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On Normality of a Family of Analytic Sets

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Introduction. In 1957, Lelong [6] defined a current associated with an analytic set in a domain of C^n as follows.

Let D be a domain in C^n, and V a purely k-dimensional analytic set in D. For a 2k-form in D, we put $t_V(\varphi)=\int_{V^*}\varphi$, where V^* is the set of all ordinary points of V. Then t_V is a closed current which is continuous of order zero in $\Omega=D-(V-V^*)$. Also it is shown that there is a norm-preserving extension of t_V to D, which is called a simple extension of t_V.

On the other hand, in 1964 Bishop [2] gave the following

Theorem. (Bishop) *Let $\{V_i\}$ be a sequence of purely k-dimensional analytic sets in a domain D of G^n which converges to a relatively closed set V in D in the sense of Hausdorff. If the 2k-dimensional volumes of V_i are uniformly bounded, then V is also an analytic set in D.*

In the present paper we shall prove the following two theorems

Therem 1. *Let F be a family of purely k-dimensional analytic sets in D. If the family of currents associated with F is bounded, then F is normal i. e., every sequence $\{V_i\}$ of analytic sets $V_i\in F$ contains a subsequence which converges to an analytic set in D.*

Theorem 2. *Suppose a sequence $\{V_i\}$ of purely k-dimensional analytic sets in D converges to a purely k-dimensional analytic set in D. If the union $\cup V_i$ is also a purely k-dimensional analytic set in D, then the sequence of currents associated with the sequence $\{V_i\}$ is bounded.*

1. Let X be a compact metric space. We shall denote by Comp(X) the set of all closed subsets of X. The Hausdorff metric in Comp(X) is defined by $D(S_1,S_2)=\max_{t_1\in S_1} d(t_1,S_2)+\max_{t_2\in S_2} d(t_2,S_1)$ for $S_2,S_2\in$ Comp(X), where $d(t,S)=\min_{s\in S} d(t,s)$, $d(t,s)$ is the distance between t and s. Then it is well-known that the space Comp (X) is

itself a compact metric space.[1]

Definition 1. Let {Si} be a sequence of closed subsets of a metric space X. We say that this sequence converges to a closed set S in X in the sense of Hausdorff, if for every compact subset K of X, $\{S_i \cap K\}$ is a convergent sequence in Comp(K) and $S = \bigcup_{K \subset X} \lim_{i \to \infty} (S_i \cap K)$.

Remark. It is easily seen that for any point $P \in \lim_{i \to \infty} (S_i \cap K)$, there is a point sequence $\{P_i\}$, of points $P_i \in S_i \cap K, \cdots\cdots$ such that $P_i \to P$. Conversely if there exists $\lim_{i \to \infty} (S_i \cap K)$, then for $P_i \in S_i \cap K$ with $P_i \to P$, it holds $P \in \lim_{i \to \infty} (S_i \cap K)$.

LEMMA 1. *If X is a σ-compact metric space, then for every sequence $\{S_i\}$ of closed subsets of X, there is a subsequence which converges to a closed set S in the sence of Hausdorff.*

Proof. Since X is σ-compact, we can exhaust X by an increasing sequence of compact subsets K_i of X. Let $X = \cup K_i, K_i \subset K_{i+1}$. Since Comp($K_1$) is compact, there is a subsequence $S_1^{(1)}, S_2^{(1)}, \cdots$ of $\{S_i\}$ such that the sequence $\{S_i^{(1)} \cap K_1\}$ converges. Further, there is a subsequence $S_1^{(2)}, S_2^{(2)}, \cdots$ of $\{S_i^{(1)}\}$ such that the sequence $S_i^{(2)} \cap K_2$ converges. We shall show that the diagonal sequence $\{S_i^{(i)}\}$ conveges in a sence of Hausdorff. Since there exists $\lim_{i \to \infty} (S_i^{(i)} \cap K)$, it is sufficient only to show that $S = \bigcup_{K \subset X} \lim_{i \to \infty} (S_i^{(i)} \cap K)$ is a closed subset of X. Let $P_i \in S$ with $P_i \to P$ in X, then for some compact set $K^{(j)}$ it holds $P_j \in \lim_{i \to \infty} (S_i^{(i)} \cap K^{(j)})$. Then by the remark after Definition 1, there is a sequence of points $Q_k^{(j)} \in S_k^{(k)} \cap K^{(j)}$ with $Q_k^{(j)} \to P_j$ ($k \to \infty$). For the positive number $\varepsilon_l = \frac{1}{2^l}$, we take a point $Q_{k_l}^{(j_l)}$ such that d $(Q_{k_l}^{(j_l)}, P) < \varepsilon_l$ and $k_l > k_{l'}$ if $l > l'$. Then evidently $Q_{k_l}^{(j_l)} \to P$ $(l \to \infty)$ and there is a compact set K containing P such that $Q_{k_l}^{(j_l)} \in S_{k_l}^{(k_l)} \cap K$ for sufficiently large l.

