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On the Minimum of Some Subharmonic Function

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1. Let $u(z)$ be a subharmonic function on the compact domain in z-plane. Then it is well-known that $u(z)$ attains its maximum on the boundary of D. But the problem where is the minimum occurs in certain circumstances.

Now let U denote a closed disc and $P_i \notin U$, $(i=1, \ldots, n)$ and $P \in U$ be n fixed points and a moving point, respectively. We consider the following function

$$(1) \qquad f_n(P) = \sum_{i=1}^{n} \frac{1}{\overline{PP_i}^k} = \sum_{i=1}^{n} \frac{1}{|z-z_i|^k}, \qquad (k>0)$$

where $\overline{PP_i}$ denotes the distance between P and P_i and z and z_i represent complex numbers corresponding to P and P_i. Since each term of (1) is a absolute value of the regular function, it is obvious that $f_n(P)$ is subharmonic in U. The former author gave some results about the minimum of (1) in the case that $k=2$ and n points are located in the special situations ([1]). Here we treat such similar problem in the case of $k=3$.

2. Let $D_{0,0}$ be a closed unit disc bounded by the unit circle $C_{0,0}$. Next we describe the six circles C_{1,j_1} $(j_1=1, \ldots, 6)$ with equal radii 1 so that C_{1,j_1} $(j_1=1, \ldots, 6)$ are tangent externally with each other around $C_{0,0}$ and hence the segments, which join the centers of C_{1,j_1} successively, constitute a regular hexagon R_1, where the center of $C_{1,1}$ has the coordinates (2,0) with respect to rectangular coordinate system. Further we describe the twelve circles C_{2,j_2} $(j_2=1,2, \ldots, 12)$ with equal radii 1 so that C_{2,j_2} $(j_2=1,2, \ldots, 12)$ are tangent externally with each other around C_{1,j_1} and hence the segments, which join the centers of C_{2,j_2} successively, constitute a regular hexagon R_2, where the center of $C_{2,1}$ has the coordinates (4,0). We continue such procedure by turns. Generally, we describe the $6n$ circles C_{n,j_n} $(j_n=1,2, \ldots, 6n)$ with equal radii 1 so that C_{n,j_n} $(j_n=1,2, \ldots, 6n)$ are tangent externally with each other around $C_{n-1,j_{n-1}}$ and hence the segments, which join the centers of C_{n,j_n} successively, constitute a regular hexagon R_n, where the center of $C_{n,1}$ has the coordinates $(2n, 0)$. We call the first index i and the second j_i of C_{i,j_i} i-th rank and j_i-th emission, respectively. It

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is obvious that the total number of circles $C_{0,0}$ and C_{i,j_i} $(i=1,2, \ldots, n; j_i=1,2, \ldots, 6i)$ is equal to $3n(n+1)+1$.

Let us denote the center of C_{i,j_i} by z_{i,j_i}. Let P be a moving point in $D_{0,0}$, which has the coordinates (x, y), $z=x+iy$. We consider the following subharmonic function

$$(2) \qquad f_{n,j_n}(z) = \sum_{i=1}^{n} \sum_{j_i=1}^{6i} \frac{1}{|z-z_{i,j_i}|^3}, \qquad \forall z \in D_{0,0}.$$

The purpose of this paper is to determine the point at which $f_{,,j_n}$ attains its minimum in $D_{0,0}$. This problem occured in the investigation whether there exist or not Kleinian groups whose singular sets have positive $\left(\frac{3}{2}\right)$-dimensional measure ([2]). It seems very easy to solve this problem, but the calculation is very difficult and complicated.

3. Now we shall give the main theorem.

Theorem A. *$f_{n,j_n}(z)$ attains its minimum at the origin.*

For the proof we prepare the following theorem as lemma.

Theorem B. *Let* $P_1=P(2, \pi)$, $P_2=P(2, \frac{\pi}{3})$ *and* $P_3=P(2, -\frac{\pi}{3})$ *be fixed points in the complex z-plane, and* $P=P(r, \theta)$ *moves in the fixed closed unit disc* U: $|z| \leqq 1$, *where* (r, θ) *denote the polar coordinates. Then the function*

$$f_3(P) = \sum_{j=1}^{3} \frac{1}{\overline{PP_j}^3} = \sum_{j=1}^{3} \frac{1}{|z-z_j|^3}, \quad z_j = 2e^{\left(\frac{5-2j}{3}\right)\pi i}, \quad (j=1,2,3)$$

attains its minimum at the origin.

If Theorem B establishes, then Theorem A is easily proved from it. For the function $f_{n,j_n}(z)$ is decomposed into pieces consisting of three terms, each of which corresponds to centers of three circles so that such centers are vertices of a equilateral triangle with centroid at the origin. We can apply Theorem B to each piece, since we can consider that the distances from the origin to the fixed points and the radius of U in Theorem B are relative.

