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On the Limit Sets of Inversion Groups

Toshio SHIMAZAKI* and Nobutaka OGI**

Department of Mathematics, Faculty of Science, Kanazawa University

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1. Introduction. The investigations on the limit sets of Kleinian groups, that is, discontinuous groups of linear transformations on the extended complex plane, have been worked by P. J. Myrberg and others, and recently by L. V. Ahlfors, T. Akaza and A. F. Beardon. It seems that most of their results are concerned with the size of such sets and there are only a few informations about the shape of such sets. We pay attention to two results about the shape. One of them is that, roughly speaking, the limit set of a finitely generated non-cyclic Schottky group is a spherical Cantor set ([4], 109-111). Another one is a Fricke-Klein's example of a Kleinian group, whose limit set is a closed Jordan curve ([3], 80-90).

In [1] Beardon investigated the property of the limit set of an inversion group on m-dimensional euclidean space R^m and showed that the limit set of such a group is a spherical Cantor set. In the present paper we shall give some results on the inversion groups. Theorem 1 is ones corresponding to the Fricke-Klein's example.

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2. Definitions and notations. Let X be a Hausdorff space, and G be a group of autohomeomorphisms of X.

A point $x \in X$ is called a *limit point* of G if there are a point $x_0 \in X$ and an infinite sequence $\{g_n\}_{n=1}^{\infty}$ of distinct elements of G such that $g_n(x_0) \to x$ as $n \to \infty$. The *limit set* of G, denoted by $L(G)$, is the totality of limit points of G. The complement of $L(G)$, that is, the set $X - L(G)$ is called the *regular set* of G and is denoted by $R(G)$. G is said to be a *discontinuous group* on X if $R(G)$ is not empty.

It is easily seen that the sets $L(G)$ and $R(G)$ are invariant under G, and that $L(G)$ is closed and consequently $R(G)$ is open. Furthermore, it follows immediately from the definition of a limit set that if G_1 is any subgroup of G, then it holds

* *Section of Applied Mathematics, Faculty of Technology, Kanazawa University.*
** *Gifu University.*

$L(G_1) \subset L(G)$. For details of discontinuous groups, we refer to Lehner's book [2].

Next we shall define an inversion group. Let S_j $(j=1,\cdots,p\,;p\geq 2)$ be an m-dimensional closed ball [1] in R^m with center $a(j)$ and radius $r(j)$. We shall consider the group generated by the inversions I_j with respect to the sphere $\mathring{S}_j$ $(j=1,\cdots,p)$, and denote it by $G=\langle I_1,\cdots,I_p\rangle$. It is easily seen that G is a discontinuous group on the extended m-dimensional euclidean space $X=R^m\cup\{\infty\}$ with the complement of $\bigcup_{j=1}^{p} S_j$ as a fundamental region. The group $G=\langle I_1,\cdots,I_p\rangle$ is called a *finitely generated inversion group* (briefly, an *inversion group*) in the case that any two of $\{S_j\}_{j=1}^{p}$ have no interior point in common [2]. Then $\{S_j\}_{j=1}^{p}$ are called the *generating balls* of G.

We denote by $I_j \circ I_k$ the composition of inversions I_j and I_k, that is, $I_j\circ I_k(x)=I_j(I_k(x))$.

Then any element g $(\neq 1)$ of G has the form

$$g=I_{j_1}\circ I_{j_2}\circ\cdots\circ I_{j_n} \tag{1}$$

where

$$j_k=1,\cdots,p \quad \text{for } k=1,\cdots,n, \text{ and } \quad j_k\neq j_{k+1} \quad \text{for } k=1,\cdots,n-1. \tag{2}$$

For convenience of the discussion we introduce some notations. For any positive integer n, we denote by $[n]$ the set of all sequences $(j_1,\cdots,j_n)$ satisfying (2).

