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One Consideration of Hydrodynamic-Approximation Dispersion Relation of Beam-Magnetoplasma System

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Abstract

Hydrodynamic-approximation dispersion relation of plasma in a magnetic field is formally deduced, in the case that its constituent species have each drift velocities and each finite temperatures. In the case of ion beam-plasma system, this equation is compared with the method used by Perulli et al. for convenience.

1. Introduction

Strict treatments of dispersion relation of magnetoplasma have been treated by Sitenko and Stepanov [1], Stepanov and Kitsenko [2], and Stix [3]. But these results are rather complicated. Recently, in the report of experiment of ion beam-plasma interaction, Perulli, Etievant, and Lutaud [4] used convenient method, introducing finite temperature effect by substituting the electron term in the Stix's theory for corresponding term in the dispersion relation of cold plasma. In this point of view, hydrodynamic-approximation dispersion relation of plasma in a magnetic field was formally deduced and then compared with the method of Perulli et al..

2. Deduction of hydrodynamic-approximation dispersion relation

Fig. 1 shows a coordinate system. External magnetic field $\boldsymbol{B}$ is applied in the z direction. Each species j has drift velocity v_{oj} in the z direction. We treat only longitudinal wave. Then, basic hydrodynamic equations are as follows

$$\frac{\partial n_j}{\partial t} + \nabla \cdot n_j \boldsymbol{v}_{dj} = 0 \qquad (1)$$

$$n_j m_j \left(\frac{\partial}{\partial t} + \boldsymbol{v}_{dj} \cdot \nabla\right) \boldsymbol{v}_{dj} = n_j q_j \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{v}_{dj} \times \boldsymbol{B}\right) - \nabla \cdot [p_j] \qquad (2)$$

$$\nabla\cdot E = 4\pi\sum_j q_j n_j \qquad (3)$$

$$\sum_j q_j n_{oj} = 0 \qquad (4)$$

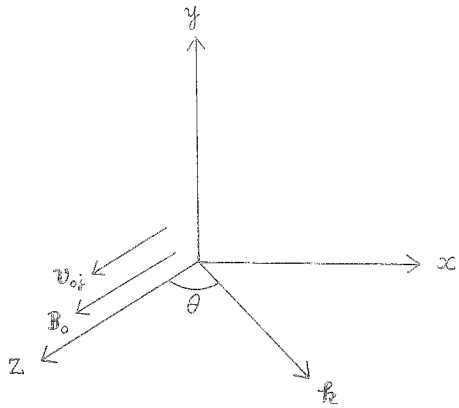

Fig. 1 Coordinate system

where n_j is particle density, n_{oj} is stationary particle density, v_{dj} is mean velocity as a whole, m_j and q_j are mass and charge respectively, $[p_j]$ is pressure tensor, E and B are electric and magnetic field respectively. Next, $n_j = n_{oj} + \Delta n_j$, $v_{dj} = v_{oj} + \Delta v_{dj}$, $B = B_o\,\hat{z} + \Delta B$, where Δn_j, Δv_{dj}, ΔB and E are small quantities of first order and assumed to be proportional to $exp\ i[k\cdot r - \omega t]$. $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors of respective directions. In the case of longitudinal wave, i. e., $k // E$, $\nabla\cdot B = 0$ from Maxwell equation. Pressure tensor is

$$[p] = \begin{bmatrix} p_\perp & 0 & 0 \\ 0 & p_\perp & 0 \\ 0 & 0 & p_{//} \end{bmatrix}$$

$$p_\perp = n\kappa T_\perp \qquad p_{//} = n\kappa T_{//}$$

where Maxwellian distribution of charged particles in the directions perpendicular and parallel to external magnetic field is assumed and $T_\perp$ and $T_{//}$ are equivalent kinetic temperature in the respective directions. Since we consider the plane wave parallel to y axis, there is not change of p in the y direction and then

$$\nabla\cdot[p] = \frac{\partial p_\perp}{\partial x}\hat{x} + \frac{\partial p_{//}}{\partial z}\hat{z}$$

Under above assumption about particle distribution, adiabatic change is difficult to be considered. Therefore, isothermal change is wholly assumed. In this point, Stix [3] also treat wholly isothermal change in his book. Then,

$$\Delta p_\perp = \kappa T_\perp \Delta n \qquad \Delta p_{//} = \kappa T_{//}\Delta n$$

$$\nabla\cdot[p] = ik_\perp \kappa T_\perp \Delta n\hat{x} + ik_{//}\kappa T_{//}\Delta n\hat{z}$$

where $k_\perp$ and $k_{//}$ are components of propagation vector k perpendicular and parallel to the external magnetic field respectively.