Therefore if each piece attains its minimum at the origin, then it is easily seen that the sum of pieces attains also its minimum at the origin.

4. Now let us prove Theorem B. For this purpose we must prepare three lemmas. We transfer P_1 to the origin O and consider the function

$$(3) \qquad f_1^*(P) = f_3(P) - \overline{PP_1}^{-3} = \overline{PP_2}^{-3} + \overline{PP_3}^{-3}.$$

From the symmetricity of the figure we may consider the behavior of $f_1^*(P)$ in the upper half plane. Denote by C the center: $(2, 0)$ of U and by H the point of contact at which the tangent OP_2 intersect with the boundary circle of U. Hereafter we use polar coordinates only. Describe a circle with the radius OH and the center at O and denote the intersecting point with OC by R. Denote by Δ_1 the closed domain bounded by segments HC, RC and the circular arc $\widehat{\mathrm{HR}}$. Extend the segment OC to the positive direction and denote the intersecting point of the extension with the boundary circle by Q. Further we denote the intersecting point of segment CP_2 with the boundary circle of U by D. Then we obtain a closed domain Δ_2 bounded by the sector CQD. Take a point P in $\Delta_1 \cup \Delta_2$. Fix r of the coordinates (r, θ) of P and rotate the radius vector OP around the origin O until OP intersects the segment CH or CP_2 and denote by $\theta_0(r)$ the rotating angle.

Then we have the following

Lemma 1. *Fix* r in $\sqrt{\frac{10}{3}} \leqq r < 2\sqrt{3}$. *Then the function* $f_1^*(\mathrm{P})$ *is a monutone increasing function of* θ *in* $0 \leqq \theta \leqq \theta_0(r)$.

Proof. It is easily seen that $f_1^*(\mathrm{P})$ is represented in the form:

$$(4) \qquad f_1^*(\mathrm{P}) = \left\{r^2+12-4\sqrt{3}\cos\left(\theta-\frac{\pi}{6}\right)\right\}^{-\frac{3}{2}} + \left\{r^2+12-4\sqrt{3}\cos\left(\theta+\frac{\pi}{6}\right)\right\}^{-\frac{3}{2}}$$
$$= (4\sqrt{3}\,r)^{-\frac{3}{2}}\left[\left\{\rho-\cos\left(\theta-\frac{\pi}{6}\right)\right\}^{-\frac{3}{2}} + \left\{\rho-\cos\left(\theta+\frac{\pi}{6}\right)\right\}^{-\frac{3}{2}}\right],$$

where $\rho = \dfrac{r^2+12}{4\sqrt{3}\,r}$, $(\geqq 1)$. Here we note that $1.3 > \rho > 1$ in the case of $\sqrt{10/3} \leqq r < 2\sqrt{3}$.

Now we put $f_1^*(\mathrm{P}) = (4\sqrt{3}\,r)^{-\frac{3}{2}} g_1(\theta)$ and differentiate $g_1(\theta)$ with respect to θ for fixed r. We have

$$\frac{\partial g_1(\theta)}{\partial \theta} = -\frac{3}{2}\left\{\rho-\cos\left(\theta-\frac{\pi}{6}\right)\right\}^{-\frac{5}{2}}\sin\left(\theta-\frac{\pi}{6}\right) - \frac{3}{2}\left\{\rho-\cos\left(\theta+\frac{\pi}{6}\right)\right\}^{-\frac{5}{2}}\sin\left(\theta+\frac{\pi}{6}\right).$$

If we put

$$(5) \qquad \varphi(\theta) = \sin\theta\,(\rho-\cos\theta)^{-\frac{5}{2}},$$

we have

$$-\frac{2}{3}\left(\frac{\partial g_1(\theta)}{\partial \theta}\right) = \varphi\left(\theta-\frac{\pi}{6}\right) + \varphi\left(\theta+\frac{\pi}{6}\right).$$

We see easily that the condition $f_1^{*\prime}(\theta) = (4\sqrt{3}\,r)^{-\frac{3}{2}} g_1'(\theta) \geqq 0$ is equivalent to the condition

$$(6) \qquad \varphi\left(\theta-\frac{\pi}{6}\right) + \varphi\left(\theta+\frac{\pi}{6}\right) \leqq 0.$$

Hence in order to prove Lemma 1, it is enough to show that (6) establishes in $0 \leqq \theta \leqq \theta_0(r)$.

