Sci. Rep. Kanazawa Univ.,
Vol. 13, No. 2, pp. 73-82
December 1968

On a Family of Pure-dimensional Analytic Sets

Chikara WATANABE

Department of Mathematics, Faculty of Science, Kanazawa University

(Recieved 23 October 1968)

Introduction. Bishop [1] defined the notion of convergence of a sequence of pure-dimensional analytic sets as follows:

Definition. Let D be a domain of C^n and $\{S_\nu\}$ be a sequence of closed sets in D. It is said that this sequence converges to a closed set S in D if for any compact subset K of D, $\{S_\nu \cap K\}$ is a convergent sequence in Comp (K)[1] and $S=\bigcup_K \lim_\nu (S_\nu \cap K)$ is a closed subset in D. Hereafter we say that $\{S_\nu\}$ converges geometrically if this sequence converges in the above sense.

On the other hand, Oka defined the notion of convergence in the sense of analyticity[2], and Nishino proved Theorem 1 of Oka[3] in the case of two variables[4].
Now Bishop proved the following

THEOREM (Bishop [1]). *Let $\{S_\nu\}$ be a sequence of purely* k*-dimensional analytic sets in a domain D of C^n. Suppose that this sequence converges geometrically to a closed subset S in D. If* 2k*-dimensional volumes of S are uniformly bounded, then S is an analytic set in D.*

Generally the notion of analytic convergence is stronger than that of geometric convergence. Moreover, even if the sequence $\{S_\nu\}$ converges geometrically, the volumes of elements of this sequence are not uniformly bounded[5]. But if $\{S_\nu\}$ converges analytically, then the volumes of elements of $\{S_\nu\}$ are locally uniformly bounded. In this paper we shall give the another proof of Theorem 1 of Oka in the case of n variables.
That is, we shall prove the following

Theorem. *Let $\mathfrak{F}$ be a family of purely λ-dimensional analytic sets in a domain*

1) Comp (K) means a compact metric space consisting of all closed subsets of K with the Hausdorff metric.
2) See [6].
3) See [6], Theorem 2, p. 95.
4) See [5], Theorem 2, p. 371.
5) See [8], p. 212.

of C^n. Then $\mathfrak{F}$ is analytically normal if and only if the 2λ-dimensional volumes of elements of $\mathfrak{F}$ are locally uniformly bounded.

§ 1. Special case.

1. In this section we consider a family of principal analytic sets in a domain of C^{n+1}, $n\geqq 1$.

Definition 1. Let $\{S_\nu\}$ be a sequence of principal analytic sets in a domain D of C^{n+1}. It is said that this sequence converges analytically at $p\in D$ if the following condition is satisfied: there is an open and connected neighbourhood U of p and holomorphic functions $f_1, f_2, \ldots$, in U such that $S_\nu \cap U=\{t\in U;\ f_\nu(t)=0\}$ and $\{f_\nu\}$ converges uniformly to a holomorphic function in U which is not identically zero.

Remark. If $\{S_\nu\}$ converges analytically at each point of D, then an analytic set S in D is uniquely determined as the limit of $\{S_\nu\}$ and if $S\neq\phi$, S is a principal analytic set in D. Moreover S has the following properties.

(i) For any $p\in S$, there is a sequence of points p_ν such that $p_\nu\in S_\nu$ and $p_\nu\rightarrow p$.

(ii) For any point $p\in S$, there is an open and connected neighbourhood U of p and holomorphic functions $f_0, f_1, f_2, \ldots$ in U such that $S\cap U=\{t\in U;\ f_0(t)=0\}$, $S_\nu\cap U=\{t\in U;\ f_\nu(t)=0\}$ and $\{f_\nu\}$ converges uniformly to f_0 in U.

(iii) For any convergent sequence of points p_ν with $p_\nu\in S_\nu$ if $p_\nu\rightarrow p$ then $p\in S$.

Conversely, if there is a principal analytic set S which has these properties, then it is unique and $\{S_\nu\}$ converges analytically at each point of D. Therefore we can say that this sequence $\{S_\nu\}$ converges analytically to S, if there is a principal analytic set S which has these properties.

LEMMA 1. *Let $\{S_\nu\}$ be a sequence of purely* k*-dimensional analytic sets in a domain D of C^n. If there is a sequence of points p_ν such that $p_\nu\in S_\nu$ and $\{p_\nu\}$ has an accumlation point p in D, then there is a subsequence of $\{S_\nu\}$ which converges geometrically to a non-empty closed subset S in D.*

Moreover for any sequence of points q_ν such that $q_\nu\in S_\nu$, $q_\nu\rightarrow q$ and $q\in D$, it holds that $q\in S$.

Proof. At first we shall prove the second part, that is, if we can choose a subsequence $\{S_{\nu_j}\}$ of $\{S_\nu\}$ which converges geometrically to a non-empty closed subset S in D, then for any sequence $\{q_\nu\}$ such that $q_\nu\in S_\nu$, $q_\nu\rightarrow q$ and $q\in D$, we have $q\in S$. Suppose that S does not contain the point q. Then there is a compact subset K such that $\lim\,(S_{\nu_j}\cap K)=S_K\neq\phi$, $q\in\mathring{K}$[6] and $q\notin S_K$. Let $\rho_q(S_K)=\min_{t\in S_K}\rho\,(q,t)$[7] $=d>0$.