Thus $P \in \lim_{l \to \infty} (S_{k_l}^{(k_l)} \cap K) = \lim_{i \to \infty} (S_i^{(i)} \cap K)$. *Q.E.D.*

Let V^* be the set of all ordinary points of a purely k-dimensional analytic set V in the domain D of C^n and let $K(\Omega)$ be the set of all differential forms of total degree $2k$ on Ω, such that the coefficients are continuous functions with compact supports. We define the norm $\| \varphi \|$ of a form φ as the maximum of absolute values of the coefficients, and for an open set G in D, the norm $\| t_V \|_G$ of a continuous linear form t_V with respect to G is defined by $\| t_V \|_G = \sup_{\substack{\| \varphi \| \leq 1 \\ \varphi \in K(G \cap \Omega)}} | t_V(\varphi) |$.

Let $\Omega = D - (V - V^*)$, where D is a domain in C^n. Then a norm-preserving extension $\tilde{t}_V$ of t_V to D is given by $\tilde{t}_V(\varphi) = \lim_{l \to 0} t_V [(1 - \alpha_l)\varphi]$ for $\varphi \in K(D)$, where the functions α_l are given as follows.

Let K be a support of φ and $E_K = K \cap (V - V^*)$. For open neighbourhoods ω_l, ω_l'

(1) Generally, if X is a compact metric space, Comp(X) is a compact metrik space for any metrik on Comp(X). See Alexandorof Hoph: Topologie, p. 115.

of E_K such that $\omega_l \supseteqq \omega_l'$, $0 < l \leqq 1$, and $\omega_l \to E_K$ $(l \to 0)$, we define the C^∞-function α_l satisfying the following conditions:

(i) $\alpha_l = 1$ on ω_l'

$\alpha_l = 0$ outside of ω_l and $0 \leqq \alpha_l \leqq 1$.

(ii) For $l < l'$, $\alpha_l \leqq \alpha_{l'}$ at each point.

The extension $\widetilde{t}_V$ of t_V is called the *current associated with* V.(2)

LEMMA 2. *Let D be a domain in C^n and V a purely k-dimensional analytic set of D. For every relatively compact open ball B in D, we have $Vol_{2k}(V \cap B) \leqq \| \widetilde{t}_V \|_B$, where $Vol_{2k}\ V$ is the $2k$-dimensional volume of V.*

Proof. For simplicity, we assume that the ball B is centered at the origin o of C^n with radius $R; B = B(o; R)$. For any r, r' with $0 < r < r' < R$, we define a C^∞-function η_r $(0 \leqq \eta_r \leqq 1)$ by

$$\eta_r = \begin{cases} 1 & \text{on } B(0:r) \\ 0 & \text{outside of } B(0:r'), \end{cases}$$

If we denote by β_k the 2k-dimensional volume element of C^n i.e., if $\beta_k = \frac{1}{k!}\left[\frac{\sqrt{-1}}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j\right]^k$,(3) then we have

$$\| \widetilde{t}_V \|_B \geqq \widetilde{t}_V(\eta_r \beta_k) = \lim_{l \to 0} \int_{V^*} (1 - \alpha_l) \eta_r \beta_k$$

$$\geqq \lim_{l \to 0} \int_{V^* \cap B(O:r)} (1 - \alpha_l) \beta_k = \int_{V^* \cap B(O:r)} \beta_k - \lim_{l \to 0} \int_{V^* \cap B(O:r)} \alpha_l \beta_k$$

Since $\alpha_l \to 0$ on V^* as $l \to 0$, we have $\lim_{l \to 0} \int_{V^* \cap B(O:r)} \alpha_l \beta_k = 0$. Thus we obtain the inequality $\| \widetilde{t}_V \|_B \geqq \mathrm{Vol}_{2k}(V \cap B(0:r))$. Since the right side increases with $r \to R$, we have $\| \widetilde{t}_V \|_B \geqq \mathrm{Vol}_{2k}(V \cap B)$. $Q \cdot E \cdot D$.