Now let us prove this fact by using the graph of (5). For this purpose it is sufficient to investigate the shape of the graph of (5) in the intervals

$$-\frac{\pi}{6} \leqq \theta \leqq -\frac{\pi}{6}+\theta_0(r) \text{ and } \frac{\pi}{6} \leqq \theta \leqq \frac{\pi}{6}+\theta_0(r) .$$

But since the solutions of $\varphi(\theta)=0$ are $\theta=n\pi$ and $\varphi(\theta)$ is an odd function, finally it is sufficient to investigate $\varphi(\theta)$ in the interval

$$\frac{\pi}{6}-\theta_0(r) \leqq \theta \leqq \theta_0(r)+\frac{\pi}{6} . \tag{7}$$

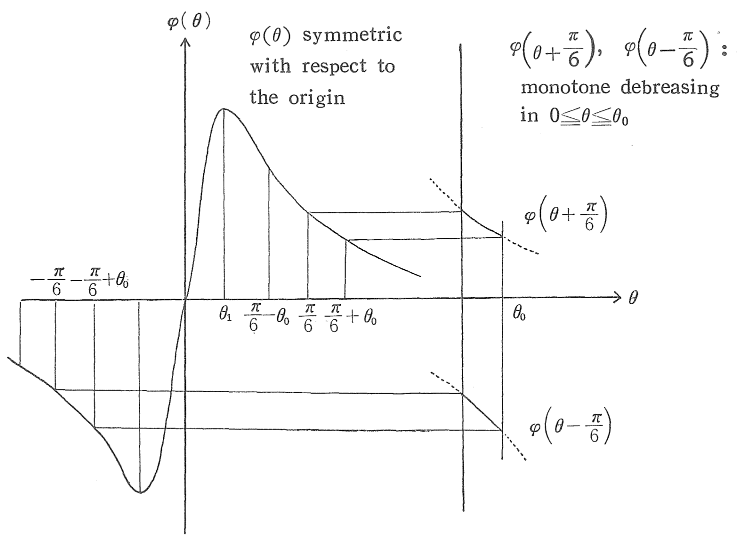


Fig. 1

5. At first we shall seek for the maximum of $\varphi(\theta)$ in $0 \leqq \theta \leqq \pi$. Differentiating $\varphi(\theta)$, we obtain

$$\varphi'(\theta)=\frac{3}{2}(\rho-\cos\theta)^{-\frac{7}{2}}\left(\cos\theta+\frac{\sqrt{\rho^2+15}+\rho}{3}\right)\left(\cos\theta-\frac{\sqrt{\rho^2+15}-\rho}{3}\right) .$$

Since $\rho>1$ from (4), it holds $(\sqrt{\rho^2+15}+\rho)/3>5/3>1$ and $1>(\sqrt{\rho^2+15}-\rho)/3>0$. Hence $\varphi(\theta)$, $(0 \leqq \theta \leqq \pi)$ attains the maximum at

$$\theta_1=\cos^{-1}\left(\frac{\sqrt{\rho^2+15}-\rho}{3}\right) . \tag{8}$$

If it holds that

$$\theta_1 \leqq \frac{\pi}{6}-\theta_0(r) , \tag{9}$$

then $\varphi(\theta)$ is monotone decreasing in the interval (7) and hence (6) establishes, where $\varphi\left(-\frac{\pi}{6}\right)+\varphi\left(\frac{\pi}{6}\right)=0$. Then $f_1{}^{*\prime}(\theta) \geqq 0$ is proved and our claim is completed.

Now since $0<\theta_1$ and $\frac{\pi}{6}-\theta_0(r)<\frac{\pi}{2}$, (9) is equivalent to

$$(10) \qquad \cos\theta_1 \geqq \cos\left(\frac{\pi}{6}-\theta_0(r)\right).$$

We can find the value of $\cos\left(\frac{\pi}{6}-\theta_0(r)\right)$ in the following two cases (a) and (b) (*Figure* 2).

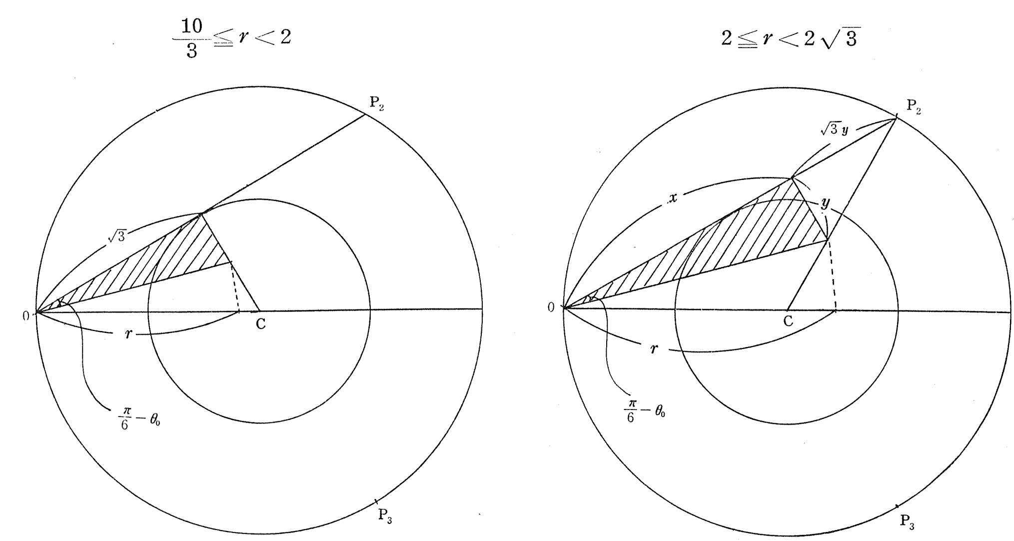
(a) The case of $\sqrt{10/3} \leqq r < 2$.