Since $\{S_{\nu_j}\cap K\}$ converges to S_K by mean of the Hausdorff metric, there is a positive integer j_0 such that $\max_{t\in S_K}\rho_t(S_{\nu_j}\cap K)+\max_{t_{\nu_j}\in S_{\nu_j}\cap K}\rho_{t_{\nu_j}}(S_K)<\frac{d}{2}$ for $j\geqq j_0$, and in particular

6) By $\mathring{K}$ we mean the interior of K.

7) $\rho(q,t)$ means the Euclid metric between q and t.

$\max_{t_{\nu_j} \in S_{\nu_j} \cap K} \rho_{t_{\nu_j}}(S_K) < \frac{d}{2}$ for $j \geqq j_0$. Thus if we put $S_K\left(\frac{d}{2}\right) = \{p' \in K;\ \rho_{p'}(S_K) < \frac{d}{2}\}$, we have $S_{\nu_j} \cap K \subset S_K\left(\frac{d}{2}\right)$ for $j \geqq j_0$. This means $\rho_q(S_{\nu_j} \cap K) \geqq \frac{d}{2}$ for $j \geqq j_0$. On the other hand, since $q \in \mathring{K}$ and $\{q_{\nu_j}\}$ also converges to q, for some positive integer j_0' we have $q_{\nu_j} \in K$ and $\rho(q, q_{\nu_j}) \geqq \rho_q(S_{\nu_j} \cap K) > \frac{d}{2}$ if $j \geqq j_0'$. This is a contradiction, and hence we can conclude that S contains the point q. Next, we shall prove the first part of Lemma 1. If for some p_{ν_0}, S_ν containing p_{ν_0} makes an infinite sequence $S_{\nu_1}, S_{\nu_2}, \ldots$, we can choose a subsequence of $\{S_{\nu_j}\}$ which converges geometrically ([7], [8]). If any p_ν is contained only in a finite number of S_ν, there is a positive number ν_1 such that $p_{\nu_1} \in S_{\nu_1}$ and $p_{\nu_1} \notin S_\nu$ if $\nu > \nu_1$. Also there is a positive number $\nu_2 > \nu_1$ such that $p_{\nu_2} \in S_{\nu_2}$ and $p_{\nu_2} \notin S_\nu$ if $\nu > \nu_2$. We continue this process. Then we have a subsequence $\{S_{\nu_j}\}$ of $\{S_\nu\}$ such that $p_{\nu_j} \in S_{\nu_j}$ and $p_{\nu_{j'}} \notin S_{\nu_j}$ if $j > j'$. Evidently we may assume that $\{S_{\nu_j}\}$ converges geometrically to a closed subset S of D and that $p_{\nu_j} \to p$. Let $K = \{p, p_{\nu_1}, p_{\nu_2}, \ldots\}$. Since K is a compact subset of D and $p_{\nu_{j'}} \notin S_{\nu_j}$ for $j > j'$, we have evidently $\lim_j (S_{\nu_j} \cap K) \neq \phi$. Since $\lim_j (S_{\nu_j} \cap K) \subset S$, we have a conclusion that S is not empty. Q.E.D.

LEMMA 2. *If a sequence $\{S_\nu\}$ of principal analytic sets in D converges analytically to a principal analytic set S, then there is a subsequence of $\{S_\nu\}$ which converges geometrically to S.*

Proof. Since we can take a sequence of points p_ν for each $p \in S$ such that $p_\nu \in S_\nu$ and $p_\nu \to p$, from Lemma 1 there is a subsequence of $\{S_\nu\}$ which converges geometrically to a non-empty closed subset $\tilde{S}$ in D. Since any subsequence of $\{S_\nu\}$ converges analytically to S, we may assume that this sequence $\{S_\nu\}$ converges itself geometrically to $\tilde{S}$. We have $\tilde{S} \subset S$ from the definition If $S - \tilde{S} \neq \phi$, then there is a point q such that $q \in S$ and $q \notin \tilde{S}$. Since it holds that $q \in \lim (S_\nu \cap K) \neq \phi$ for some compact subset K of D such that $q \in \mathring{K}$, any sequence of points p_ν with $p_\nu \in S_\nu \cap K$ does not converge to q. On the other hand, from definition there is a sequence of points q_ν such that $q_\nu \in S_\nu$ and $q_\nu \to q$. Since $q \in \mathring{K}$ from the assumption, we have $q_\nu \in K$ for sufficiently large ν. Thus we have $q_\nu \in S_\nu \cap K$, $q_\nu \to q$. This is a contradiction. Thus we obtain $S = \tilde{S}$. Q.E.D.

Remark. Under the assumption of Lemma 2, the sequence $\{S_\nu\}$ is itself not always geometrically convergent.

Example. Let $S_1 = \{z_1, z_2) \in C^2;\ z_1 = \frac{1}{2}\}$, $S_2 = \{(z_1, z_2) \in C^2;\ z_1 = 1 + \frac{1}{2}\}$ $\ldots$, $S_{2\nu} = \{(z_1, z_2) \in C^2;\ z_1 = 1 + \frac{1}{2^\nu}\}$, $S_{2\nu+1} = \{(z_1, z_2) \in C^2;\ z_1 = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{\nu+1}}\}$, ... Then S_ν is a principal analytic set in C^2, and evidently converges analytically to a principal analytic set $S=\{(z_1, z_2); z_1=1\}$ in C^2. But for a compact subset K of C^2 such that $K=\{(z_1, z_2)\in C^2; |z_1-2|\leqq 1, |z_2|\leqq 1\}$, $S_{2\nu+1}\cap K=\phi$ and $S_{2\nu}\cap K\neq\phi$, and thus $\{S_\nu\cap K\}$ is not a convergent sequence in the space Comp(K). However, the subsequence $\{S_{2\nu}\}$ of $\{S_\nu\}$ converges geometrically to $S=\{(z_1, z_2); z_1=1\}$.

