \leq k)$.*

Proof. Tanaka [3], lemma 3 - 3 .

Now we put

$$G(x\ ;\ t_1,\dots,t_k) = A\{n \leq x\ ;\ \omega'\{f_i(n)\} = t_i \text{ for } 1 \leq i \leq k\},$$

$$G(x\ ;\ t_1,\dots,t_k) = A\{n \leq x : \omega'\{f_i(n)\} = t_i,\ p^2 \nmid f_i(n) \text{ for } 1 \leq i \leq k,\ p \in \pi\},$$

$$H(x\ ;\ m_1,\dots,m_k) = A\{n \leq x\ ;\ m_i \mid f_i(n),\ p \nmid f_i(n)/m_i \text{ for } 1 \leq i \leq k,\ p \in \pi\}.$$

Then we have

$$(1) \qquad G(x\ ;\ t_1,\dots,t_k) = G'(x\ ;\ t_1,\dots,t_k) + O\left(\frac{x}{\ln x}\right)$$

and

$$(2) \quad G'(x\ ;\ t_1;\dots,t_k) = \sum_{m_i \in \mathfrak{M}(t_i)}{}' H(x\ ;\ m_1,\dots,m_k).$$

We denote by $p_1,\dots,p_h$ the primes which belong to the set π and do not divide $m_1,\dots,m_k$.

Lemma 6.

$$H(x\ ;\ m_1,\dots,m_k)\ =\ x\frac{\varphi(m_1)\dots\varphi(m_k)\nu_1(m_1)\dots\nu_k(m_k)}{m_1^2\dots m_k^2}\prod_{j=1}^{h}(1-\frac{\nu(p_j)}{p_j})\left\{1+O\left(\frac{1}{(\ln x)^{C_{10}}}\right)\right\},$$

where $\varphi(m)$ *ts Euler's function.*

The proof is similar to that of [T]. We have

$$(3)\qquad H(x;\ m_1,\dots,m_k)=\sum_{\sigma=1}^{s}F(x\ ;\ a_\sigma\ ;\ m_1^2\dots m_k^2\ ;\ p_1,\dots,p_k),$$

where $s=\varphi(m_1)\dots\varphi(m_k)\ \nu_1(m_1)\dots\nu_k(m_k)$. If we take sufficiently large x, and we put $z=x^{1/(8r\ln\ln x)}$, then we have

$$\max\ (r_1,l)<e^{(\ln\ln x)^2}<p_j\ (1\leq j\leq h),$$
$$m_1^2\dots m_k^2<(x^{1/8r\ln\ln x})^{4r\ln\ln x}=\sqrt{x},$$
$$\ln\ln z\leq c_5\ln x.$$

From lemma 2, we have

$$F(x\ ;\ a_\sigma,\ m_1^2\dots m_k^2\ ;\ p_1,\dots,p_k)$$
$$=\frac{x}{m_1^2\dots m_k^2}\prod_{j=1}^{h}(1-\frac{\nu(p_j)}{p_j})\left\{1+O(\frac{1}{(\ln x)c_{10}})\right\}.$$

From this and (3) we obtain.

Lemma 7. *Let* $t_1,\dots,t_k$ *be positive integers such that* $t_i<2r_i\ln\ln x\ (1\leq i\leq k)$, *then we have*

$$G(x\ ;\ t_1,\dots,t_k)=x\frac{e^{-(y_1+\dots+y_k)}y_1^{t_1}\dots y_k^{t_k}}{t_1!\dots t_k!}+O(\frac{x}{(\ln x)^{C_{11}}}).$$

Proof. By (1), (2), lemma 5 and lemma 6 we have

$$G(x\ ;\ t_1,\dots,t_k)\ =\ x\prod_{p\in\pi}(1-\frac{\nu(p)}{p})\left\{\frac{y_1^{t_1}\dots y_k^{t_k}}{t_1!\dots t_k!}+O(\frac{1}{\ln x})\right\}\left\{1+O(\frac{1}{(\ln x)^{C_{10}}})\right\}.$$

We derive from simple calculations

$$\prod_{p\in\pi}(1-\frac{\nu(p)}{p})=e^{-(y_1+\dots+y_k)}\left\{1+O(\frac{1}{\ln x})\right\}.$$

From this we obtain the lemma.