From the figure it is easily seen that

$$(11) \qquad \cos\left(\frac{\pi}{6}-\theta_0(r)\right)=\frac{\sqrt{3}}{r}.$$

(b) The case of $2 \leqq r < 2\sqrt{3}$.

From the figure we get the following simultaneous equations :



(a) (b)

Fig. 2

$$(12) \qquad \begin{cases} x+\sqrt{3}y=2\sqrt{3} \\ x^2+y^2=r^2 \end{cases}$$

Solving (12) and considering the relation $\cos\left(\frac{\pi}{6}-\theta_0\right)=\frac{x}{r}$, we obtain

$$(13) \qquad \cos\left(\frac{\pi}{6}-\theta_0(r)\right)=\frac{\sqrt{3}(1+\sqrt{r^2-3})}{2r}.$$

We shall prove from (8), (10), (11) and (13) the following inequality

$$(14) \qquad \frac{\sqrt{\rho^2+15}-\rho}{3} \geqq \begin{cases} \dfrac{\sqrt{3}}{r}, & \text{for } \sqrt{\dfrac{10}{3}} \leqq r < 2\,, \\[2ex] \dfrac{\sqrt{3}(1+\sqrt{r^2-3})}{2r}, & \text{for } \quad 2 \leqq r < 2\sqrt{3}\,, \end{cases}$$

where $\rho = \dfrac{r^2+12}{4\sqrt{3}\,r}$, $(\geqq 1)$.

(i) The case of $\sqrt{\dfrac{10}{3}} \leqq r < 2$.

It holds

$$\frac{\sqrt{\rho^2+15}-\rho}{3} - \frac{\sqrt{3}}{r} = \frac{1}{3r}\left\{ r\sqrt{\rho^2+15} - (r\rho + 3\sqrt{3})\right\}$$

$$= \frac{54\left(r^2-\dfrac{10}{3}\right)}{\sqrt{3}\,r\,\{\sqrt{(r^2+12)^2+48\times 15r^2}+(r^2+48)\}} \geqq 0\,.$$

(ii) The case of $2 \leqq r < 2\sqrt{3}$.

We obtain from (14)

$$\frac{\sqrt{\rho^2+15}-\rho}{3} - \frac{\sqrt{3}(1+\sqrt{r^2-3})}{2r}$$

$$(15) \qquad = \frac{1}{12\sqrt{3}\,r}\,\{\sqrt{(r^2+12)+48\times 15r^2} - (r^2+30+18\sqrt{r^2-3})\}$$

$$= \frac{\sqrt{3}\{(10r^2+6)-(r^2+30)\sqrt{r^2-3}\}}{r\,\{\sqrt{(r^2+12)^2+48\times 15r^2}+(r^2+30+\sqrt{r^2-3})\}}\,.$$

If $\sqrt{r^2-3}$ is replaced by t in (15), then the right hand side of (15) is modified into the following :

$$(16) \qquad \frac{-\sqrt{3}(t-3)^2(t-4)}{r\,\{\sqrt{(r^2+12)^2+48\times 15r^2}+(r^2+30+\sqrt{r^2-3})\}}\,.$$

Since $2 \leqq r < 2\sqrt{3}$ corresponds to $1 \leqq t < 3$ from $t=\sqrt{r^2-3}$, (16) and hence the left hand side of (15) is positive. Thus we can prove the inequalities in (14) from the above and the proof of Lemma 1 is complete. *q.e.d.*

6. Next we must prove the following

Lemma 2. *If* P *moves on the interval* $[0, 3]$, *the function*

$$(17) \qquad f_3(\mathrm{P}) = \overline{\mathrm{OP}}^{-3}+\overline{\mathrm{P}_2\mathrm{P}}^{-3}+\overline{\mathrm{P}_3\mathrm{P}}^{-3} = r^{-3}+2\{3+(3-r)^2\}^{-\frac{3}{2}}$$

does not attain its minimum on $[2.557, 3]$, (Fig. 3).