Let $D=\Delta\times\{|w|<R\}$ be a domain of C^{n+1} whose coordinate system is $(z_1, z_2, \ldots, z_n, w)$, where $R>0$ and Δ is a domain of the $(z_1, z_2, \ldots, z_n)$ -space. Let S be a principal analytic set in D. If S does not intersect $\Delta\times\{R_0\leqq|w|<R\}$ for some positive number $R_0<R$, then by Weierstrass preparation theorem, the projection π: $(z_1, \ldots, z_n, w)\rightarrow(z_1, z_2, \ldots, z_n)$ is a proper mapping on S. Then $\mathfrak{A}=(S, \pi, \Delta)$ is an analytic cover[8]. Let m be a sheet number of S. Then there is a pseudo-polynomial $F(z, w) =w^m+A_1(z)w^{m-1}+\ldots+A_m(z)$ such that S is given by the zeros of $F(z, w)$, where $A_j(z)$ is holomorphic in Δ for all j. Because, for each point $z^{(0)}\in\Delta$, let $w_1, w_2, \ldots, w_l$ be points of S which lie over $z^{(0)}$. Then there is a positive number r such that $\Omega_k=\{(z, w); |z_j-z_j^{(0)}|<r, |w-w_k|<r, j=1,2, \ldots, n\}\subset D$ and $\Omega_k\cap\Omega_{k'}=\phi$ if $k\neq k'$. For each k, we may assume $S\cap\Omega_k=\{(z,w)\in\Omega_k; (w-w_k)^{m_k}+A_1^{(k)}(z)(w-w_k)^{m_k-1}+\ldots +A_{m_k}^{(k)}(z)=0\}$. If we choose sufficiently small r, then $\bigcup_k S\cap\Omega_k=S\cap\widetilde{D}_0$, where $\widetilde{D}_0 =\{(z, w); |z_j-z_j^{(0)}|<r, |w|<R, j=1,2, \ldots, n\}$. Thus $S\cap\widetilde{D}_0$ is given by the zeros of a pseudo-polynomial $F(z, w)=\prod_{k=1}^{l}\{(w-w_k)^{m_k}+A_1^{(k)}(z)(w-w_k)^{m_k-1}+\ldots+A_{m_k}^{(k)}(z)\} =w^m+B_1(z)^{m-1}+\ldots+B_m(z)$, where $B_j(z)$is holomorphic in the polydisc $\{z; |z_j-z_j^0| <r, j=1,2, \ldots, n\}$. We associate the polydisc $\{z; |z_j-z_j^{(0)}|<r, j=1,2, \ldots, n\}$, $F(z, w)$ and $\widetilde{D}_0$ to each point $z^{(0)}\in\Delta$.

Let$V=\{z; |z_j-z_j^{(0)}|r; j=1,2, \ldots, n\}\cap\{z; |z_j-z_j'|<r'; j=1,2, \ldots, n\}$ be not empty and δ, δ' be discriminants of F, F' respectively. Then for each $z\in G=\{z; |z_j-z_j^{(0)}| <r; j=1,2,\ldots,n\}-\{z; \delta(z)=0\}$ the roots of the equation $F(z, w)=0$ are given by $w =w_1(z), \ldots, w=w_m(z)$ and they are holomorphic and bounded in G. Hence they are continued holomorphically to the polydisc $\{z;|z_j-z_j^{(0)}|<r;j=1,2, \ldots, n\}$. Similary, the roots of the equation $F'(z, w)=0$ are given by $w=w_1'(z), \ldots, w=w_m'(z)$ and holomorphic in $\{z;|z_j-z_j'|<r';j=1,2, \ldots, n\}$. Since it holds that $w_1(z)=w_{j_1}'(z)$, $w_2(z)=w_{j_2}'(z), \ldots,$ and $w_m(z)=w_{j_m}'(z)$ on V for some permutation $(j_1, j_2, \ldots, j_m)$ of $(1,2, \ldots, m)$, we can define a pseudo-polynomial $F(z, w)=w^m+A_1(z)w^{m-1}+\ldots +A_m(z)$ such that $A_k(z)=B_k(z)$ on $\{z;|z_j-z_j^{(0)}|<r;j=1,2, \ldots, n\}$ for each point $z^{(0)}\in\Delta$ and for all k.

Thus we have the following

LEMMA 3. *Let $D=\Delta\times\{|w|<R\}$ be a domain of C^{n+1} and S be a principal analytic set in D. If S does not intersect $\Delta\times\{R_0\leqq|w|<R\}$, then S is given by the zeros of a pseudo-polynomial $F(z, w)=w^m+A_1(z)w^{m-1}+\ldots+A_m(z)$, where m is a sheet number of an analytic cover (S, π, Δ) and $A_j(z)$ is holomorphic in Δ for all j.*

8) See [3], p. 101-109.

The condition in Lemma 3 is very important. Thus we describe as a definition.