2. Before we prove our theorem, we shall recall following

Definition 3. A subset Φ of $K(D)$ is said to be bounded in $K(D)$ if the norms of forms in Φ are uniformly bounded and their supports are contained in a compact set of D.

Definition 4. A family B of currents on D is said to be bounded if for every bounded subset Φ of $K(D)$, $\sup_{\substack{t \in B \\ \varphi \in \Phi}} |t(\varphi)| < \infty$. By the above two lemmas and by Bishop's Theorem, we obtain the following

THEOREM 1. *Let F be a family of purely k-dimensional analytic sets in D. If*

(2) $\widetilde{t}_V$ is independent of the choice of α_e.

(3) We denote by φ^k the k-th exterior power of φ, where φ is a differential form. See [4] Chapter I,

the family of currents associated with F is bounded, then F is normal, i. e., *for every sequence* $\{V_i\}$ *in F, there is a subsequence which converges to an analytic set in D.*

Remark. The converse of Bishop's Theorem is not true. For example let D be an open ball of radius 3 with center at the origin in C^2. The sequence $\{A_n\}$ of purely 1-dimensional analytic sets in D given by

$$A_n = \bigcup_{i=0}^{n-1}\left\{ z\in D \ ;\ z_1 = 1 + \frac{1}{2} + \cdots\cdots + \frac{1}{2^{\frac{1}{2}n(n-1)+i}} \right\}$$

converges to the purely 1-dimensional analytic set $A=\{z\in D\ ;\ z_1=2\}$ in D. But we have evidently $\mathrm{Vol}_2 A_n \to \infty$ as $n\to\infty$.

PROPOSITION 1. *For any relatively compact domain G of a domain D in* C^n, *there is a constant c depending only on the pure dimension k of an analytic set V such that* $\|\tilde{t}_V\|_G \leqq c Vol_{2k}(V\cap G)$.

Proof. Since $\|\tilde{t}_V\|_G = \|t_V\|_G$,(4) it is sufficient only to prove the inequality $\|t_V\|_G \leqq c\mathrm{Vol}_{2k}(V\cap G)$. With every point of $G\cap V^*$, we associate a neighbourhood U such that $U\cap V^*$ is contained in a coordinate neighbourhood of the manifold V^*. Also with every point of $G\cap\Omega - V^* = G - V$, we associate the relatively compact neighbourhood in $G-V$. Then we obtain an open covering $\{U_j\}$ of $G\cap\Omega$. By taking the refinement of $\{U_j\}$ if necesssary, we may assume that this covering is locally finite. Let $\Sigma\gamma_j$ be a partition of unity subordinate to this covering. For $\varphi\in K(\Omega\cap G)$, $\varphi=\Sigma\varphi\gamma_j$, which is essentially a finite sum since the support of φ is compact. Thus $\varphi = \sum_1^N \varphi\gamma_j$ and

$$t_V(\varphi) = \sum_1^N t_V(\varphi\gamma_j) = \sum_1^N \int_{V^*}\varphi\gamma_j = \sum_1^N \int_{V^*\cap U_j}\varphi\gamma_j.$$

Since $V^*\cap U_j$ is contained in a coordinate neighbourhood of the complex k-dimensional manifold V^* we may assume that $V^*\cap U_j$ is an open set in the space $C^k=\{z_{k+1}=0,\cdots\cdots z_n=0\}$. Then since φ is written as $\varphi=\varphi_I dz_1\wedge dz_2\wedge\cdots\wedge dz_k\wedge d\bar{z}_1\cdots\wedge d\bar{z}_k$ with $|\varphi_I|\leq 1$, we have $\int_{V^*\cap U_j}\gamma_j\varphi = \int_{V^*\cap U_j\cap K}\gamma_j\varphi = \left(\frac{2}{\sqrt{-1}}\right)^k \int_{V^*\cap U_j\cap K}\gamma_j\varphi_I d\omega$, where K is the support of φ and $d\omega = \left(\frac{\sqrt{-1}}{2}\right)^k dz_1\wedge dz_2\cdots\wedge dz_k\wedge d\bar{z}_1\wedge d\bar{z}_2\wedge\cdots\wedge d\bar{z}_k$ is the volume element of $V^*\cap U_j$. Thus we obtain

$$\left|\int_{V^*\cap U_j}\gamma_j\varphi\right| \leq 2^k\int_{V^*\cap U_j\cap K}|\gamma_j\varphi_I|\,d\omega \leq 2^k\int_{V^*\cap U_j\cap K}\gamma_j d\omega \quad .$$