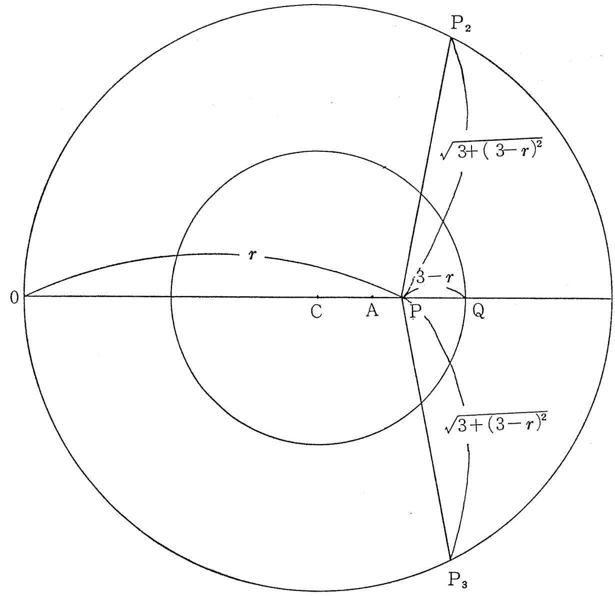


Fig. 3

Proof. If we differentiate $f_3(\mathrm{P})$ with respect to r, we have

$$(18)\qquad f_3'(r)=\frac{3\{2r^4(3-r)-(\sqrt{3+(3-r)^2})^5\}}{r^4(\sqrt{3+(3-r)^2})^5}\,.$$

Further we differentiate $f_3'(r)$ with respect to r, so we obtain

$$(19)\qquad \frac{1}{3}f_3''(r)=\frac{2[2\{3+(3-r)^2\}\{\sqrt{3+(3-r)^2}\}^5+r^5\{4(3-r)^2-3\}]}{r^5(\sqrt{3+(3-r)^2})^7}$$

If $f(r)$ attains its minimum at r, then it hold

$$f_3'(r)=0 \quad\text{and}\quad f_3''(r)\geqq 0\,.$$

Hence we have from (18) and (19)

$$(20)\qquad (\sqrt{3+(3-r)^2})^5=2r^4(3-r)$$

and

$$(21)\qquad 2\{3+(3-r)^2\}\{\sqrt{3+(3-r)^2}\}^5+r^5\{4(3-r)^2-3\}\geqq 0\,.$$

Substituting (20) into (21) and dividing by r^4, we obtain

$$4(3-r)^2+5(3-r)-3\geqq 0\,.$$

Therefore we have

$$r\leqq\frac{29-\sqrt{73}}{8}<\frac{20.456}{8}=2.557\,,$$

since $r \geqq \dfrac{29+\sqrt{73}}{8}$ is not satisfied by the assumption.

q.e.d.