Definition 2. Let S be a principal analytic set in $\Delta\times\{|w|<R\}$. If S does not intersect $\Delta\times\{R_0\leq|w|<R\}$ for some positive number $R_0<R$, then S is said to satisfy a condition (β). If for a family $\mathfrak{F}$ of principal analytic sets in $\Delta\times\{|w|<R\}$, there is a positive number $R_0<R$ such that any $S\in\mathfrak{F}$ does not intersect $\Delta\times\{R_0\leq|w|<R\}$, then $\mathfrak{F}$ is said to satisfy a condition (β).

Definition 3. Let $\mathfrak{F}$ be a family of principal analytic sets in a domain D of C^n. $\mathfrak{F}$ is said to be analytically normal at $p\in D$ if for any sequence $\{S_\nu\}$ in $\mathfrak{F}$, there is a subsequence of $\{S_\nu\}$ which converges analytically at p. If $\mathfrak{F}$ is analytically normal at each point of D, then $\mathfrak{F}$ is said to be analytically normal in D.

The lemma obtained by O. Fujita [2] is stated as follows:

Lemma 4. (Fujita). *Let $\mathfrak{F}$ be a family of principal analytic sets in the domain $D=\Delta\times\{|w|<R\}$ of C^{n+1}. Suppose that $\mathfrak{F}$ satisfies the condition (β). Then $\mathfrak{F}$ is analytically normal if and only if the sheet numbers of elements of $\mathfrak{F}$ are uniformly bounded.*

Lemma 5. *Let $\{S_\nu\}$ be a sequence of principal analytic sets in a domain $D=\Delta\times\{|w|<R\}$ of C^{n+1}. Suppose that this sequence satisfies the condition (β). If the* 2n-*dimensional volumes of elements of this sequence are uniformly bounded, then $\{S_\nu\}$ is analytically normal in D.*

Proof. Let m_ν be the sheet number of S_ν. Then from Lemma 3, S_ν is given by the zeros of a pseudo-polynomial $F_\nu(z, w)=w^{m_\nu}+A_1^{(\nu)}(z)\,w^{m_\nu-1}+\ldots+A_{m_\nu}^{(\nu)}(z)$.

Let δ_ν be a descriminant of $F_\nu(z, w)$. For each $z\in\Delta-\{z;\ \delta_\nu(z)=0\}$, the roots $w=w_1^{(\nu)}(z), \ldots, w=w_{m_\nu}^{(\nu)}(z)$ are all distinct, and these functions $w=w_j(z)$ are all holomorphic and bounded in $\Delta-\{z;\ \delta_\nu(z)=0\}$. Thus from Riemann's theorem, these are holomorphic in Δ. We denote by $\sum_j^{(\nu)}$ the principal analytic set $w=w_j^{(\nu)}(z)$ in Δ. Then $\mathrm{Vol}_{2n}(S_\nu)=\sum_{j=1}^{m_\nu}\mathrm{Vol}_{2n}(\sum_j^{(\nu)})$. Let z^0 be any fixed point in Δ, then $|w_j^{(\nu)}(z^0)|<R_0$ for all ν, j. Since for any ν, j, the disc $\{w;\ |w-w_j^{(\nu)}(z^0)|<R-R_0\}$ is contained in $|w|<R$, for some positive number ρ independent of ν,j, the ball $B_j^{(\nu)}$ of radius ρ with center $(z^0, w_j^{(\nu)}(z^0))$ is contaned in D. Then from the theorem of Bishop, we have $\mathrm{Vol}_{2n}(\sum_j^{(\nu)})>\alpha(2n)\rho^{2n}$ 9). Thus we obtain $m_\nu\cdot\alpha(2n)\rho^{2n}<\mathrm{Vol}_{2n}(S_\nu)$. Since there holds $\mathrm{Vol}_{2n}(S_\nu)<M$ for some constant M, m_ν is uniformly bounded. Then from Lemma 4, we obtain the conclusion. Q.E.D.

Lemma 6. *Let $\mathfrak{F}$ be a family of principal analytic sets in a domain D of C^{n+1}. If the* 2n-*dimensional volumes of elements of $\mathfrak{F}$ are locally uniformly bnunded,* i.e., *if for each point $p\in D$, there is a neighbourhood U of p such that the* 2n-*dimensional volumes of $S\cap U$ are uniformly bounded for all $S\in\mathfrak{F}$, then $\mathfrak{F}$ is analytically normal in D.*

Proof. We shall prove that $\mathfrak{F}$ is analytically normal at each point p of D. For

9) See [1], lemma 3, p. 290 or [7], p. 16, Theorem B.