For any $(j_1,\cdots,j_n)\in[n]$, $n\geq 2$, we put

$$S(j_1,\cdots,j_n)=I_{j_1}\circ\cdots\circ I_{j_{n-1}}(S_{j_n}), \tag{3}$$

namely $S(j_1,\cdots,j_n)$ is the image ball of S_{j_n} by $g=I_{j_1}\circ\cdots\circ I_{j_{n-1}}$, and especially for $n=1$, $S(j)=S_j$ $(j=1,\cdots,p)$ are generating balls of G.

Lastly we define

$$F_n=\bigcup_{(j_1,\cdots,j_n)\in[n]} S(j_1,\cdots,j_n) \qquad \text{for } n=1,2,\cdots, \tag{4}$$

where the union is taken over all of $[n]$.

It follows immediately from the definition that for any positive integer n,

$$S(j_1,\cdots,j_n)\supset S(j_1,\cdots,j_n,j_{n+1}) \tag{5}$$

for $\quad j_{n+1}=1,\cdots,p$ and $j_{n+1}\neq j_n$,

and

$$F_n\supset F_{n+1}. \tag{6}$$

3. Lemmas. Following two lemmas are used in the proofs of theorems in the next section. These were proved by Beardon [1] in the case of generating balls being mutually disjoint.

At first we have the following

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1) In the present paper, a ball means always an m-dimensional *closed* ball, i.e. $S_j=\{x : |x-a(j)|\leq r(j)\}$, and a sphere means an $(m-1)$-dimensional sphere i. e. $\mathring{S}_j=\{x : |x-a(j)|=r(j)\}$.

2) cf. Beardon [1].

LEMMA 1. *Let G be the inversion group defined in the above. Denote by $r(j_1,\cdots,j_n)$ the radius of $S(j_1,\cdots,j_n)$. Then it holds*

$$r(j_1,\cdots,j_n)\to 0 \quad as\ n\to\infty.$$

Proof. Putting

$$r_n=\max\{r(j_1,\cdots,j_n)\ ;\ (j_1,\cdots,j_n)\in[n]\}\ ,$$

by (5) and (6) we have that $\{r_n\}_{n=1}^{\infty}$ is a strictly monotone decreasing sequence of positive numbers. Suppose that

$$\lim_{n\to\infty} r_n=r>0.$$

Then we can determine a positive integer N depending only on any given ε (>0) so that it holds

$$(7)\qquad r<r_n<r+\varepsilon$$

for all $n\geqq N$.

Next choose a $(j_1,\cdots,j_{N+1})\in[N+1]$ such that $r(j_1,\cdots,j_{N+1})=r_{N+1}$, and fix it. Then by the elementary geometrical properties of inversions, we have

$$\frac{r_{N+1}}{r_N}\leqq\frac{r(j_1,\cdots,j_{N+1})}{r(j_2,\cdots,j_{N+1})}=\frac{a(j_1)-a(j_1\cdots,j_{N+1})}{a(j_1)-a(j_2,\cdots,j_{N+1})}$$

$$\leqq\frac{r(j_1)-r(j_1,\cdots,j_{N+1})}{r(j_1)+r(j_2,\cdots,j_{N+1})}<1-\frac{r}{r_1}$$

where $a(j_1,\cdots,j_{N+1})$ denotes the center of $S(j_1,\cdots,j_{N+1})$ and $r_1=\max_{1\leqq j\leqq p} r(j)$.

Noting that $0<\frac{r}{r_N}<1$, we have

$$\frac{r_{N+1}}{r_N}<1-\frac{r^2}{r_1r_N}$$

and hence

$$r_{N+1}<r_N-\frac{r^2}{r_1}.$$

It follows from (7) that for any ε with $0<\varepsilon<\frac{r^2}{r_1}$, it holds

$$r_{N+1}<r_N-\varepsilon<r.$$

This contradicts with the fact that r is the limit value of monotone decreasing sequence. Therefore we have $r=0$, and thus we have proved our lemma.