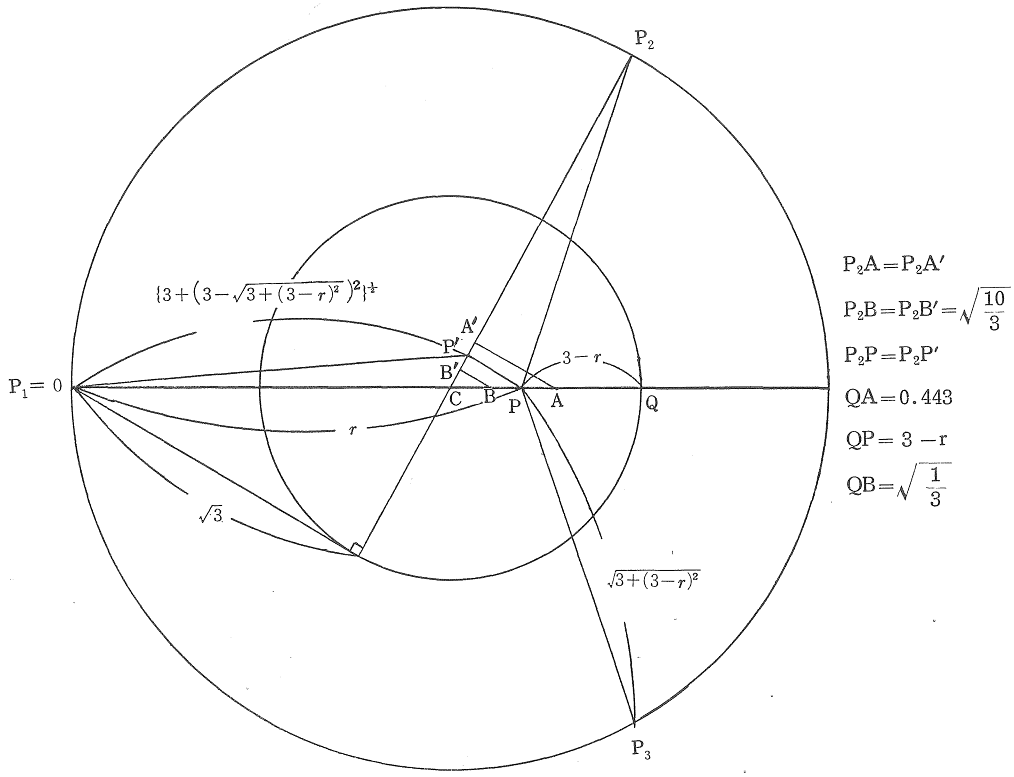


Fig. 4

Take any point P on the closed interval [O, Q] with distance r from the origin $P_1=O$. It is easily seen that $\overline{PP_2}=\overline{PP_3}=\sqrt{3+(3-r)^2}$, (Fig. 4). Describe a circle with radius PP_2 and the center at P_2 and denote by P' the intersecting point with CP_2. It is obvious that $\overline{P'P_1}=\overline{P'P_3}=\sqrt{3+\{3-\sqrt{3+(3-r)^2}\}^2}$. We consider the following quantity $D(r) = f_3(P)-f_3(P')=(\overline{PP_1}^{-3}+\overline{PP_2}^{-3}+\overline{PP_3}^{-3})-(\overline{P'P_1}^{-3}+\overline{P'P_2}^{-3}+\overline{P'P_3}^{-3})=(\overline{PP_1}^{-3}+\overline{PP_2}^{-3})-(\overline{P'P_1}^{-3}+\overline{P'P_3}^{-3})=[r^{-3}+\{3+(3-r)^2\}^{-\frac{3}{2}}]-2[3+\{3-\sqrt{3+(3-r)^2}\}^2]^{-\frac{3}{2}}$. Then we have the following

Lemma 3. *$D(r)$ is positive on* $I^*=[3-\dfrac{1}{\sqrt{3}}, 2.557]$ *and it holds* $f_3(P)>f_3(P')$ *on* I^*.

Proof. It holds for $3-\dfrac{1}{\sqrt{3}} \leqq r \leqq 2.557$

$$(22)\quad \begin{cases} r^2 \leqq 2.557^2 < \dfrac{19.62}{3} \\ 3+(3-r)^2 \leqq 3+\left(\dfrac{1}{3}\right)^2 = \dfrac{10}{3} \\ 3+\{3-\sqrt{3+(3-r)^2}\}^2 \geqq 3+\left(3-\sqrt{\dfrac{10}{3}}\right)^2 > \dfrac{46-32.87}{3} = \dfrac{13.13}{3}\,. \end{cases}$$

Therefoee we have from (22)

$$D(r)\times[3+\{3-\sqrt{3+(3-r)^2}\}^2]^{\frac{3}{2}}$$

$$=\left[\frac{3+\{3-\sqrt{3+(3-r)^2}\}^2}{r^2}\right]^{\frac{3}{2}}+\left[\frac{3+\{3-\sqrt{3+(3-r)^2}\}^2}{3+(3-r)^2}\right]^{\frac{3}{2}}-2$$

$$\geqq\left(\frac{13.13}{19.62}\right)^{\frac{3}{2}}+\left(\frac{13.13}{10}\right)^{\frac{3}{2}}-2>(0.64)^{\frac{3}{2}}+1.313\sqrt{1.313}-2$$

$$>0.8^3+1.313\times1.145-2=0.015385>0.$$

q.e.d.

7. Now let us prove Theorem B by using the above Lemmas.

Proof of Theorem B.

It is obvious that $f_3(P)$ is symmetric with respect to lines P_iC $(i=1,2,3)$, that is, the values $f_3(P_i^*)$, $(i=1,2,\ldots,6)$ in *Figure* 5 are the same.