any sequence $\{S_\nu\}$ in $\mathfrak{F}$, we may assume that $S_\nu \cap U \neq \phi$ for sufficiently large ν, where U is any neighbourhood of p. Since the volumes of $S_\nu \cap U$ are uniformly bounded for some neighbourhood U of p, we can choose a subsequence $\{S_{\nu_j}\}$ which converges geometrically to an analytic set S in U. For simplicity we may assume that $S_\nu \cap U$ converges geometrically to S and p is the origin. S is not empty and the dimension of S at the origin is at most n. Then there is a polydisc $\Omega=\{(z, w);\ |z_j|<r,\ |w|<\rho,\ j=1,2,\ \dots,\ n\} \Subset U$ such that $S \cap \Omega=\{(z, w) \in \Omega;\ f_1(z, w)=0,\ \dots,\ f_l(z, w)=0\}$, where $f_1, \dots, f_l$ are holomorphic in Ω and not identically zero. We put $S^*=\{(z, w) \in \Omega;\ f_1(z, w)=0\}$. Evidently we may assume $f_1(0, w) \neq 0$ and $f_1(0,0)=0$. Then from Weierstrass preparation theorem, we may assume $f_1(z, w)=w^m+A_1(z)w^{m-1}+\dots+A_m(z)$, where $A_1(z), \dots, A_m(z)$ are holomorphic in $\{z; |z_j|<r,\ j=1,2,\ \dots,\ n\}$ and $A_1(0)=\dots=A_m(0)=0$. By the same way as the proof of Lemma 3, the roots $w=g_1(z), \dots, w=g_m(z)$ of $f_1(z, w)=0$ are holomorphic in $\{z; |z_j|<r,\ j=1,2,\ \dots,\ n\}$. By the continuity of $g_k(z)$ we may assume $|g_k(z)|<\frac{\rho}{2}$ for all k and for all z with $|z_j|<r$, $j=1,2,\ \dots,\ n$. Then S^* is a principal analytic set in Ω and does not intersect $\{(z, w);\ |z_j|<r,\ \frac{\rho}{2} \leq |w|<\rho,\ j=1,2,\ \dots,\ n\}$. Since $S_\nu \cap \bar{\Omega} \to S \cap \bar{\Omega}$ by the Hausdorff metric[10], $\{S_\nu \cap \Omega\}$ satisfies the condition (β). Then from Lemma 5, we obtain a conclusion. Q.E.D.

2. Next we shall consider volumes of principal analytic sets S_ν when a sequence $\{S_\nu\}$ converges analytically.

LEMMA 7. *Let $\{S_\nu\}$ be a sequence of principal analytic sets in a domain D of C^{n+1}. If this sequence converges analytically to a principal analytic set S in D, then the* 2n-*dimensional volumes of elements of this sequence are locally uniformly bounded.*

Proof. For $p \in D-S$, there exists a neighbourhood V of p such that $S_\nu \cap V=\phi$ for sufficiently large ν. Thus we have only to consider points of S. Suppose that for some point $p \in S$ and for any neighbourhood U of p, there is a subsequencce $\{S_{\nu_j}\}$ such that $\mathrm{Vol}_{2n}(S_{\nu_j} \cap U)$ is not uniformly bounded. Since $\{S_{\nu_j}\}$ also converges analytically to S, from Lemma 1, we may assume that this sequence converges geometrically to S. On the other hand, since S is a principal analytic set in D, there are some positive numbers r, ρ, such that $\Delta=\{z;\ |z_j|<r,\ j=1,2,\ \dots,\ n\}$, $\Omega_1=\Delta \times \{|w|<\rho\} \Subset D$ and $S \cap \Omega_1=\{(z, w) \in \Omega_1;\ f(z, w)=0\}$, where p is the origin, f is holomorphic and not identically zero in Ω_1. Moreover from Weierstrass preparation theorem, we may assume $f(z, w)=w^m+A_1(z)w^{m-1}+\dots+A_m(z)$, where $A_k(z)$ is holomorphic in Δ, and $A_k(0)=0$, $k=1,2,\ \dots,\ m$. Evidently we may assume $m=1$ and $f(z, w)=w-A(z)$. Then from the continuity of $w=A(z)$, we can take r, ρ such that $|A(z)|<\rho$

10) Generally, it does not follows that $S_\nu \cap K \to S \cap K$ by the Hausdorff metric (See [7] p. 23). But for any relatively compact neighbourhood U of $p \in S$, it follows that $S_\nu \cap \bar{U} \to S \cap \bar{U}$ by the Hausdorff metric.

for $z\in\Delta$. Then, since $\{S_\nu\cap\overline{\Omega}_1\}$ converges to $S\cap\overline{\Omega}_1$ by the Hausdorff metric, we can choose positive numbers ρ_1, ρ' with $\rho_1<\rho'<\rho$ such that the polydisc $\Omega_2=\Delta\times\{|w|<\rho'\}\Subset D$ and $S_\nu\cap\Omega_2$ does not intersect $\Delta\times\{\rho'\leqq|w|<\rho\}$ for sufficiently large ν. Then from Lemma 3, $S_\nu\cap\Omega_2$ is given by the zeros of a pseudo-polynomials $F_\nu(z, w)=w^{m_\nu}+A_1^{(\nu)}(z)w^{m_\nu-1}+\dots+A_{m_\nu}^{(\nu)}(z)$ in Ω_2. We denote by $\Sigma_k^{(\nu)}$ the principal analytic set given by the equation $w=w_k^{(\nu)}(z)$ in Ω_2, where $w_k^{(\nu)}(z)$ is a root of $F_\nu(z, w)=0$. Also, since $\{S_\nu\cap\overline{\Omega}_2\}$ converges to $S\cap\overline{\Omega}_2$ by mean of the Hausdorff metric, for any $\varepsilon>0$ there is a positive integer ν_0 such that $\Sigma_k^{(\nu)}\subset\{(z, w);\ z\in\Delta,\ |w-A(z)|<\delta\}$ for each k and $\nu\geqq\nu_0$. This means that $w_k^{(\nu)}(z)$ converges uniformly to $A(z)$ in Δ for each k. On the other hand, from Lemma 4, m_ν is bounded in Ω_2, that is, there exists a positive constant m such that $m_\nu\leqq m$. We put $z_1=x_1+\sqrt{-1}\,y_1, \dots, z_n=x_n+\sqrt{-1}\,y_n$, $w=u+\sqrt{-1}\,v$. For a $(2n, 2n+2)$ matrix

$$M=\begin{pmatrix} 1 & & & 0 & \frac{\partial}{\partial x_1}u & \frac{\partial}{\partial x_1}v \\ & \ddots & & & \vdots & \\ & & \ddots & & & \\ 0 & & & 1 & \frac{\partial}{\partial y_n}u & \frac{\partial}{\partial y_n}v \end{pmatrix}$$