On the other hand, let α be a C^∞-fundtion $(0\leq\alpha\leq 1)$ defined by

(4) See [6] §1.

$$\alpha = \begin{cases} 1 & \text{on } K \\ 0 & \text{outside } G', \text{ where } K \Subset G' \Subset G. \end{cases}$$

Then $\mathrm{Vol}_{2k}(V \cap G) = \int_{V^* \cap G} d\omega \geqq \int_{V^* \cap G} \alpha d\omega$. Since $\alpha d\omega \in K(G \cap \Omega)$, it holds $\alpha d\omega = \sum_1^{N'} \alpha \gamma_j d\omega$. As the support of $\alpha d\omega$ contains K, we have $N \leqq N'$.

Hence $\int_{V^* \cap G} \alpha d\omega = \sum_1^{N'} \int_{V^* \cap G \cap U_j} \alpha \gamma_j d\omega \geqq \sum_1^{N} \int_{V^* \cap G \cap U_j} \alpha \gamma_j d\omega \geqq \sum_1^{N} \int_{V^* \cap G \cap K} \gamma_j d\omega \geqq \frac{1}{2^k} \sum_1^{N} \left| \int_{V^* \cap U_j \cap K} \gamma_j \varphi \right| \geqq \frac{1}{2^k} \, | t_V(\varphi) |$.

Thus we obtain $\mathrm{Vol}_{2k}(V \cap G) \geqq \frac{1}{2^k} \, | t_V(\varphi) |$

These inequalities hold for any $\varphi \in K(G \cap \Omega)$, and hence $2^k \mathrm{Vol}_{2k}(V \cap G) \geqq \| t_V \|_G$. $Q \cdot E \cdot D$.

THEOREM 2. *Let a sequence $\{V_i\}$ of purely k-dimensional analitic sets in D converges to a purely k-dimensional analytic set in D. If the union $\cup V_i$ is also a purely k-dimensional analytic set in D, then the set of currents associated with the family $\{V_i\}$ is bounded.*

Proof. By the definition of the boundedness of the set of currents, it is sufficient only to prove that $\{ \| t_{V_i} \|_G \}$ is bounded for any relatively compact domain G in D. Since $\mathrm{Vol}_{2k}(\cup V_j \cap G) < \infty$, $\mathrm{Vol}_{2k}(V_j \cap G)$ is bounded. Then by Proposition 1, $\{ \| \tilde{t}_{V_i} \|_G \}$ is bounded. $Q \cdot E \cdot D$.

Remark. Let F be a family of purely k-dimensional analytic sets. We assume that whenever a sequence $\{V_i\}$ in F converges to a purely k-dimensional analytic set, the union $\cup V_i$ is also purely k-cimensional analytic set. Then the family of currents associated with F is bounded.

Proof. If not true, there should be a sequence $\{V_k\}$ in F such that $\| \tilde{t}_{V_k} \|_G \geqq k$. But by Lemma 1, $\{V_k\}$ contains a convergent subsequence $\{V_{k_l}\}$, and hence from Theorem 2, the set of current $\{\tilde{t}_{V_{K_l}}\}$ is bounded. This is a contradiction.

Reference

[1] Yasuo Akizuki: Theory of harmonic integral, (Japanese) I, II. Iwanami, Japan, 1958.
[2] Bishop: Condition for the analyticity of certain sets. Mich. Math. Jour., 11 (1964), 289–304.
[3] De Rham: Variété differentiables. Herman Paris, 1955.
[4] Gabriel Stolzenberg: Volume, limits, and extension of analytic varieties. Lecture note in Math., no. 19, 1966. Springer Verlag.
[5] Lelong: Integration of a differential form on an anayltic subvariety. Proc. Nat. Acad., 43 (1957)., 46–49.
[6] Lelong: Intégration sur un ensemble analytique complexe. Bull. Soc. Math. France, 85 (1957).