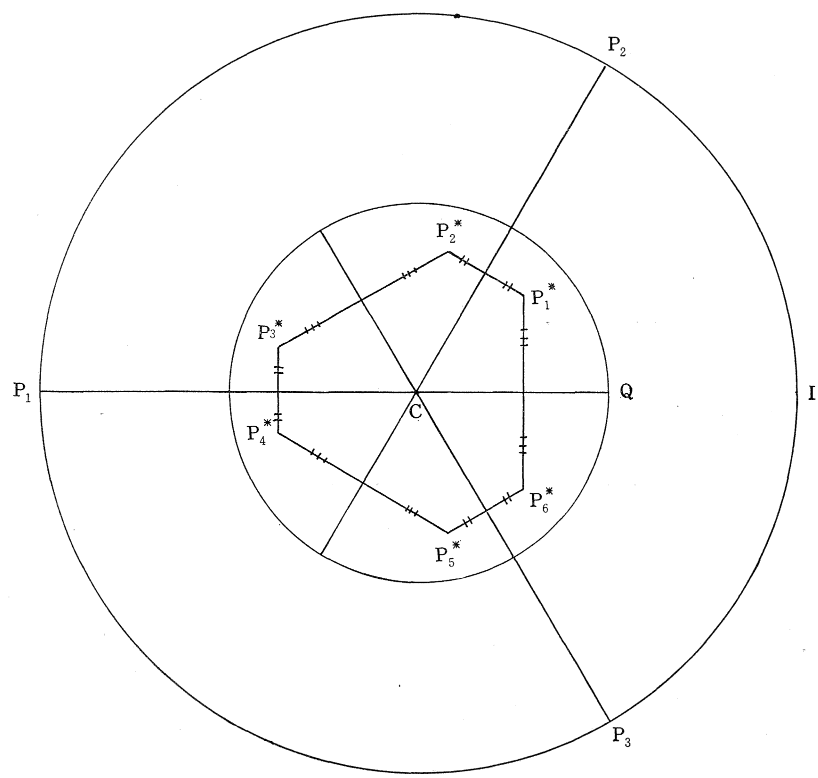


Fig. 5

Hence it is enough to investigate $f_3(P)$ in the angular domain ICP_2. Let P_1 be the origin O of polar coordinates and $P_1C=OC$ be the basic line. (See Fig. 6)

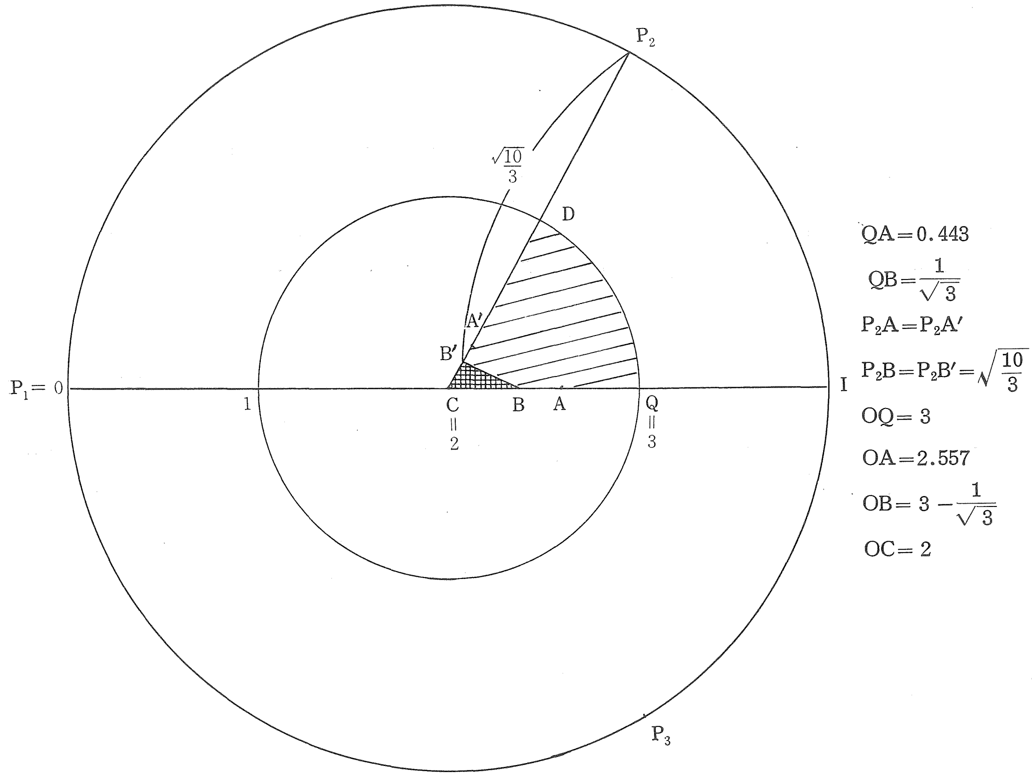


Fig. 6

Let the moving point $P=re^{i\theta}$ restrict in the closed interval [2 , 3]. Then the function

$$f_3(P) = f_1^*(P)+r^{-3}$$

is monotone increasing as the function of the variable θ in $0 \leqq \theta \leqq \theta_0(r)$ from Lemma 1. Hence there exists a point P_0 only on the segment CQ at which $f_3(P)$ attains its minimum in the angular domain QCD. P_0 is restricted on the closed interval [C, A] from Lemma 2, where $\overline{OA}=2.557$ and $\overline{AQ}=0.443$.