When basic equations are linearized about small quantities of first order and operators $\frac{\partial}{\partial t} \to -i\omega$, $\nabla\cdot \to ik\cdot$ are taken in, following equations are obtained from eqs. (1), (2), (3) and (4)

$$(\omega - k_{//}v_{oj})\Delta n_j - n_{oj}(k_\perp \Delta v_{djx} + k_{//}\Delta v_{djz}) = 0 \qquad (5)$$

$$-i(\omega - k_{//}v_{oj})\ \Delta v_{dj} = \frac{q_j}{m_j}E + \omega_{cj}(\Delta v_{dj}\times\hat{z})$$

$$-\frac{ik_\perp \kappa T_\perp^{(j)}}{n_{oj}\ m_j}\Delta n_j\hat{x} - \frac{ik_{//}\kappa T_{//}^{(j)}}{n_{oj}\ m_j}\Delta n_j\,\hat{z} \qquad (6)$$

$$ikE = 4\pi \sum_j q_j \Delta n_j \tag{7}$$

$$\omega_{cj} = \frac{q_j B_0}{m_j c} \tag{8}$$

From x, y, and z components of eq. (6), we obtain following equations (hereafter we omit notation Δ and suffix d)

$$v_{jx} = \frac{iq_j E_\perp}{m_j} \frac{(\omega - k_{//} v_{oj})}{\{(\omega - k_{//} v_{oj})^2 - \omega_{cj}^2\}} + \frac{k_\perp \kappa T_\perp^{(j)}}{n_{oj} m_j} \frac{(\omega - k_{//} v_{oj}) n_j}{\{(\omega - k_{//} v_{oj})^2 - \omega_{cj}^2\}} \tag{9}$$

$$v_{jz} = \frac{iq_j E_{//}}{m_j} \frac{1}{(\omega - k_{//} v_{oj})} + \frac{k_{//} \kappa T_{//}^{(j)}}{n_{oj} m_j} \frac{n_j}{(\omega - k_{//} v_{oj})} \tag{10}$$

When eq. (9) and eq. (10) are substituted into eq.(5), we obtain next eqation

$$n_j = \frac{1}{1 - \left[\frac{\kappa T_{//}^{(j)}}{m_j} \frac{k_{//}^2}{(\omega - k_{//} v_{oj})^2} + \frac{\kappa T_\perp^{(j)}}{m_j} \frac{k_\perp^2}{\{(\omega - k_{//} v_{oj})^2 - \omega_{cj}^2\}}\right]}$$

$$\times \left[\frac{in_{oj} q_j E_{//} k_{//}}{m_j} \frac{1}{(\omega - k_{//} v_{oj})^2} + \frac{in_{oj} q_j E_\perp k_\perp}{m_j} \frac{1}{\{(\omega - k_{//} v_{oj})^2 - \omega_{cj}^2\}}\right] \tag{11}$$

When eq. (11) is substituted into (7) and angle between $\boldsymbol{k}$ vector and z axis is θ, we obtain finally dispersion relation

$$1 = \sum_j \frac{1}{1 - \left[\frac{\kappa T_{//}^{(j)}}{m_j} \frac{k_{//}^2}{(\omega - k_{//} v_{oj})^2} + \frac{\kappa T_\perp^{(j)}}{m_j} \frac{k_\perp^2}{\{(\omega - k_{//} v_{oj})^2 - \omega_{cj}^2\}}\right]}$$

$$\times \left[\frac{\omega_{pj}^2 \cos^2\theta}{(\omega - k_{//} v_{oj})^2} + \frac{\omega_{pj}^2 \sin^2\theta}{(\omega - k_{//} v_{oj})^2 - \omega_{cj}^2}\right] \tag{12}$$

$$\omega_{pj}^2 = \frac{4\pi n_{oj} q_j^2}{m_j} \tag{13}$$

3. Dispersion relation of ion beam-magnetoplasma system

In the case of ion beam-plasma system, of which experiment interests us, if suffix e,i, and bi are used for plasma electron, ion and beam ion respectively and only electron temperature is taken into account, general dispersion relation (12) reduces to

$$1 = \frac{1}{1 - \left[\frac{\kappa T_{//}^{(e)}}{m_e} \frac{k_{//}^2}{\omega^2} + \frac{\kappa T_\perp^{(e)}}{m_e} \frac{k_\perp^2}{\omega^2 - \omega_{ce}^2}\right]} \left\{\frac{\omega_{pe}^2 \cos^2\theta}{\omega^2} + \frac{\omega_{pe}^2 \sin^2\theta}{\omega^2 - \omega_{ce}^2}\right\}$$

$$+ \frac{\omega_{pi}^2 \cos^2\theta}{\omega^2} + \frac{\omega_{pi}^2 \sin^2\theta}{\omega^2 - \omega_{ci}^2} + \frac{\omega_{bi}^2 \cos^2\theta}{(\omega - k_{//} v_0)^2} + \frac{\omega_{bi}^2 \sin^2\theta}{(\omega - k_{//} v_0)^2 - \omega_{ci}^2} \tag{14}$$

When we take $T_{//}^{(e)}=T_{\perp}^{(e)}\equiv T$ and attach factor 3 of adiabatic change to T, this equation coincides with Hasegawa's[5] formula.