Secondly we shall prove the following

LEMMA 2. $$L(G)=\bigcap_{n=1}^{\infty}F_n.$$

Proof. Noting that the complement of F_1 is a fundamental region of G, we obtain that $L(G)\subset F_1$. Suppose that $L(G)\subset F_k$. Then we have from the invariance of $L(G)$ under G

$$L(G)=\bigcap_{j=1}^{p} I_j(L(G))\subset\bigcap_{j=1}^{p} I_j(F_k)=F_{k+1}.$$

By induction for k we have $L(G)\subset F_n$ $(n=1,2,\cdots)$, and hence it holds

$$L(G)\subset\bigcap_{n=1}^{\infty}F_n.$$

To obtain the converse inclusion, take any point x contained in $\bigcap_{n=1}^{\infty} F_n$. Then there exists from (4) and (5) an infinite sequence $\{S(j_1,\cdots, j_n)\}_{n=1}^{\infty}$ of shrinking balls $S(j_1,\cdots, j_n)$ containing x. We have from Lemma 1

$$(8) \qquad \bigcap_{n=1}^{\infty} S(j_1,\cdots, j_n)=x.$$

Now putting

$$g_n=I_{j_1}\circ\cdots\circ I_{j_n} \quad (n=1,2,\cdots),$$

we get

$$g_n(x_0)=I_{j_1}\circ\cdots\circ I_{j_n}(x_0)\in I_{j_1}\circ\cdots\circ I_{j_{n-1}}(S_{j_n})=S(j_1,\cdots, j_n),$$

for any x_0 belonging to the complement of $F_1=\bigcup_{j=1}^{p} S_j$.

Consequently we obtain

$$\lim_{n\to\infty} g_n(x_0)\in\bigcap_{n=1}^{\infty} S(j_1,\cdots, j_n),$$

and hence by (8)

$$x\in L(G).$$

This completes the proof.

4. **Theorems.** Before we state our theorems, we shall give another definition. We shall say that a finite set of balls $\{S_j\}_{j=1}^{p}$ *forms a closed chain* in the case that the set has the following two properties: (i) any two of $\{S_j\}_{j=1}^{p}$ have no interior point in common, and (ii) S_j has a common boundary point with S_{j+1} for $j=1,\cdots, p$ (where $S_{p+1}=S_1$).

Making use of Fricke-Klein's method ([3], 80-90), we have the following

THEOREM 1. *If a set of generating balls $\{S_j\}_{j=1}^{p+1}$ $(p\geqq 2)$ forms a closed chain, then the limit set $L(G)$ of the inversion group $G=\langle I_1,\cdots, I_{p+1}\rangle$ is a simple closed curve in R^m, namely $L(G)$ is a one-to-one continuous image of a unit circle.*

Proof. From our assumption, it is clear that the set of balls $\{S(j_1,\cdots,j_n)\ ;\ (j_1,\cdots, j_n)\in[n]\}$, denoted by $\mathcal{S}_{n-1}$, forms a closed chain for $n=1,2,\cdots$. To prove our theorem, we shall renumber the index $(j_1,\cdots, j_n)$ of $S(j_1,\cdots, j_n)$ and denote the renumbering balls by $S^*(i_0, i_1,\cdots,i_{n-1})$. Noting that $S(j_1,\cdots, j_n, j_{n+1})\subset S(j_1,\cdots, j_n)$ for $j_{n+1}\neq j_n$, we define it by induction for n. First we set

$$S^*(i_0)=S(i_0) \quad \text{for } i_0=1,\cdots, p+1.$$

Next assuming that we can define the renumbering balls $S^*(i_0, i_1,\cdots,i_{n-1})$ for any element of $\mathcal{S}_{n-1}$, where $i_0=1,\cdots, p+1$ and $i_k=1,\cdots, p$ for $k=1,\cdots,n-1$, we shall define $S^*(i_0, i_1,\cdots, i_{n-1}, i_n)$ for any element of $\mathcal{S}_n$. For a fixed $(i_0,i_1,\cdots, i_{n-1})$, we define $S^*(i_0, i_1,\cdots, i_{n-1}, i_n)$, $i_n=1,\cdots, p$, belonging to $\mathcal{S}_n$ such that