$\mathrm{Vol}_{2n}(S\cap\Omega_2)$ is given by the integral

$$\mathrm{Vol}_{2n}(S\cap\Omega_2)=\int_\Delta\sqrt{|D_1|^2+\dots+|D_l|^2}\;dx_1dy_1\cdots dx_ndy_n$$

where D_j is a 2n-dimensional minor of M and $|D_j|$ is its determinant.

Let $w_k^{(\nu)}(z)=u_k^{(\nu)}+\sqrt{-1}\,u_k^{(\nu)}$, then $\frac{\partial}{\partial x_1}u_k^{(\nu)}, \frac{\partial}{\partial y_1}u_k^{(\nu)}, \dots,$ converge uniformly to $\frac{\partial}{\partial x_1}u, \frac{\partial}{\partial y_1}u\ \dots,$ respectively.

Thus $\mathrm{Vol}_{2n}(\Sigma_k^{(\nu)})$ converges to $\mathrm{Vol}_{2n}(S\cap\Omega_2)$ for each k. Since $m_\nu\leqq m$, we obtain $\mathrm{Vol}_{2n}(S_\nu\cap\Omega_2)<2m\ \mathrm{Vol}_{2n}(S\cap\Omega_2)$ for sufficiently large ν. This is a contradiction. Q.E.D.

Corollary. *Let $\mathfrak{F}$ be a family of principal analytic sets in D. If this family is analytically normal in D, then the* 2n-*dimensional volumes of elements of $\mathfrak{F}$ are locally uniformly bounded.*

Combining Lemma 6 with the above corollary, we obtain the following

Proposition 1. *Let $\mathfrak{F}$ be a family of principal analytic sets in D. Then $\mathfrak{F}$ is analytically normal in D if and only if the* 2n-*dimensional volumes of elements of $\mathfrak{F}$ are locally uniformly bounded.*

Remark. Let $A_n=\bigcup_{i=0}^{n-1}\{z\in C^2;\ z_1=1+\frac{1}{2}+\dots+\frac{1}{2^{\frac{1}{2}n(n-1)+i}}$. Then evidently $\{A_n\}$ converges geometrically to an analytic set $\{z\in C^2;\ z_1=2\}$. But the 2-dimensional volumes of A_n are not uniformly bounded, and since the sheet numbers of A_n are not bounded, any subsequende of $\{A_n\}$ does not converge analytically.

§ 2. General case.

1. Let $\{S_\nu\}$ be a sequence of purely λ-dimensional analytic sets in a domain D of C^n. $\{S_\nu\}$ is said to converge analytically at $p \in D$ if there are a polydisc (γ) containing p and l functions $f_1^{(\nu)}, \dots f_l^{(\nu)}$ holomorphic in (γ) satisfying the following properties, where l is independent of ν.

(i) $S_\nu \cap (\gamma) = \{z \in (\gamma);\ f_1^{(\nu)}(z) = 0, \dots, f_l^{(\nu)}(z) = 0\}$.

(ii) $f_j^{(\nu)} (j = 1, 2, \dots,)$ converges uniformly to a holomorphic function $f_j^{(0)}$ in (γ).

(iii) If S_0' is an analytic set in (γ) defined by $f_1^{(0)}(z) = 0, \dots, f_l^{(0)}(z) = 0$, then S_0' is also purely λ-dimensional.

(iv) For any point $p \in S_0'$ and for any neighbourhood V of p, there is a positive integer ν_0 such that $S_\nu \cap V \neq \phi$ if $\nu \geqq \nu_0$.

This definition of analytic convergence is a generalization of that in §1. Moreover it is easily seen that the above definition is equivalent to the following

Definition. Let $\{S_\nu\}$ be a sequence of purely λ-dimensional analytic sets in D. $\{S_\nu\}$ is said to converge analytically if there is a purely λ-dimensional analytic set S_0 in D satisfying following properties (i), (ii) and (iii).

(i) For any point $p \in S_0$, there is a neighbourhood U of p and holomorphic functions $f_1^{(\nu)}, f_2^{(\nu)}, \dots, f_l^{(\nu)}, f_1^{(0)}, f_2^{(0)}, \dots, f_l^{(0)}$ in U such that $S_\nu \cap U = \{z \in U;\ f_j^{(\nu)}(z) = 0\ j = 1, 2, \dots, l\}$, $S_0 \cap U = \{z \in U;\ f_j^{(0)}(z) = 0\ j = 1, 2, \dots,\}$ and $f_j^{(\nu)}$ $(j = 1, 2, \dots, l)$ converges uniformly to $f_j^{(0)}$ in U.

(ii) For any $p \in S_0$, there is a sequence of points p_ν such that $p_\nu \in S_\nu$ and $p_\nu \to p$.