For given x, the system of real numbers $(\alpha_1,\ \beta_1,\dots,\alpha_k,\ \beta_k)$ is said to be admissible if there exists at least one positive integer n satisfying

$$y_i(x)+\alpha_i\sqrt{y_i(x)}<n<y_i(x)+\beta_i\sqrt{y_i(x)}\quad(1\leq i\leq k).$$

Lemma 8. *We put* $\mu=\max_{1\leq i\leq k}(1,|\alpha_i|,|\beta_i|)$. *Let* x *be a sufficiently large real number. Let* $(\alpha_1,\ \beta_1,\dots,\ \alpha_k,\ \beta_k)$ *be an admissible system with* $\mu^6<y_i(x)\ (1\leq i\leq k)$. *Let* $t_i=y_i(x)+u_i\sqrt{y_i(x)}\ (1\leq i\leq k)$ *be arbitrary positive integers with* $\alpha_i<u_i<\beta_i$ $(1\leq i\leq k)$. *Then we have*

$$G(x\ ;\ t_1,\dots,t_k)=(2\pi)^{-\frac{k}{2}}(y_1\dots y_k)^{-\frac{1}{2}}xe^{-\frac{1}{2}(u_1^2+\dots+u_k^2)}+O\left(\frac{\mu^3\ x}{(\ln\ln\ x)^{k/2}(\ln\ln\ x)^{1/2}}\right).$$

The O-term is uniform with respect to $t_1,\dots,t_k$, *that is,* $u_1,\dots,u_k$.

Proof. Tanaka [3], Lemma 3.7.

When t is a positive integer, we have Stirling's formula

$$t\,! = 2\pi\ t^{t+\frac{1}{2}}e^{-t}\left(1+O\left(\frac{1}{t}\right)\right).$$

Here we put $t=y+u\sqrt{y}$, we obtain

$$t\,! = 2\pi\ y^t\ e^{-y+\frac{1}{2}u^3}\left(1+O\left(\frac{u^3}{y}\right)\right) \qquad (|u^3|<\sqrt{y}).$$

From this and lemma 7, $G(x\ ;\ t_1,\dots,t_k)$ may also be written

$$G(x;t_1,\dots,t_k)=(2\pi)^{-\frac{k}{2}}(y_1\dots y_k)^{-\frac{1}{2}}xe^{-\frac{1}{2}(u_1^2+\dots+u_k^2)}\prod_{i=1}^{k}\left(1+O\left(\frac{u_i^3}{\sqrt{y_i}}\right)\right)$$

$$+O\left(\frac{1}{(\ln\ x)^{C_{12}}}\right)$$

$$=(2\pi)^{-\frac{k}{2}}(y_1\dots y_k)^{-\frac{1}{2}}x\ e^{-\frac{1}{2}(u_1^2+\dots+u_k^2)}+O\left(\frac{\mu^3\ x}{(\ln\ln\ x)^{k/2}(\ln\ln\ x)^{1/2}}\right)$$

Lemma 9. *We put*

$$B^*(x) = A\{n\le x; y_i+\alpha_i\sqrt{y_i}<\omega'\{f_i(n)\}<y_i+\beta_i\sqrt{y_i}\ \textit{for}\ 1\le i\le k\},$$

then we have

$$B^*(x)=(2\pi)^{-\frac{k}{2}}\,x\prod_{i=1}^{k}\int_{\alpha_i}^{\beta_i}e^{-\frac{1}{2}u_i^2}du_i+O\left(\frac{\mu^{3+k}\ x}{\sqrt{\ln\ln\ x}}\right).$$