i) $S^*(i_0, i_1, \cdots, i_{n-1}, i_n) \subset S^*(i_0, i_1, \cdots, i_{n-1})$ for $i_n = 1, \cdots, p$,

ii) $S^*(i_0, i_1, \cdots, i_{n-1}, j)$ circumscribes with $S^*(i_0, i_1, \cdots, i_{n-1}, j+1)$ for $j = 1, \cdots, p-1$

and

iii) $S^*(i_0, i_1, \cdots, i_{n-1}, 1)$ circumscribes to $S^*(i_0, i_1, \cdots, i_{n-1}-1)$ and $S^*(i_0, i_1, \cdots, i_{n-1}, p)$ does so to $S^*(i_0, i_1, \cdots, i_{n-1}+1)$.

Now let us prove our theorem. In virtue of Lemma 2 and the above definition of renumbering, we have that for any $x \in L(G)$ there exists uniquely a sequence of closed balls $\{S^*(i_0, i_1, \cdots, i_n)\}_{n=1}^{\infty}$ such that

(9) $$S^*(i_0) \supset S^*(i_0, i_1) \supset \cdots \supset S^*(i_0, i_1, \cdots, i_n) \supset \cdots,$$

and $$x = \bigcap_{n=0}^{\infty} S^*(i_0, \cdots, i_n).$$

Consequently we can associate every $x \in L(G)$ with a uniquely determined infinite sequence $(i_0, i_1, \cdots, i_n, \cdots)$ of positive integers where $i_0 = 1, 2, \cdots, p+1$ and $i_n = 1, \cdots, p$ for any $n \neq 0$.

Next we consider the totality of such sequences $(i_0, i_1, \cdots, i_n, \cdots)$ and define an equivalence relation on the set. We say that $(i_0, i_1, \cdots i_n, \cdots)$ is *equivalent* to $(i_0', i_1', \cdots, i_n', \cdots)$ if there exists a positive integer N such that

i) $i_n = i_n'$ for $n = 0, 1, \cdots, N-1$,

ii) $i_N' = i_N - 1$

and

iii) $i_n = 1$, $i_n' = p$ for any $n \geqslant N+1$.

Denoting the totality of the equivalence classes by $[P]$, we have by (9) that $L(G)$ corresponds to $[P]$ in one-to-one manner by the mapping $\varphi : x \to (i_0, i_1, \cdots, i_n, \cdots)$. Moreover $[P]$ also corresponds to a unit circle $C = \{e^{i\theta};\ 0 \leqslant \theta < 2\pi\}$ in one-to-one manner by the mapping $\psi : (i_0, i_1, \cdots, i_n, \cdots) \to e^{i\theta}$, where

$$\theta = \frac{2\pi}{p+1}\{(i_0 - 1) + \sum_{n=1}^{\infty} (i_n - 1)\, p^{-n}\}.$$

Consequently the composition Φ of φ and ψ is a one-to-one onto mapping of $L(G)$ to C. In virtue of Lemma 1 we can easily show that $\Phi = \psi \circ \varphi$ is continuous and hence so is the inverse of Φ. This completes the proof.

Remark. It is easily shown that $L(G)$ is contained in an $(m-1)$-dimensional plane Π (an $(m-1)$-dimensional sphere Σ) in the case that all of the generating balls in the above theorem are orthogonal to Π (Σ respectively).

Beardon [1] proved that *$L(G) \cap S_j$ is a spherical Cantor set for $j = 1, \cdots, p+1$ if $\{S_j\}_{j=1}^{p+1}$ are mutually disjoint.* As an application of Theorem 1 it follows immadiately that there exists a simple closed curve passing through all points of $L(G)$.