(iii) For any sequence of points p_ν with $p_\nu \to p$, there holds $p \in S_0$.

In the general case the following lemma is essential.

LEMMA 8 (Fujita [2]). *Let $D = \Delta \times \{|w_1| < R\} \times \cdots \times \{|w_\mu| < R\}$ be a domain of $C^{\lambda+\mu}$, where Δ is a domain of the $(z_1, z_2, \dots, z_\lambda)$-space. Let $\mathfrak{F}$ be a family of purely λ-dimensional analytic sets in D satisfying the condition (β').*

Condition (β'). There is a positive number $R_0 < R$ such that for any $S \in \mathfrak{F}$, S is contained in $\Delta \times \{|w_1| < R_0\} \cdots \times \{|w_\mu| < R_0\}$.

Then $\mathfrak{F}$ is analytically normal if and only if the projection $\underline{\mathfrak{F}}^{(j)}$[11] *of $\mathfrak{F}$ on the $(z_1, z_2, \dots, z_\lambda, w_j)$-space is analytically normal in $\Delta \times \{|w_j| < R\}$.*

Remark. In the above case, it is well-known that $\underline{\mathfrak{F}}^{(j)}$ is a family of principal analytic sets in $\Delta \times \{|w_j| < R\}$[12]. Let S be a purely λ-dimensional analytic set in D. If S satisfies the condition (β'), then from Lemma 3, $\underline{S}^{(j)}$ is given by the zeros of a pseudo-polynomial $F^{(j)}(z; w_j) = w_j^{m_j} + A_1^{(j)}(z) w^{m_j - 1} + \dots + A_{m_j}^{(j)}(z)$, since $\underline{S}^{(j)}$ is a principal analytic set in $\Delta \times \{|w_j| < R\}$ and satisfies the condition (β). As in the proof of Lemma 3, the roots $w = w_1^{(j)}(z), \dots, w = w_{m_j}^{(j)}(z)$ of $F^{(j)}(z; w^j) = 0$ are holomorphic in Δ. Let $S_{k_1 k_2 \dots k_\mu} = \{(z, w_{k_1}^{(1)}, \dots, w_{k_\mu}^{(\mu)}(z)) \in C^{\lambda+\mu};\ z \in \Delta\}$, $1 \leqq k_1 \leqq m_1, \cdots$,

11) We denote by $\underline{S}(j)$ the projection of S on the $(z_1, \dots, z_\lambda, w_j)$- space. Then $\underline{\mathfrak{F}}(j) = \{\underline{S}(j);\ S \in \mathfrak{F}\}$.

12) For example, see [2], PROPOSITION 1, p. 390.

$1 \leqq k_\mu \leqq m_\mu$, then $S_{k_1 k_2 \dots k_\mu}$ is a purely λ-dimensional analytic set and $S = \bigcup S_{k_1 k_2 \dots k_\mu}$. For simplicity we assume that $m_1 = m_2 = \dots = m_\mu = 1$.
It is easily seen that in our case this assumption does not lose a generality. As in the proof of Lemma 7, there hold

$$\mathrm{Vol}_{2\lambda}(S) = \int_\Delta \sqrt{|D_1|^2 + \dots + |D_l|^2}\, dx_1 dy_1 \dots dx_\lambda dy_\lambda$$

and

$$\mathrm{Vol}_{2\lambda}(\underline{S}^{(j)}) = \int_\Delta \sqrt{|D_1^{(j)}|^2 + \dots + |D_{l_j}^{(j)}|^2}\, dx_1 dy_1 \dots dx_\lambda dy_\lambda,$$

where D_k is a 2λ-dimensional minor of

$$M = \begin{pmatrix} 1 & & 0 & \frac{\partial}{\partial x_1} u_1, & \frac{\partial}{\partial x_1} v_1, & \cdots \frac{\partial}{\partial x_1} u_\mu, & \frac{\partial}{\partial x_1} v_\mu \\ & \ddots & & \vdots & & & \\ 0 & & 1 & \frac{\partial}{\partial y_\lambda} u_1, & \frac{\partial}{\partial y_\lambda} v_1, & \dots \frac{\partial}{\partial y_\lambda} u_\mu, & \frac{\partial}{\partial y_\lambda} v_\mu \end{pmatrix}$$

and $D_{k'}^{(j)}$ is a 2λ-dimensionl minor of

$$\underline{M}^{(j)} = \begin{pmatrix} 1 & & 0 & \frac{\partial}{\partial x_1} u_j, & \frac{\partial}{\partial x_1} v_j \\ & \ddots & & \vdots & \\ 0 & & 1 & \frac{\partial}{\partial y_\lambda} u_j, & \frac{\partial}{\partial y_\lambda} v_j \end{pmatrix}$$

$z_1 = x_1 + \sqrt{-1}\, y_1$, $z_2 = x_2 + \sqrt{-1}\, y_2$, $\dots$, $z_\lambda = x_\lambda + \sqrt{-1}\, y_\lambda$, $w_1 = u_1 + \sqrt{-1}\, v_1$, $\dots$, $w_\mu = u_\mu + \sqrt{-1}\, v_\mu$.
For any k' $(1 \leqq k' \leqq l_j)$, there is an integer k $(1 \leqq k \leqq l)$ such that $D_{k'}^{(j)} = D_k$.
Since $\sqrt{|D_1^{(j)}|^2 + \dots + |D_{l_j}^{(j)}|^2} \leqq \sqrt{|D_1|^2 + \dots + |D_l|^2}$, we have $\mathrm{Vol}_{2\lambda}(\underline{S}^{(j)}) \leqq \mathrm{Vol}_{2\lambda}(S)$.
Thus from Lemma 5 and 8 we easily obtain the following