Proof. First we consider the case when the system $(\alpha_1,\ \beta_1,\dots,\alpha_k,\ \beta_k)$ is an admissible with $\mu^6<y_i$ $(1\le i\le k)$. We denote by s_i $(1\le i\le k)$ the number of distinct positive integers t_i satisfying the following inequality $y_i+\alpha_i\sqrt{y_i}<t_i<y_i+\beta_i\sqrt{y_i}$. Then we have

$$s_i = O(\mu\sqrt{\ln\ln\ x}) \qquad (1\le i\le k).$$

We denote by t_{ij} $(1\le j\le s_i)$ the integral values of t_i taken in the above interval. Further we put $t_{ij}=y_i+u_{ij}\sqrt{y_i}$ and $u_{i,j+1}-u_{i,j}=(y_i)^{-1/2}>0$ $(1\le j\le s_i)$. By lemma 8, we have

$$B^*(x)=\sum_{y_i+\alpha_i\sqrt{y_i}<t_i<y_i+\beta_i\sqrt{y_i}}G(x;t_1,\dots,t_k)$$

$$=\sum_{l_1=1}^{s_i}\dots\sum_{l_k=1}^{s_k}G(x\ ;\ t_{1,l_1},\dots,t_{k,l_k})$$

$$=\sum_{l_1=1}^{s_i}\cdots\sum_{l_k=1}^{s_k}\left\{(2\pi)^{-\frac{k}{2}}(y_1\dots y_k)^{-\frac{1}{2}}x\,e^{-\frac{1}{2}(u^2_{1,l_1}+\dots+u^2_{k,l_k})}\right.$$

$$\left.+O\left(\frac{\mu^3\ x}{(\ln\ln\ x)^{k/2}(\ln\ln\ x)^{1/2}}\right)\right\}$$

$$=(2\pi)^{-\frac{k}{2}}x\prod_{i=1}^{k}\sum_{l_i=1}^{s_i}e^{-\frac{1}{2}u^2_{i,l_i}}(u_{i,l_{i+1}}-u_{i,l_i})+O\left(\frac{\mu^{3+k}\ x}{\sqrt{\ln\ln\ x}}\right).$$

We may write

$$\sum_{l_i=1}^{s_i}e^{-\frac{1}{2}u^2_{i,l_i}}(u_{i,l_i+1}-\mathrm{u}_{i,l_i})=\int_{\alpha i}^{\beta i}e^{-\frac{1}{2}u_i^2}\,du_i+O\left(\frac{1}{\sqrt{y_i}}\right).$$

and therefore we have

$$(4)\qquad B^*(x)=(2\pi)^{-\frac{k}{2}}x\prod_{i=1}^{k}\int_{\alpha i}^{\beta i}e^{-\frac{1}{2}u_i^2}du_i+O\left(\frac{\mu^{3+k}\ x}{\sqrt{\ln\ln\ x}}\right).$$

Secondly we consider the case when $(\alpha_1,\ \beta_1,\dots,\alpha_k,\ \beta_k)$ is not admissible. $B^*(x)$ is equal to o and there exists i such that $0<\beta_i-\alpha_i\leq(y_i)^{-1/2}$. Hence

$$x(2\pi)^{-\frac{k}{2}}\prod_{i=1}^{k}\int_{\alpha i}^{\beta i}e^{-\frac{1}{2}u_i^2}du_i=O\left(x\prod_{i=1}^{k}(\beta_i-\alpha_i)\right)$$

$$=O\left(\frac{\mu^{k-1}\ x}{\sqrt{\ln\ln\ x}}\right).$$

and the result follows.

The proof of the main terorem. We put

$B_1(x\,;\alpha_i,\ \beta_i)=A\{\sqrt{x}<n\leq x\,;y_i+\alpha_i\sqrt{y_i}<\omega'\{f_i(n)\}<y_i+\beta_i\sqrt{y_i}$ for $(1\leq i\leq k)\}$,

$B_2(x\,;\alpha_i,\ \beta_i)=A\{\sqrt{x}<n\leq x\,;y_i+\alpha_i\sqrt{y_i}<\omega'\{f_i(n)\}<y_i+\beta_i\sqrt{y_i}$

$\omega\{f_i(n)\}-\omega'\{f_i(n)\}<g(x)$ for $1\leq i\leq k\}$,

$A_1(x\,;\alpha_i,\ \beta_i)=A\{\sqrt{x}<n\leq x\,;r_i\ \ln\ln\ n+\alpha_i\sqrt{r_i\mathrm{l\ nln}\ n}<\omega\{f_i(n)\}$