Take a point B on the closed interval [C, A] so that $\overline{P_2B}=\sqrt{\frac{10}{3}}$. Further we take any point P in [B, A] and describe a circle with radius P_2P and the center P_2 and denote by P′ the intersecting point with the segment CP_2. Then we have from Lemma 3 $D(r)=f_3(P)-f_3(P')>0$, and hence P_0 is not on the segment AB, but necessarily on the segment $\overline{CB}$.

Now it holds that $\overline{P_2P} \geqq \sqrt{\frac{10}{3}}$ for any point $P \neq C$ on CB. Considering the point P_2 as the origin and the line segment P_2C as the basic line and using Lemma 1 for $2 > r \geqq \sqrt{\frac{10}{3}}$ again, we obtain the result that $f_3(P')<f_3(P)$ for the point P′ on the line segment P_2C so that $\overline{P_2P'}=\overline{P_2P}$. Hence the point P_0 in which $f_3(P)$ attains its minimum is not P. On the other hand there exists such point P_0, since $f_3(P)$ is continuous on the closed interval [C, B]. Thus P_0 must coincide with C.

q.e.d.

Bemark. (1) We can prove that Theorem B establishes in the case of the exponent α ($3<\alpha<4$) by the similar manner.
(2) Does the function $f_{n,j_n}(z)$ converge absolutely and uniformly in $D_{0,0}$, as n tends to infinity ? If it is true, then $f(z)=\lim_{n\to\infty} f_{n,j_n}(z)$ will be also subharmonic in $D_{0,0}$ and attains its minimum at the origin
(3) When the points P_{n,j_n} are located at the vertices of the general normal polygon and the distances from P_{n,j_n} to the origin are more complicated, it is conjectured to obtain the interesting results.

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On the Number of Prime Factors of Integers

to Professor IKUZO YAMAMOTO on his 70th Birthday

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1. Introduction. Throughout the paper, the letters p, p_1, p_2, ... will be reserved for prime numbers. Let $\omega(n)$ be the number of distinct prime factors of a positive integer n, and let x be a positive real number. Let $f_i(\xi)$ $(1\leqq i\leqq k)$ be polynomials in ξ, satisfying the following conditions :

(c_1) Each $f_i(\xi)$, $(1\leq i\leq k)$ has integral coefficients ;
(c_2) Each $f_i(\xi)$, $(1\leq i\leq k)$ is of positive degree ;
(c_3) Each $f_i(\xi)$, $(1\leq i\leq k)$ is positive for $\xi\geq 1$;
(c_4) $f_1(\xi)$, ..., $f_k(\xi)$ are relatively prime in pairs.

Let r_i $(1\leq i\leq k)$ be the number of the primitive and irreducible factors of $f_i(\xi)$. Let c_1, c_2,..., be positive absolute constants. We put $g(x)=(\mathrm{lnln}x)^{1/4k}(\mathrm{lnlnln}x)^{1/2k}$. Let A{…} denote the number of positive integers with some conditions...... We put, for integers $n\geq 3$ and for $1\leq i\leq k$,

$$\frac{\omega\{f_i(n)\}-r_i\ \mathrm{lnln}\ n}{\sqrt{r_i\ \mathrm{lnln}\ n}}=u_i(n)$$

To each integer $n\geq 3$, there corresponds a point $(u_1(n),...,u_k(n))$ in a k-dimensional space R^k. Let E be a Jordan-measurable set, bounded or unbounded, in R. Let A(x;E) denote the number of integers n $(3\leq n\leq x)$, for which the points $(u_1(n),...,u_k(n))$ belong to the set E. Tanaka obtained the following Theorem A :

Theorem A. $$\lim_{x\to\infty}\frac{A(x;E)}{x}=(2\pi)^{-k/2}\int_E\exp\left(-\frac{1}{2}\sum_{i=1}^{k}u_i^2\right)du_1...du_k.$$
The integral is the sence of Riemann.[3].

Similarly, by using the sieve method of A. Selberg [2], [1] and Tanaka's method [3], we shall prove the following main theorem :

Main Theorem. *Let* α_i, β_i $(1\leq i\leq k)$ *be any real numbers with* $\alpha_i<\beta_i$ $(1\leq i\leq k)$. *We put*

$$A(x)=A\left\{3\leq n\leq x\ ;\ r_i\ \mathrm{lnln}\ n+\alpha_i\sqrt{r_i\ \mathrm{lnln}\ n}<\omega\{f_i(n)\}<r_i\ \mathrm{lnln}\ n+\beta_i\sqrt{r_i\ \mathrm{lnln}\ n}\right\},\quad (1\leq i\leq k).$$

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