LEMMA 9. *Let $\mathfrak{F}$ be a family of purely λ-dimensional analytic sets in a domain $D = \Delta \times \{|w_1| < R\} \times \dots \times \{|w_\mu| < R\}$ of $C^{\lambda+\mu}$. Suppose that $\mathfrak{F}$ satisfies the condition (β'). If 2λ-dimensional volumes of elements of $\mathfrak{F}$ are uniformly bounded, then $\mathfrak{F}$ is analytically normal in D.*

2. Let S be a purely λ-dimensional analytic set in a pomain $D = \Delta \times \{|w_1| < R\} \times \dots \times \{|w_\mu| < R\}$ of $C^{\lambda+\mu}$ which satisfies the condition (β'). For each point $p \in D$, we take as a neighbourhood of p, $\widetilde{D} = \widetilde{\Delta} \times \{|w_1| < \widetilde{R}\} \times \dots \times \{|w_\mu| < \widetilde{R}\}$, where $\widetilde{\Delta} \Subset \Delta$, $R_0 < \widetilde{R} < R$. We put $\widetilde{S} = S \cap \widetilde{D}$, $\underline{S}^{(j)} = \underline{\widetilde{S}}^{(j)} \cap [\widetilde{\Delta} \times \{|w_j| < \widetilde{R}\}]$. If $\mathrm{Vol}_{2\lambda}(\underline{\widetilde{S}}^{(j)}) < M$ for each j, then by using the same notation as in No. 1, it is easily seen that for any k and j, $\left|\frac{\partial}{\partial x_k} u_j\right|^2$, $\left|\frac{\partial}{\partial y_k} v_j\right|^2$, $\left|\frac{\partial}{\partial x_k} v_j\right|^2$, $\left|\frac{\partial}{\partial y_k} u_j\right|^2$ are bound above by $\left(\frac{M}{\mathrm{Vol}_{2\lambda}\widetilde{\Delta}}\right)^2$ on $\widetilde{\Delta}$. Thus using Lemma 8 we obtain the following

LEMMA 10. *Let $\mathfrak{F}$ be a family of purely λ-dimensional analytic sets in a domain $D = \Delta \times \{|w_1| < R\} \times \dots \times \{|w_\mu| < R\}$ of $C^{\lambda+\mu}$ which satisfies the conditin (β'). If $\mathfrak{F}$ is analytically normal in D, then the volumes of elements of $\mathfrak{F}$ are locally uniformly* *bounded.*

Thus combining Lemma 9 with Lemma 10, we obtain the following

PROPOSITION 2. *Let $\mathfrak{F}$ be a family of purely λ-dimensional analytic sets in a domain $D=\Delta\times\{|w_1|<R\}\times\ldots\times\{|w_\mu|<R\}$ of $C^{\lambda+\mu}$ which satisfies the condition (β'). Then $\mathfrak{F}$ is analytically normal if and only if the volumes of elements of $\mathfrak{F}$ are locally uniformly bounded.*

Now, let S be a purely λ-dimensional analytic set in a domain D of C^n. Then by the local description theorem[13], it is easily seen that for any point $p=(p', p_{\lambda+1}, \ldots, p_n)\in S$, $p'=(p_1, \ldots, p_\lambda)$, we may assume that there is a polydisc $\Omega=\{z; |z_j-p_j|<r, j=1,2,\ldots\lambda, |z_{\lambda+l}-P_{\lambda+l}|<\rho, l=1,2,\ldots, n-\lambda\}\Subset D$ such that $S\cap\Omega$ satisfies the condition (β'). Since Lemma 2 is true for a sequence of pure-dimensional analytic sets, by the same way as in §1 we obtain the folowing

THEOREM. *Let $\mathfrak{F}$ be a family of purely λ-dimensional analytic sets in a domain D of C^n. Then $\mathfrak{F}$ is analytically normal if and only if the 2λ-volumes of elements of $\mathfrak{F}$ are locally uniformly bounded.*

References

[1] Bishop, Conditions for the analyticity of certain sets, Mich. Math. Jour., 11 (1964), 289–304.
[2] O. Fujita, Sur les familles d' ensembles analytiques, J. Math. Soc. Japan, 16 (1964), 379–405.
[3] R. C. Gunning and H. Rossi, Analytic functions of several complex variables, Prentice-Hall, 1965.
[4] S. Hitotumatu, Theory of analytic functions of several complex variables (in Japanese), Baihūkan, Japan, 1960.
[5] T. Nishino, Sur les familles de surfaces analytiques, J. Math. Kyoto Univ., 1 (1962), 357–377.
[6] K. Oka, Note sur les familles de fonctions analytiques multiformes etc., J. Sci. Hiroshima Univ., 4 (1934), 93–98.
[7] G. Stolzenberg, Volumes, lemits, and extensions of analytic varieties, Lecture Notes in Math., No. 19, Springer Verlag, 1966.
[8] Ch. Watanabe, On normality of a family of analytic sets, Sci. Rep. Kanazawa Univ., 12 (1967), 209–213.

13) See [4], p. 152.