In the case of $\omega_{ce}\gg\omega$, we neglect the terms including ω_{ce} and obtain

$$1=\frac{\omega_{pe}^2\cos^2\theta}{\omega^2-\frac{\kappa T_{//}^{(e)}}{m_0}k_{//}^2}+\frac{\omega_{pi}^2\cos^2\theta}{\omega^2}+\frac{\omega_{pi}^2\sin^2\theta}{\omega^2-\omega_{ci}^2}$$

$$+\frac{\omega_{bi}^2\cos^2\theta}{(\omega-k_{//}v_0)^2}+\frac{\omega_{bi}^2\sin^2\theta}{(\omega-k_{//}v_0)^2-\omega_{ci}^2} \qquad (15)$$

4. Comparison with the method of Perulli et al.

According to Stix[3], dispersion relation for the electrostatic approximation is as follows

$$1+\sum_j\sum_{n=-\infty}^{\infty}\left[\frac{\omega_p^2me^{-\lambda}I_n(\lambda)}{k^2\kappa T_{\perp}}A_n\right]_j=0 \qquad (16)$$

On the other hand, if we take $T_{//}=0$ in eq. (15), then we obtain dispersion relation for ion beam-cold plasma system. In this equation Perulli et al.[4] substitute for the electron term $-\frac{\omega_{pe}^2\cos^2\theta}{\omega^2}$ the electron term

$$A=\sum_{n=-\infty}^{\infty}\frac{\omega_{pe}^2m_ee^{-\lambda_e}I_n(\lambda_e)}{k^2\kappa T_{\perp}^{(e)}}A_{ne} \qquad (17)$$

in the Stix's equation (16). We will make comparison of both electron terms in the two limitting cases.

i) $v_{th}^{(e)}\approx\left(\frac{\kappa T_{//}^{(e)}}{m_e}\right)^{\frac{1}{2}}\gg\frac{\omega}{k_{//}}$

$v_{th}^{(e)}$ is mean thermal velocity of electrons. We consider that $\left(\frac{\kappa T_{//}^{(e)}}{m_e}\right)^{\frac{1}{2}}$ and $\left(\frac{\kappa T_{\perp}^{(e)}}{m_e}\right)^{\frac{1}{2}}$ are quantities of about same order. Furthermore, we assume that $\omega_{ce}\gg\omega$ and $\frac{\omega_{ce}}{k_{//}}$, $\frac{\omega_{ce}}{k_{\perp}}\gg v_{th}^{(e)}$. In this case, $\lambda_e\ll1$ and we take up only o-th order and first order terms of λ_e in eq. (16). Then we otain

$$A=\frac{\omega_{pe}^2m_e}{k^2\kappa T_{//}^{(e)}}-\lambda_e\frac{\omega_{pe}^2m_e}{k^2\kappa T_{\perp}^{(e)}}\left\{\frac{T_{\perp}^{(e)}}{T_{//}^{(e)}}-1\right\} \qquad (18)$$

We notice that Landau or cyclotron damping term does not appear in eq. (18). The term of λ^0 coincides with the electron term of eq. (15) in the same approximation.

ii) $v_{th}^{(e)} \ll \frac{\omega}{k_{//}}$

In this case also, we assume that $\omega_{ce} \gg \omega$. Then we obtain

$$A = -\frac{k_{//}^2}{k^2}\frac{\omega_{pe}^2}{\omega^2} + \lambda_e\left\{\frac{k_{//}^2}{k^2}\frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{pe}^2\, m_e}{k^2\kappa T_{\perp}^{(e)}}\right\}$$

$$+ i\pi^{\frac{1}{2}}(1-\lambda_e)\frac{\omega_{pe}^2\, m_e}{k^2\kappa\, T_{//}^{(e)}}\frac{\omega}{k_{//}}\left(\frac{m_e}{2\,\kappa T_{\perp}^{(e)}}\right)^{\frac{1}{2}} exp\left[-\left\{\frac{\omega}{k_{//}}\left(\frac{m_e}{2\,\kappa T_{//}^{(e)}}\right)^{\frac{1}{2}}\right\}^2\right] \quad (19)$$

The term of λ_e^0 is

$$-\frac{\omega_{pe}^2\, cos^2\theta}{\omega^2} + i\pi^{\frac{1}{2}}\frac{\omega_{pe}^2\, m_e}{k^2\kappa\, T_{//}^{(e)}}\frac{\omega}{k_{//}}\left(\frac{m_e}{2\,\kappa T_{\perp}^{(e)}}\right)^{\frac{1}{2}} exp\left[-\left\{\frac{\omega}{k_{//}}\left(\frac{m_e}{2\,\kappa T_{//}^{(e)}}\right)^{\frac{1}{2}}\right\}^2\right] \quad (20)$$

which includes Landau or cyclotron damping term other than the electron term of eq. (15) in the same approximation, i.e., the electron term of cold plasma.

5. Conclusion

It was shown that in the case of ion beam-plasma system, hydrodynamic-approximation dispersion relation leads to the same results in some approximation as the method used by Perulli et al. for convenience, except Landau or cyclotron damping term. Furthermore, hydrodynamic-approximation dispersion relation formally deduced here allows us to take into account of finite temperatures of each species. This dispersion relation gives a substitute, being able to be easily treated, of the strict theories by Sitenko and Stepanov [1], Stepanov and Kitsenko [2], and Stix [3].

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