Namely we have

COROLLARY. *Assume that the generating balls* $\{S_j\}_{j=1}^{p+1}$ *are mutually disjoint. Then there is a simple closed curve* γ *in* R^m *such that* $\gamma \supset L(G)$.

Proof. By adding to $\{S_j\}_{j=1}^{p+1}$ a new finite number of suitable balls $\{S'_j\}_{j=1}^{q}$, we can obtain a set of generating balls $\{S_j^*\}_{j=1}^{p+q+1}$ that contains the set $\{S_j\}_{j=1}^{p+1}$ and forms a closed chain. By means of Theorem 1 we have a simple closed curve $L(G^*)$, where $G^* = \langle I_1^*, \cdots, I_{p+q+1}^* \rangle$. That $G \subset G^*$ implies that $L(G) \subset L(G^*)$. Setting $\gamma = L(G^*)$, we have our corollary.

Lastly we consider the case of $m=2$, that is, $X = R^2 \cup \{\infty\}$. Then a closed Jordan curve $L(G)$ in Theorem 1 devides X into two components D and D^*, where $D^* \ni \infty$. We have the following

THEOREM 2. *Assume that a set of generating balls (namely, closed discs)* $\{S_j\}_{j=1}^{p}$ $(p \geqq 3)$ *forms a closed chain. Then the interior domain* D *and the exterior domain* D^*, *bounded by* $L(G)$ *are invariant under* $G = \langle I_1, \cdots, I_p \rangle$, *that is, it holds* $g(D) = D$ *and* $g(D^*) = D^*$ *for any* $g \in G$.

Proof. Since the set $\{S(j_1, \cdots, j_n)\ ;\ (j_1, \cdots, j_n) \in [\![n]\!]\}$ forms also a closed chain, the complement of F_n consists of two components D_n and D_n^*, where $D_n^* \ni \infty$. We shall prove that D is invariant under G. It follows from Lemma 2 that

$$D_n \subset D_{n+1} \quad \text{for } n = 1, 2, \cdots,$$

and

$$D = \bigcup_{n=1}^{\infty} D_n.$$

Consequently we have

$$D = D_1 + \sum_{n=1}^{\infty} (D_{n+1} - D_n), \text{ (disjoint union).} \tag{10}$$

To prove our theorem, it suffices to show that

$$I_j(D) \subset D$$

for $j = 1, \cdots, p$, because by (1) $g = I_{j_1} \circ \cdots \circ I_{j_n}$ for any $g \in G$ with $g \neq 1$ and $I_j \circ I_j = 1$. Let x be any point of D, and then by (10) $x \in D_1$ or else $x \in D_{n+1} - D_n$ for some n. It is clear that $I_j(x) \in D_2 - D_1$ for $j = 1, \cdots, p$ when $x \in D_1$. If x be contained in $D_{n+1} - D_n$ for some n, then there is a $(j_1, \cdots, j_n) \in [\![n]\!]$ such that

$$x \in S(j_1, \cdots, j_n) \text{ and } x \notin S(j_1, \cdots, j_n, j_{n+1}) \text{ for any } j_{n+1} \neq j_n.$$

By use of the fact (which follows from (3)) that

$$I_j(S(j_1, \cdots, j_n)) = S(j, j_1, \cdots, j_n) \quad \text{for } j \neq j_1,$$

and

$$I_j(S(j_1, \cdots, j_n)) = S(j_2, \cdots, j_n) \quad \text{for } j = j_1,$$

we obtain that

$$I_j(x) \in D_{n+2} - D_{n+1} \qquad \text{for } j \neq j_1,$$

and

$$I_j(x) \in D_n - D_{n-1} \qquad \text{for } j = j_1,$$

and hence in every case it holds $I_j(x) \in D$ for $j = 1, \cdots, p$. Therefore D is invariant under G and so is D^*. This completes the proof.

Note. Recently, we know the paper by A. Leutbecher [5], one of whose results is closely related to Theorem 1 of ours in the case of $m = 2$.

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