$<r_i\ \ln\ln\ n+\beta_i\sqrt{r_i\ \ln\ln\ n}$ for $1\leq i\leq k\}$,

$A_2(x\,;\alpha_i,\ \beta_i)=A\{\sqrt{x}<n\leq x\,;r_i\ \ln\ln\ n+\alpha_i\sqrt{r_i\ \ln\ln\ n}<\omega\{f_i(n)\}$

$<r_i\ \ln\ln\ n+\beta_i\sqrt{r_i\ \ln\ln\ n}$,

$\omega\{f_i(n)\}-\omega'\{f_i(n)\}<g(x)$ for $1\leq i\leq k\}$,

then we have

$$(5)\qquad A(x)=A_1(x\ ;\ \alpha_i,\ \beta_i)+O(\sqrt{x})$$

and

$$(6)\qquad B^*(x)=B_1(x\ ;\ \alpha_i,\ \beta_i)+O(\sqrt{x}).$$

By lemma 3 we have

$$(7) \qquad A_1(x\,;\,\alpha_i,\,\beta_i) = A_2(x\,;\,\alpha_i,\,\beta_i)+O\left(\frac{x\ \mathrm{lnlnln}\ x}{g(x)}\right),$$

and

$$(8) \qquad B_1(x\,;\,\alpha_i,\,\beta_i) = B_2(x\,;\,\alpha_i,\,\beta_i)+O\left(\frac{x\ \mathrm{lnlnln}\ x}{g(x)}\right).$$

For the positive number n such that

$$\sqrt{x}<n\leq x,\ \omega\{f_i(n)\}-\omega'\{f_i(n)\}<g(x),$$

the inequality

$$r_i\ \mathrm{lnln}\ n+\alpha_i\sqrt{r_i\ \mathrm{lnln}\ n}<\omega\{f_i(n)\}<r_i\ \mathrm{lnln}\ n+\beta_i\sqrt{r_i\ \mathrm{lnln}\ n}$$

may be written

$$y_i+(\alpha_i+O(w_i))\sqrt{y_i}<\omega'\{f_i(n)\}<y_i+(\beta_i+O(w_i))\sqrt{y_i},$$

where

$$w_i=\frac{\mu\ \mathrm{lnlnln}\ x+g(x)}{\sqrt{y_i}}\ .$$

From these we have

$$(9) \qquad A_2(x\,;\,\alpha_i,\,\beta_i) = B_2(x\,;\,\alpha_i+O(w_i),\ \beta_i+O(w_i)).$$

By (5), (6), (7), (8) and (9) we have

$$\begin{aligned}A(x) &= A_2(x\,;\,\alpha_i,\,\beta_i)+O\left(\frac{x\ \mathrm{lnlnln}\ x}{g(x)}\right)+O(\sqrt{x})\\
&= B_2(x\,;\,\alpha_i+O(w_i),\ \beta_i+O(w_i))+O\left(\frac{x\ \mathrm{lnlnln}\ x}{g(x)}\right)+O(\sqrt{x})\\
&= B^*(x\,;\,\alpha_i+O(w_i),\ \beta_i+O(w_i))+O\left(\frac{x\ \mathrm{lnlnln}\ x}{g(x)}\right)+O(\sqrt{x})\\
&= (2\pi)^{-\frac{k}{2}}x\prod_{i=1}^{k}\int_{\alpha_i}^{\beta_i}e^{-\frac{1}{2}u_i^2}du_i+O\left(\frac{x\mu^k(\mu\ \mathrm{lnlnln}\ x+g(x))^k+\mu^{3+k}x}{\sqrt{\mathrm{lnln}\ x}}\right)\\
&\qquad\qquad +O\left(\frac{x\ \mathrm{lnlnln}\ x}{g(x)}\right)\\
&= (2\pi)^{-\frac{k}{2}}x\prod_{i=1}^{k}\int_{\alpha_i}^{\beta_i}e^{-\frac{1}{2}u_i^2}du_i+O\left(\frac{x\mu^{2(k+1)}(\mathrm{lnlnln}\ x)^{1-1/2k}}{(\mathrm{lnln}\ x)^{1/4k}}\right).\end{aligned}$$

This proves the theorem.

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